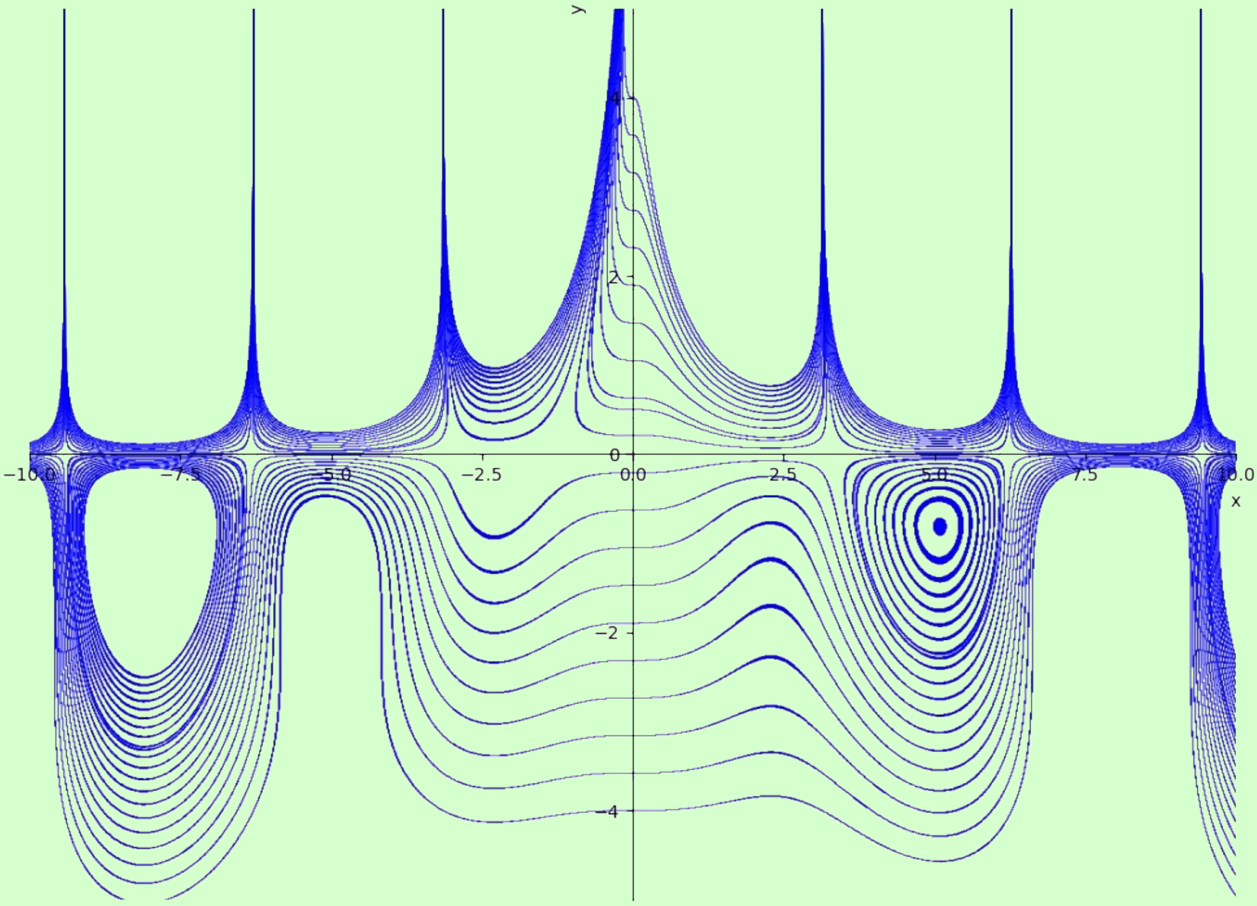
Ordinary Differential Equations Course



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Preface

This booklet was originally written in Hebrew in 2018, following many years of experience teaching the course "Ordinary Differential Equations" at the Technion Faculty of Mathematics. It covers the fundamental material in this subject required for engineering courses in the various faculties of the Technion, and may also be suitable for similar courses at colleges and other academic institutions. It has undergone many revisions and improvements and has continued to evolve over the years. The idea of translating it into English, thereby making it accessible to a much larger audience than the Hebrew-speaking population, was not realized because of the considerable effort and time required. Recently, artificial intelligence agents such as ChatGPT have demonstrated a remarkable ability to translate text with exceptional speed and efficiency. We therefore decided to conduct a feasibility test on several chapters of the Complex Functions course booklet (Hebrew). The immediate result was excellent: within a few hours, the entire booklet had been translated, and it was published within several days, after a small number of corrections and adjustments to the English. The result can be viewed at the following link: Complex Functions Course (English). We are very thankful for the Technion Mathematics department for lending us a ChatGPT (version 5.6 Sol) subscription which enabled us to do this work.

In view of the exceptional success of the translation of the Complex Functions course booklet, we decided to translate the Ordinary Differential Equations course booklet (Hebrew) from Hebrew into English using ChatGPT. The translation was completed in less than two hours! Several more hours were devoted to corrections and adjustments to the English language, and we are pleased to present the English version of the booklet here. Since this is the first English version, we would be pleased to receive feedback from readers.

The best, most effective, and almost the only way to learn mathematics is through repeated study of the material, accompanied by solving exercises. At the end of each chapter, we have included a varied collection of exercises selected from a large body of exercise sheets and examinations given in this course over the years at the Technion and at other institutions of higher education. A comprehensive and thorough understanding of the material depends greatly on the number of exercises solved by the student.

We would be pleased to receive corrections, ideas, and suggestions for improving the booklet. We do not intend to publish it on paper or in any other printed form, but rather to continue enriching it with additional links to relevant content on the Internet. We assume that all students have access to electronic devices that allow them to view and interact with the content presented in it.

We very much hope that the material is presented in a way that will enable you, the reader, to study it effectively and thoroughly, and we wish you success in this endeavor.

The booklet is updated from time to time and can be downloaded from several websites. The official website on which the latest update can be found is: `https://www.samyzaf.com/technion/ode/ode_eng.pdf`.

Samy Zafrany, September 2026

Contents

Chapter 1

General Introduction to Ordinary Differential Equations

Special thanks to Professor Haggai Katriel of the Department of Mathematics at ORT Braude College for granting permission to use his ODE course lecture notes. A large portion of the examples and exercises in this chapter were taken from his notes.

1.1 What Are Differential Equations?

- These are equations in which the unknown is a real-valued function.
- A differential equation contains at least one derivative of an unknown function.
- **Example 1:**
  (a) $f'(x) = 2x$
  (b) $f'(x) = f(x)$
  (c) $f''(x) - f'(x) + x^2 = 5f(x)$
  (d) $f''(x)^4 + f'(x)^2 + 1 = 0$
  (e) $y''' + 5xy' - y\sin x = 3e^x$
  (f) $t^3y'''(t) - t^2y''(t) + ty'(t) - 2y(t) = 1 + \ln t$
  (g) $\dfrac{y'}{1 + yy''} = xy^3$
  (h) Newton's law of cooling
  $$T'(t) = k[T(t) - C]$$
  In this equation, the function $T(t)$ describes the temperature of a cup of hot coffee at time t in a room whose temperature is C (the assumption is that the temperature C is constant), and k is a physical constant that depends on the type of object being cooled.

Problem: Newton enters a room whose temperature is 25° with a cup of coffee whose initial temperature is 90° ($T(0) = 90$). After 10 minutes, the temperature of the coffee was 70° Celsius ($T(10) = 70$). When will the temperature of the coffee reach 30°?

Fig. 1.1: Newton's cup of coffee

Solution: This is a differential equation with an initial condition

$$\begin{cases} T'(t) = k[T(t) - 25] \\ T(0) = 90 \end{cases}$$

At present, we have no idea how to solve the problem. After another lesson or two, we will be better prepared to do so.

1.2 Applications of Differential Equations

- **Physics:** Laws of motion for bodies and particles, astronomy, waves, heat, fluid and gas dynamics, electromagnetism, thermodynamics, weather and earthquake prediction, radioactivity, nuclear energy, and more ...
- **Communications:** Signal processing, image processing, information flow (cell phones, cables, routers, television, displays)
- **Electronics:** Laws of electricity, current, voltage, capacitance, inductance, dependability (reliability)

Example 2: A Skylake processor: 4.5 billion transistors and, by a rough estimate, about 20 billion electrical wires, inside a package smaller than 1 square centimeter. Each wire is governed by several differential equations (current, capacitance, resistance, inductance, heat, durability, environmental effects). How can one handle 20 billion differential equations of the following kind?

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = V'(t)$$

C - capacitance, L - inductance, R - resistance, $I(t)$ - current at time t, $V(t)$ - voltage at time t.

And that is without saying a word about the complicated dependencies among them ... (every electrical wire in the processor may be affected by its neighboring wires).

- **Economics:** Population growth, economic growth and decline, pricing, inflation, employment and

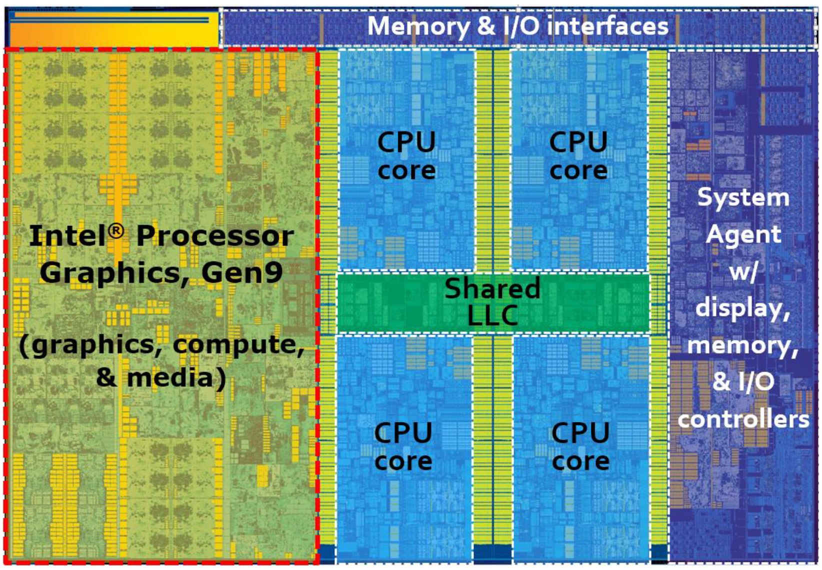

Fig. 1.2: An Intel Skylake processor

unemployment levels, banking, the stock market, and stocks.

- **Ecology:** Effects of ecological changes, growth and decline of animal populations, bacteria, viruses, and cell types. Climate effects, diseases and epidemics, vaccinations, weather prediction, and climate change.
- This is only a partial list. Further information can easily be obtained by entering the desired topic together with the words "differential equation" in a Google search.

1.2.1 A Mathematical Model for Predicting Population Size

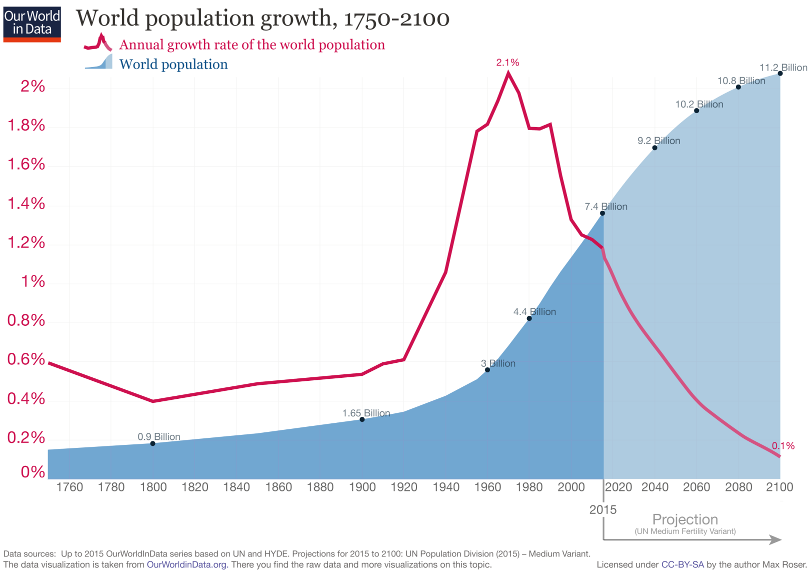

Fig. 1.3: Growth of the world population 1750-2100

- Predicting population size is very important for the future planning of housing, transportation, medical services, and similar infrastructure. It is therefore very important to construct a mathematical model that can predict (to a good approximation, if not exactly) the size of the population at every future time (at least in the near future), and, if possible, also its distribution among age groups (a more difficult problem).

- Suppose that our population consists of one million people, and that at the end of 2020 the Central Bureau of Statistics reports 21000 births and 5000 deaths. For now, we will begin with a simple, imprecise model.
- Based on these data, can we find a function $P(t)$ that predicts the population size at time t? Assume that t is the number of years since the beginning of 2020.
- **Question 1:** What will the population size be in 2060?
- According to the data, the growth rate of our population is

$$r = \frac{21000 - 5000}{1000000} = 0.016$$

Therefore, a possible formula for the population size in year n is

$$P(n) = 1000000 \cdot (1 + r)^n$$

- The answer to Question 1 is therefore:

$$P(40) = 1000000 \cdot (1 + r)^{40} = 1886897.538427$$

That is, after 40 years the population will have grown by nearly 89%.

- **Question 2:** What will the population size be after half a year? After a third of a year? A quarter of a year? A month? A day? An hour? ...
- The preceding formula assumes that the population size is updated once a year, whereas people are born and die continuously, at every instant throughout the year. This is called a discrete solution, whereas we seek a continuous solution that gives an answer for every time t, where t is a real number.
- An equation describing a more continuous model is

$$P(t + \Delta t) = P(t) + rP(t)\Delta t$$

or, equivalently,

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = rP(t)$$

Since Δt may be arbitrarily small, we obtain

$$\lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = P'(t) = rP(t)$$

We have therefore obtained a differential equation

$$P'(t) = rP(t)$$

The derivative of the population function $P'(t)$ is indeed the most precise expression of the **population growth rate**

- Later, we will prove that the general solution of the equation is

$$P(t) = Ke^{rt} \qquad (*)$$

where K is the population size at time $t = 0$:

$$P(0) = Ke^{r\cdot 0} = K = 1000000$$

- The population growth rate computed earlier ($r = 0.016$) is not exact, because it assumes a discrete growth model. We can compute the exact rate by substituting $t = 1$ into formula (*):

$$P(1) = 1000000e^{r} = 1016000$$

and therefore

$$r = \ln\frac{1016000}{1000000} = 0.01587335$$

- This formula agrees well with the preceding answer to Question 1:

$$P(40) = 1000000e^{40r} = 1886897.538427$$

But now we can also use it to compute the population growth after half a year or a quarter of a year

$$P(0.25) = 1000000e^{0.25r} = 1003976.221$$

- **Exercise 1.1:** The birth rate in Germany in 2014 was 0.00842, whereas the death rate was 0.0129. What is expected to happen to Germany's population by 2050?
- **Exercise 1.2:** Continuing the preceding exercise: When will Germany's population reach half its size in 2014?
- The same differential equation also arises in phenomena involving the decay (disintegration) of radioactive material (the number of radioactive atoms at time t), the calculation of interest on loans or bank deposits, the breakdown of a drug in the body, and the rate of a chemical reaction (for example, a chemical reaction that produces a compound C from two substances A, B).
- **Exercise 1.3:** Radioactive materials tend to decay because of the instability of atoms whose

nuclei contain an excess of neutrons, which tend to repel one another. The number of atoms that decay at any given instant is proportional to the total number of atoms. If $Q(t)$ is the quantity of radioactive material at time t, then the differential equation is

$$Q'(t) = rQ(t)$$

where r is the decay coefficient appropriate to the type of radioactive material. The term "half-life" denotes the time required for half the original quantity of the radioactive material to decay. It is known that for Carbon-14, the half-life is 5568 years. Calculate the decay coefficient r.

For further reading on radioactive decay, click the following link:

https://en.wikipedia.org/wiki/Radioactive_decay

Radioactive decay (also known as nuclear decay or radioactivity) is the process by which an unstable atomic nucleus loses energy (in terms of mass in its rest frame) by emitting radiation, such as an alpha particle, beta particle with neutrino or only a neutrino in the case of electron capture, gamma ray, or electron in the case of internal conversion. A material containing such unstable nuclei is considered radioactive. Certain highly excited short-lived nuclear states can decay through neutron emission, or more rarely, proton emission.

1.2.2 A Mathematical Model for the Motion of a Simple Pendulum

In principle, two forces act on a spherical bob of mass m (see figure 1.4)

- The force of gravity, whose magnitude is mg
- The tension force, equal to $mg\cos\theta$ (counterbalancing gravity)
- The force acting on m in the direction of the pendulum's motion is $mg\sin\theta$
- For now, we ignore air resistance, which is negligible
- We make the practical assumption that the angle θ is small, so the arc along which the mass m moves is very close to a straight line.
- Let $\theta(t)$ denote the angle between the string and the vertical at time t.

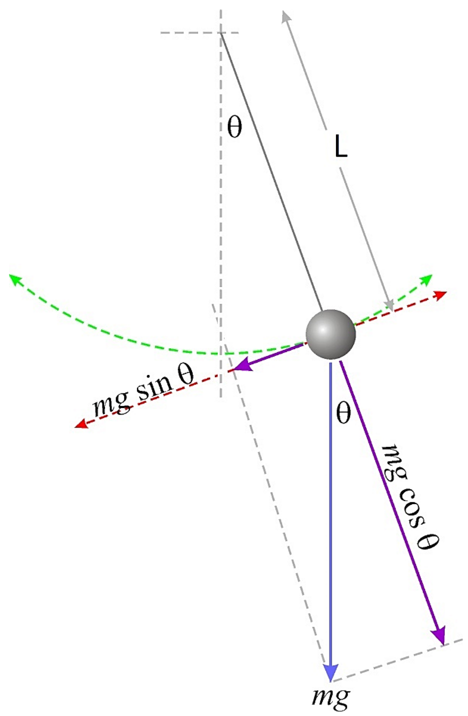

Fig. 1.4: A simple pendulum (link to Wikipedia)

- The distance traveled by m is therefore $L\cdot\theta(t)$, and hence the acceleration of m is $L\theta''(t)$.
- By Newton's second law,

$$mL\theta''(t) = -mg\sin\theta$$

(Note that the positive direction of motion is from left to right)

- We have obtained a differential equation (of order 2)

$$\theta''(t) + \frac{g}{L}\sin\theta = 0$$

Assuming that the angle θ is small, the difference between it and $\sin\theta$ is negligible, and therefore we can simplify the equation to a linear one (which is easier to solve)

$$\theta''(t) + \frac{g}{L}\theta = 0$$

We will learn how to solve linear differential equations later. For now, we note that the general solution of the equation is

$$\theta(t) = c_1 \sin\sqrt{\frac{g}{L}}t + c_2 \cos\sqrt{\frac{g}{L}}t$$

- It is customary to write $\omega = \sqrt{\frac{g}{L}}$. This is the value of the **period** (the time required to complete one oscillation). Thus, a more convenient form of the general solution is

$$\theta(t) = c_1 \sin\omega t + c_2 \cos\omega t$$

This is a two-parameter family of infinitely many distinct solutions. Of course, it can be checked directly that all these functions solve the equation by substituting them into it.

- **Question:** Is there a relationship between the number of parameters in the family of solutions and the order of the equation?

- **Programming Corner:** Programmers among us can verify the results by using SymPy (Python):

http://live.sympy.org

Mathematical programming in Python is neither a component nor a requirement of the course, but it may help in understanding the material and solving exercises.

```
from sympy import *
   t, g, L = symbols("t g L", real=True, positive=True)
   T = symbols("T", cls=Function)
de = Eq(T(t).diff(t,t) + g/L * T(t), 0)
dsolve(de, T(t))
>>> Eq(T(t), C1*sin(sqrt(g)*t/sqrt(L)) + C2*cos(sqrt(g)*t/sqrt(L)))
```

1.2.3 A Mathematical Model for Free Fall with Air Resistance

- Two forces act on a body falling freely from a certain height

A. The force of gravity, which causes the body to move toward the center of the Earth

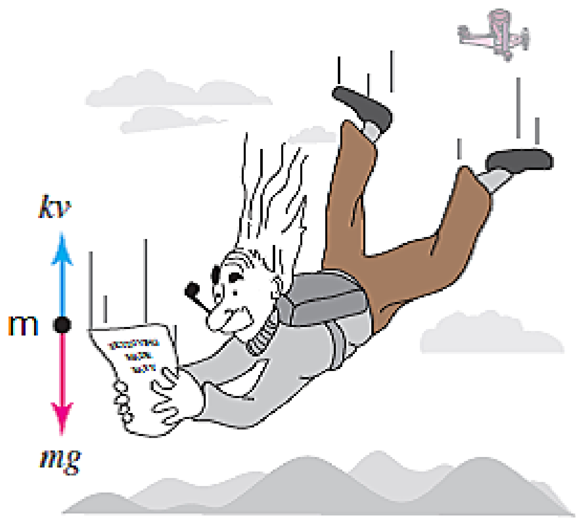

Fig. 1.5: Free fall with air resistance

B. Air resistance (friction), which acts in the direction opposite to that of the fall

- **Assumption:** The magnitude of the resistance is directly proportional to the speed of the body.
- That is: If $y(t)$ is the distance function of the falling body, and if $A(t)$ is the magnitude of the resistance force acting on the body at time t, then

$$A(t) = -kv(t) = -ky'(t)$$

where k is a physical constant that may depend on the type and shape of the material and on the air density. We note that the positive direction is the direction of the fall (downward), and therefore the air resistance $A(t)$ acts in the opposite direction.

- The total force acting on the body of mass m at time t is therefore

$$F(t) = mg - ky'(t)$$

By Newton's second law,

$$F(t) = ma(t) = my''(t)$$

We have therefore obtained a differential equation

$$my''(t) = mg - ky'(t) \tag{1.1}$$

which we will learn to solve later.

- **Exercise 1.4:** In other contexts, it is more customary to assume that the resistance force is proportional to the square of the body's speed. What will the differential equation be in this case?
- **Programming Corner:** For now, equation (1.1) can be solved using **SymPy**:

http://live.sympy.org

```
from sympy import *
   t, m, g, k = symbols("t m g k", real=True, positive=True)
   y = symbols("y", cls=Function)
de = Eq(m*y(t).diff(t,t) - m*g + k*y(t).diff(t), 0)
dsolve(de, y(t))
>>> Eq(y(t), C1 + C2*exp(-k*t/m) + g*m*t/k)
```

Thus, **SymPy** offers us the following general solution:

$$c_1 + c_2 e^{-\frac{kt}{m}} + \frac{mgt}{k}$$

As expected, the general solution is a two-parameter family of infinitely many distinct solutions. This result should be cross-checked against the result we obtain in the lesson on solving nonhomogeneous linear equations. In many cases, much can be learned from such cross-checks.

1.2.4 The Escape Velocity of a Satellite from a Gravitational Field

Fig. 1.6: An Indian record: launching 104 satellites with a single rocket

- Escape velocity is the minimum initial velocity required to break free from Earth's gravitational field and enter a stable orbit around it (or escape into outer space).
- It turns out that this velocity does not depend on the mass of the escaping body, but only on the mass of the attracting body (the Earth, in our case). The escape velocity for a body at the surface of the Earth is 11.186km/s. Naturally, we must obtain this result by solving a differential equation.

Assumptions

(a) The body has an aerodynamic structure that minimizes the effects of air resistance

(b) The body moves perpendicular to the surface of the Earth

(c) Favorable weather conditions

Notation

M Mass of the Earth

R Radius of the Earth

G Universal gravitational constant:

$$G = 6.67428 * 10^{-11}[m^3/s^2kg]$$

m Mass of the escaping satellite

$y(t)$ Distance between the center of the Earth and the satellite at time t

$v(t)$ Velocity of the satellite at time t ($v(t) = y'(t)$)

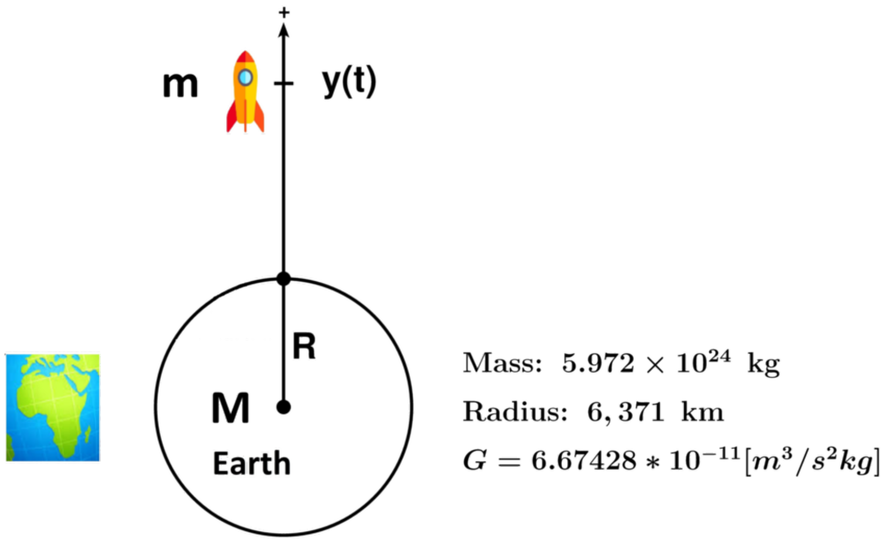

Fig. 1.7

- The force of gravity acting on a body of mass m located at a distance $y(t)$ from the center of the Earth is given by the formula
$$F(t) = -\frac{GMm}{y(t)^2}$$
(The positive direction of motion is upward, whereas gravity acts downward!)

- By Newton's second law,

$$F(t) = m \cdot a(t) = my''(t) = -\frac{GMm}{y(t)^2}$$

- We have obtained a differential equation of order 2

$$y''(t) = -\frac{GM}{y(t)^2}$$

Note that the equation does not depend on the mass m of the satellite!

- Two initial conditions must be added to this equation:

$$\begin{cases} y'' = -\dfrac{GM}{y^2} \\ y(0) = R \\ y'(0) = v_0 \end{cases}$$

where R is the radius of the Earth (the initial position of the satellite), and v_0 is the initial velocity of the satellite, which is supposed to ensure its escape from the gravitational field.

- **Question:** Is there a relationship between the order of the equation and the number of initial conditions?
- The key to solving the equation is the following equality, which allows us to reduce the order of the equation from 2 to 1 (later, we will learn similar techniques for reducing the order of an equation).
- At this stage, the solution of the equation may be skipped (we will return to it after several lessons, once we have gained more experience and knowledge)

$$y''(t) = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = v\frac{dv}{dy}$$

which leads to a new differential equation for the velocity function $v(y)$ (velocity as a function of distance!)

$$vv'(y) = -\frac{GM}{y^2}$$

which leads to the general solution

$$v(y)^2 = \frac{2GM}{y} + C$$

- Substituting $y = R$ gives

$$v(R)^2 = v_0^2 = \frac{2GM}{R} + C$$

and hence we can determine C

$$C = v_0^2 - \frac{2GM}{R}$$

- We have obtained a particular solution that depends only on the initial velocity v_0

$$v^2 = \frac{2GM}{y} + v_0^2 - \frac{2GM}{R}$$

- To ensure escape velocity, we must ensure that $v(y) \geq 0$ for every y !
- The expression $\frac{2GM}{y}$ is positive and tends to zero as y tends to infinity, so it is sufficient to ensure that

$$v_0^2 \geq \frac{2GM}{R}$$

Substituting the known physical constants G, M, R leads to the solution

$$v_0 \geq 11.186 \ [km/s]$$

- Conclusion: To ensure escape from the gravitational field, the satellite must be launched with an initial velocity of at least 11.186 kilometers per second.
- For now, the solution may be ignored (it will most likely not be understood in the first lesson). We will return to this example when we reach the topic of order reduction (if I forget, please remind me!).

1.2.5 Computing the Fixed Monthly Payment on a Mortgage Loan

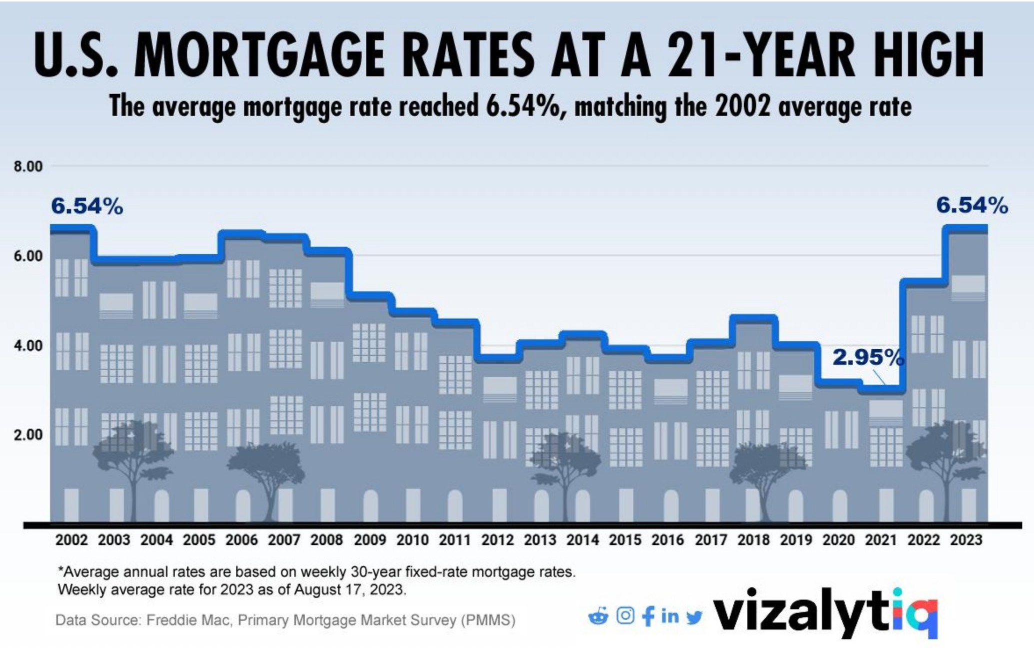

Fig. 1.8

- Suppose we took out a mortgage of one million dollars at an interest rate of 4.2% for 20 years. What is the fixed monthly payment? And what is the total amount we will repay to the bank by the end of the loan period?

- For now, we ignore the effects of indexation on the monthly payment and other incidental expenses (fees, administrative charges, and so forth)
- **Notation:**
  - t is the time variable (measured in years)
  - $M(t)$ is the size of the loan at time t
  - r is the interest rate as a real number in the interval $[0,1]$ (in the present example, $r = 0.042$)
  - P is the fixed payment per unit of time (annually, in our case)
  - n is the duration of the mortgage (in years)
- Our working assumption is that interest accrues and loan payments are made continuously (although interest is generally calculated monthly, and payments are also monthly). The resulting inaccuracy is negligible (at least in comparison with the cumulative damage).
- During the time interval from t to $t+\Delta t$, the fixed payment $P \cdot \Delta t$ must be subtracted, and the interest for the period Δt must be added

$$M(t+\Delta t) = [M(t) - P\Delta t] + [M(t) - P\Delta t] \cdot r \cdot \Delta t$$

  For example, if $\Delta t = \frac{1}{12}$ (a monthly payment), then

$$M\left(t + \frac{1}{12}\right) = \left[M(t) - \frac{P}{12}\right] + \left[M(t) - \frac{P}{12}\right] \cdot \frac{r}{12}$$

  Therefore

$$\frac{M(t+\Delta t) - M(t)}{\Delta t} = r[M(t) - P\Delta t] - P$$

  If we let Δt tend to zero, we obtain a differential equation

$$M'(t) = r \cdot M(t) - P$$

  which we will learn to solve in the next lesson.
- For now, we present the general solution without proof:

$$M(t) = Ce^{rt} + \frac{P}{r}$$

  To find the particular solution for our specific problem, we use the data

$$r = 0.042, \quad M(0) = 1000000, \quad n = 20$$

Substitute $t = 0$ and $t = 20$ into the general solution

$$\begin{cases} M(0) = C + \dfrac{P}{r} = 1000000 \\ M(20) = Ce^{20r} + \dfrac{P}{r} = 0 \end{cases}$$

Subtracting the two equalities immediately gives

$$C = \frac{M(0)}{1 - e^{20r}}$$

and substituting C into the second equality gives

$$P = \frac{rM(0)e^{20r}}{e^{20r} - 1}$$

In our specific case, we obtain

$$P = \frac{0.042 \cdot 10^6 \cdot e^{20 \cdot 0.042}}{e^{20 \cdot 0.042} - 1} = 73905.996$$

This is the fixed payment for each year. On a monthly basis, this is 6158.833.

- The total repayment to the bank is

$$20 \cdot 73905.996 = 1478119.927$$

That is, the bank earned about 48% on the investment of funds that were not its own ... (and this still excludes indexation, fees, and various insurance charges)

- Our solution is pretty close to that of the American Bank mortgage calculator, and the difference most likely arises from the discrete model (which works against the borrower?) versus the continuous model. See figure 1.9. You may find many mortgage calculators like this in the web.
- We can use the formulas obtained above to create payment tables or graphs. See table 1.1.
- **Exercise 1.5:** George from the neighborhood grocery store entered his bank branch and proposed a mortgage in which the annual (or monthly) payment decreases at a rate of 5% per year. For example, if the payment in the first year is 100,000 dollars, then the payment in the second year will be 95,000, and in the 20th year the payment will reach

$$100000 \cdot (0.95)^{19} = 35,848.59$$

If the data are as in the preceding exercise, what will the differential equation be this time? What is the annual payment schedule? And what will the total repayment to the bank be at the end

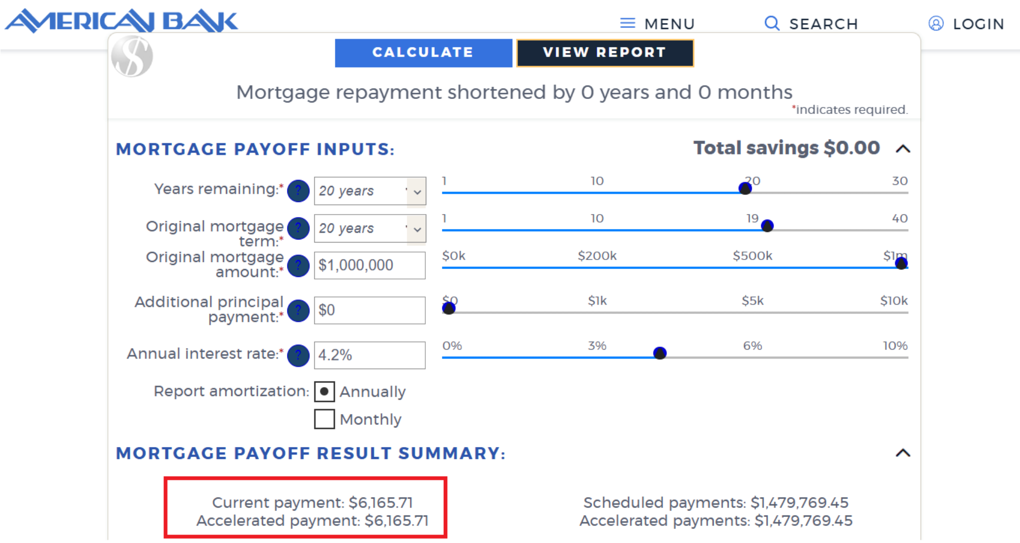

Fig. 1.9: American Bank mortgage calculator for 1 million dollars loan, 20 years, 4.2% interest

Table 1.1: Repayment table for the years 2020-2040

Year	Payment	Mortgage
2020	73906.00	1000000.00
2021	73906.00	967414.50
2022	73906.00	933431.26
2023	73906.00	897990.32
2024	73906.00	861029.17
2025	73906.00	822482.59
2026	73906.00	782282.57
2027	73906.00	740358.19
2028	73906.00	696635.49
2029	73906.00	651037.32
2030	73906.00	603483.25
2031	73906.00	553889.37
2032	73906.00	502168.18
2033	73906.00	448228.45
2034	73906.00	391974.99
2035	73906.00	333308.57
2036	73906.00	272125.69
2037	73906.00	208318.40
2038	73906.00	141774.13
2039	73906.00	72375.47
2040	73906.00	0.00

of the loan period?

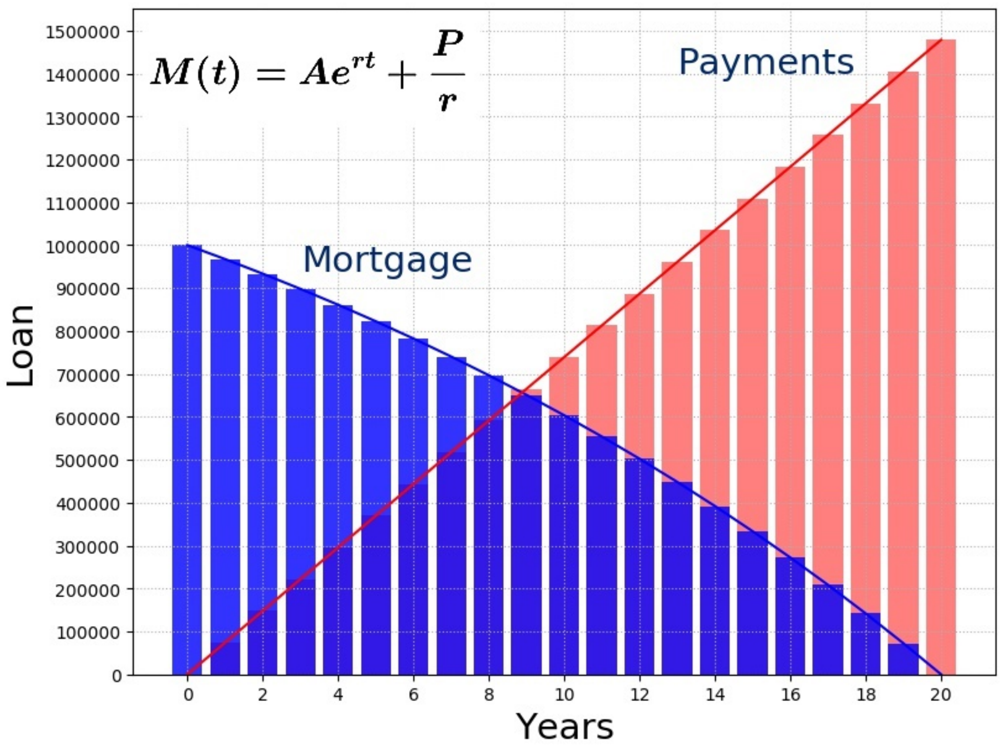

Fig. 1.10: Bar chart of the mortgage balance and repayment amount over the years

1.3 Classification of Differential Equations

- The primary classification of differential equations is determined by the order of the highest derivative appearing in the equation.
- If the highest derivative has order n, then the equation is called **a differential equation of order n**. In general, the difficulty of solving an equation increases with its order, and this primary classification is therefore necessary.
- A differential equation contains at least one derivative of an unknown function, which we will usually denote by $y(x)$ (or by $y(t)$). The symbol y denotes two things in different contexts: a generic function name (the unknown), and the dependent variable y.
- The general form of a differential equation of order n is represented by

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

In certain contexts, a constant is left on the right-hand side

$$F(x, y, y', y'', \dots, y^{(n)}) = C$$

This general form is also called the ”implicit form“.

- In introductory courses on the subject, it is customary to restrict attention to the normal form in which the highest derivative can be isolated as a function of all the other derivatives and x

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)})$$

Therefore, the general normal form of a first-order differential equation considered in this course is

$$y' = f(x, y)$$

Examples of first-order differential equations in normal form

A. $y' = xy^2$

B. $y' = \dfrac{xy}{3 + x \sin xy}$

C. $y' = y^3$

Exercise 1.6: Find the order of each of the following equations and put it into normal form, if possible:

A. $f'(x) = 2x$

B. $f'(x) = f(x)^5$

C. $y'' - (y')^7 + x^2 = 5y$

D. $f''(x)^4 + f'(x)^2 + 1 = \ln f''(x)$

E. $\dfrac{y'}{1 + y^8 y''} = xy^5$

F. $5xy' - y \sin x - x^8 y''' = 3e^x$

The second classification of differential equations is based on their algebraic characterization. The simplest algebraic form is the linear form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

or, equivalently,

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x) \tag{1.2}$$

An equation of the form (1.2) is called a **linear differential equation of order n**

Example 3: The equation

$$x^2 y'' - 3xy' + 4y = 0$$

is a linear equation of order 2.

Example 4: The equation

$$y''' + 2e^x y'' + yy' = x^4$$

is not linear!

1.4 What Is a Solution of a Differential Equation?

The second concept we must define is what constitutes a solution of a differential equation. We consider the explicit form

$$y^{(n)} = f(x, y', y'', \dots, y^{(n-1)}) \tag{1.3}$$

A function $\varphi(x)$ is called a **solution of equation (1.3) on the interval** (α, β) if at every point x in the interval (α, β), the function $\varphi(x)$ is n times differentiable and satisfies the equality

$$\varphi^{(n)}(x) = f(x, \varphi'(x), \varphi''(x), \dots, \varphi^{(n-1)}(x))$$

We emphasize that this is an open interval, whose endpoints may be $\pm\infty$.

Example 5: The function $\varphi(x) = x^2 \ln x$ is a solution of the equation

$$x^2 y'' - 3xy' + 4y = 0$$

on the interval $(0, \infty)$. This can be checked simply by substitution.

Example 6: The functions $\varphi_1(x) = \sin x$, $\varphi_2(x) = \cos x$ are solutions of the equation

$$y'' + y = 0$$

on the entire real line $(-\infty, \infty)$. In fact, every linear combination of $\varphi_1(x)$ and $\varphi_2(x)$ solves the equation

$$\varphi(x) = c_1 \sin x + c_2 \cos x$$

This is easily checked by substitution. We have therefore found infinitely many distinct solutions of the equation. Such a set of solutions is also called a **two-parameter family of infinitely many solutions**.

Given a differential equation

$$y^{(n)} = f(x, y', y'', \dots, y^{(n-1)})$$

we wish to answer the following questions:

A. Does the equation have any solution?

B. If so, how many solutions are there?

C. How can we find <u>all</u> the solutions?

D. If there are infinitely many solutions, how can we represent all of them in a minimal form?

- A general solution of an ordinary differential equation is usually an expression of the form

$$F(x, y, c_1, c_2, \ldots, c_n) = 0 \tag{1.4}$$

A. where c_1 , c_2 , $\ldots$, c_n are free parameters.

B. Every arbitrary choice of c_1 , c_2 , $\ldots$, c_n determines a specific solution of the equation. The expression (1.4) thus defines an **n-parameter family** of infinitely many solutions.

C. In most cases, the resulting expression is in "implicit" form, and additional effort is required to transform it into "explicit form":

$$y(x) = G(x, c_1, c_2, \ldots, c_n)$$

The implicit function theorem from Calculus 2 guarantees the existence and uniqueness of an explicit solution on a certain domain, but the solution cannot always be isolated explicitly (most algebraic problems still cannot be solved). In such cases, we must settle for implicit solutions and use numerical and other methods to compute the function's values at desired points, differentiate it, graph it, and so forth.

D. We note that the family of solutions (1.4) may "miss" some of the solutions, and in some cases two families of this kind (or more) will be required to express **<u>all the solutions</u>** of the equation.

- **Example 7:** A general solution of a differential equation of order 2 may look like this:

$$\sin(3x + c_1 y) + y^2 e^{c_2 x} \cos(2x + y) = 1$$

It does not appear possible to isolate an explicit solution (at least not by the methods known to us at this stage).

Exercises for chapter 1

1. What is the order of each of the following differential equations?

(a) $f''(x) - [f'(x)]^3 + x^2 = 5f(x)$

(b) $f''(x)^4 + f'(x)^2 + 1 = 0$

(c) $y''' + 5xy' - y\sin x = 3e^x$

(d) $t^3y'''(t) - t^2y''(t) + ty'(t) - 2y(t) = 1 + \ln t$

(e) $\dfrac{y'}{1+yy''} = xy^3$

2. For each differential equation, determine whether it is linear.

(a) $y''' + 2x = y$

(b) $y'' + 2y = \frac{1}{y} + \pi$

(c) $(x+y')^2 = (x-y')^2 + 1$

(d) $\sqrt{x}y'' - (1+x)y' + y\sin x = e^x$

3. Prove that $y(x) = \sqrt{x^2+C}$ is a solution of the differential equation $yy' = x$ on the entire real line.

4. Prove that $y(x) = xe^{-x}$ is a solution of the differential equation

$$y'' + 2y' + y = 0$$

Are there any other solutions? (Try multiplying the preceding solution).

5. Find a general solution of the following differential equations.

(a) $y' = 5$

(b) $y' = 6x^2 + 1\,, \quad -\infty < x < \infty$

(c) $\dfrac{y'}{x} = \dfrac{2}{x^2+1}\,, \quad 0 < x < \infty$

(d) $(y')^2 = 8y' - 15$

(e) $\dfrac{y'}{\sqrt{1-x^2}} = 2x\,, \quad -1 < x < 1$

(f) $y'' = 1 + \sqrt{y''}$

6. Solve the initial-value problem

(a) $\begin{cases} y' = 6x^2 + 1 \\ y(3) = 1 \end{cases}$

(b) $\begin{cases} y'' = 6x + 12 \\ y(0) = 2 \\ y'(0) = 3 \end{cases}$

(c) $\begin{cases} (y')^2 = 8y' - 15 \\ y(2) = 10 \end{cases}$

In the last part, make sure that you obtain two distinct solutions!

Chapter 2

First-Order Linear Differential Equations

- The general form of a first-order linear equation is

$$a_1(x)y' + a_2(x)y = g(x) \tag{2.1}$$

or, in normal form,

$$y' + a(x)y = b(x) \tag{2.2}$$

Working assumption: $a(x)$ and $b(x)$ are continuous functions on the open interval (α, β).

- The simplest linear equation is

$$y' = g(x)$$

and its general solution is

$$y(x) = \int g(x)dx + C$$

History: On November 7, 1675, Leibniz wrote the integral sign for the first time: $\int y\,dy = \frac{y^2}{2}$

- Before proceeding to the general solution of equation (2.2), we illustrate the idea behind the solution with an example:

$$y' + \frac{1}{x}y = 3x$$

After multiplying both sides by x, we obtain

$$xy' + y = 3x^2$$

The left-hand side reminds us of the product rule from Calculus 1

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'x()$$

In our case: $u(x) = x$, $v(x) = y(x)$

$$[xy]' = y + xy' = 3x^2$$

and therefore

$$xy = x^3 + C$$

We have obtained a general solution (a one-parameter family of infinitely many solutions) of our equation:

$$y(x) = x^2 + \frac{C}{x}$$

At this stage, it is not clear whether our equation has additional solutions not included in this family.

2.1 The Integrating Factor Method

- The preceding example outlines the procedure for finding the general solution of equation (2.2): We seek a function $\mu(x)$ with the following property:

$$[\mu(x)y]' = \mu(x)[y' + a(y)y] \tag{2.3}$$

A function $\mu(x)$ satisfying the last equality is called an ***integrating factor*** of equation (2.2).

- If we can find at least one integrating factor (of course, one is all we need), then equation (2.2) becomes

$$[\mu(x)y]' = \mu(x)b(x)$$

which is easy to solve.

- We continue to develop equality (2.3),

$$\mu'(x)y + \mu(x)y' = \mu(x)y' + \mu(x)a(x)y$$

and therefore

$$\mu'(x)y = \mu(x)a(x)y$$

and therefore

$$\frac{\mu'(x)}{\mu(x)} = a(x)$$

- To continue the search, we add a working assumption: $\mu(x) > 0$ on the interval (α, β).
As noted, our goal is to find one of the integrating factors, not all of them! Therefore, if we can find such a factor despite our restrictive working assumptions, our problem will be solved.

- Under the working assumption that our integrating factor is positive on the interval (α, β), we may write

$$[\ln \mu(x)]' = a(x)$$

and therefore our integrating factor is

$$\mu(x) = e^{\int a(x)\,dx}$$

(Again, the expression $\int a(x)\,dx$ has infinitely many solutions, but one of them is sufficient for our purposes)

- From the equality

$$[\mu(x)y]' = \mu(x)b(x)$$

it follows that

$$\mu(x)y = \int \mu(x)b(x)\,dx + C$$

and therefore the general solution of the first-order linear equation (2.2) is

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)b(x)\,dx + C\right] \tag{2.4}$$

All the solutions are defined on the interval (α, β) because the functions $a(x)$ and $b(x)$ are continuous there and, in addition, $\mu(x) > 0$ throughout the interval. ∎

- Question: Is this really the general solution? Could there be additional solutions not included in it that might be obtained by other methods? We will prove later that the solution found above is indeed general. There are, however, differential equations for which this question has no answer, but even then, finding an infinite family of solutions is a significant achievement.

- **Example 1:** Find the general solution of the equation

$$xy' + y = x^3$$

Solution: We put the equation into normal form:

$$y' + \frac{1}{x}y = x^2$$

Therefore $a(x) = \frac{1}{x}$, $b(x) = x^2$, and the integrating factor is

$$\mu(x) = e^{\int a(x)\,dx} = e^{\int \frac{1}{x}} = e^{\ln x} = x$$

Therefore

$$\begin{aligned} y(x) &= \tfrac{1}{\mu(x)}\left[\int \mu(x)b(x)\,dx + C\right] \\ &= \tfrac{1}{x}\left[\int x^3\,dx + C\right] \\ &= \tfrac{1}{x}\left[\tfrac{x^4}{4} + C\right] = \tfrac{x^3}{4} + \tfrac{C}{x} \end{aligned}$$

Thus the general solution is

$$y(x) = \frac{x^3}{4} + \frac{C}{x}$$

- It is advisable to verify that the general solution obtained does indeed solve the equation by direct substitution (especially in examinations).

$$y'(x) = \frac{3x^2}{4} - \frac{C}{x^2}$$

Therefore

$$xy' + y = \left(\frac{3x^3}{4} - \frac{C}{x}\right) + \frac{x^3}{4} + \frac{C}{x} = x^3$$

The following is a graph of the family of solutions obtained:

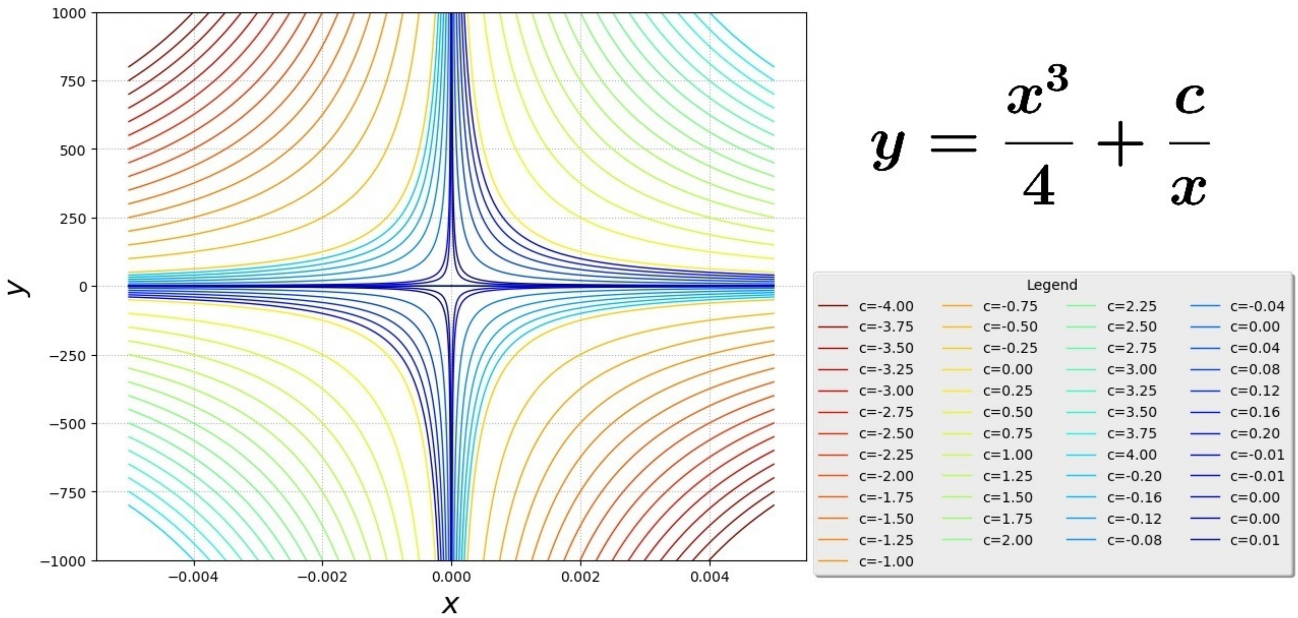

Fig. 2.1: The family of solutions of the equation $xy' + y = x^3$

- Question: What is the domain of existence of the general solution?

- Note the following details, which will become clearer later when we reach the ***existence and uniqueness theorem***

  (a) Exactly one solution passes through every point in the plane (within the domain of existence of the equation)

(b) No two distinct solutions intersect at any point

(c) The family of solutions covers the entire plane (except for points at which the equation is not defined)

Example 2: Find a particular solution of the following initial-value problem:

$$\begin{cases} xy' + 2y = \sin x, \quad (0 < x < \infty) \\ y(\pi) = 0 \end{cases}$$

Solution: We begin by finding the general solution of the equation and only then attempt to find the particular solution. Normal form:

$$y' + \frac{2}{x} y = \frac{\sin x}{x}$$

Therefore $a(x) = \frac{2}{x}$, $b(x) = \frac{\sin x}{x}$, and the integrating factor is

$$\mu(x) = e^{\int a(x)\,dx} = e^{\int \frac{2}{x}} = e^{2\ln x} = x^2$$

We substitute into the general-solution formula (2.4)

$$\begin{aligned} y(x) &= \tfrac{1}{\mu(x)} \left[\int \mu(x) b(x)\,dx + C \right] = \tfrac{1}{x^2} \left[\int x \sin x\,dx + C \right] \\ &= \tfrac{1}{x^2} \left[-x\cos x + \sin x + C \right] = \tfrac{1}{x^2} \left[\sin x - x\cos x + C \right] \end{aligned}$$

Thus the general solution of the equation is

$$y(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} + \frac{C}{x^2}$$

Particular solution:

$$y(\pi) = \frac{\sin \pi}{\pi^2} - \frac{\cos \pi}{\pi} + \frac{C}{\pi^2} = 0 - \frac{-1}{\pi^2} + \frac{C}{\pi^2} = 0$$

Therefore $C = -\pi$, and hence the particular solution is

$$y_p(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} - \frac{\pi}{x^2}$$

The following is a graph of the family of solutions obtained:

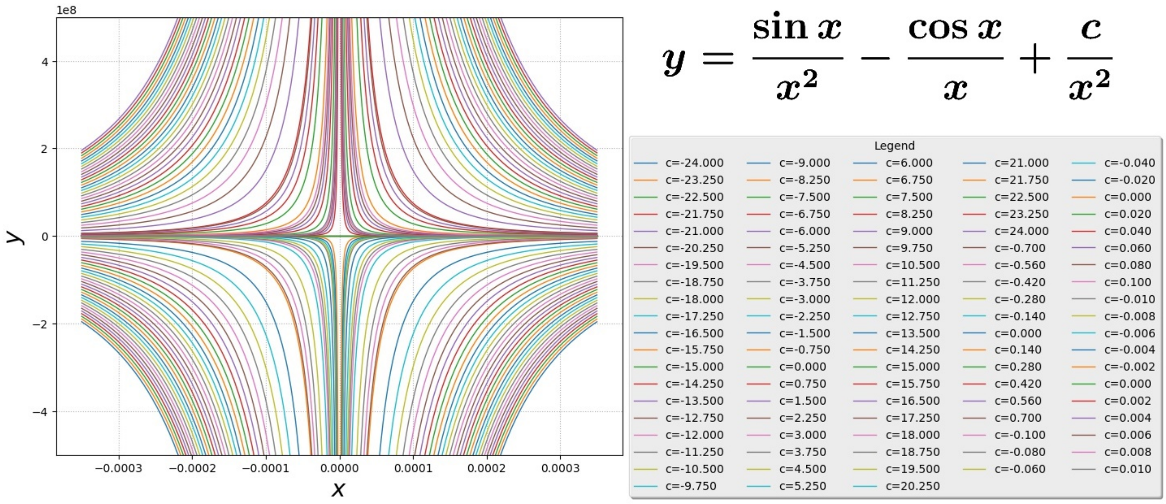

Fig. 2.2: The family of solutions of the equation $xy' + 2y = \sin x$

2.2 The Existence and Uniqueness Theorem for a First-Order Linear Equation

Theorem 2.1: Consider a first-order linear equation with an initial condition:

$$\begin{cases} y' + a(x)y = b(x), & (\alpha < x < \beta) \\ y(x_0) = y_0 \end{cases} \tag{2.5}$$

where

A. $a(x)$ and $b(x)$ are continuous functions on the interval (α, β)

B. $\alpha < x_0 < \beta$

Then there exists a ***unique*** function $y(x)$ that solves the equation on the interval (α, β) and also satisfies $y(x_0) = y_0$.

Proof: The existence of a general solution was already proved above by the integrating-factor method.

$$y(x) = \frac{1}{\mu(x)} \left[\int \mu(x) b(x)\, dx + C \right]$$

We proceed to find a particular solution. Let

$$B(x) = \int \mu(x) b(x)\, dx$$

Then

$$y(x) = \frac{1}{\mu(x)} \left[B(x) + C \right]$$

Substitute the point $x = x_0$ to obtain

$$y_0 = \frac{1}{\mu(x_0)}\,[B(x_0) + C]$$

Therefore, if we take

$$C_0 = y_0\mu(x_0) - B(x_0)$$

we obtain a particular solution y_p satisfying $y_p(x_0) = y_0$

$$y_p(x) = \frac{1}{\mu(x)}\,[B(x) + C_0]$$

It remains to prove the ***uniqueness of the solution***. Suppose that $z = z(x)$ is another solution of the initial-value problem (2.5), obtained by a new method that was previously unknown to us. Then $z(x)$ must satisfy the equality

$$[\mu(x)z(x)]' = \mu(x)[z'(x) + a(x)z(x)]$$

because we proved above that the integrating factor $\mu(x)$ must satisfy this equality with ***every solution*** of the equation $y' + a(x)y = b(x)$, and therefore there must be a constant C_1 such that

$$z(x) = \frac{1}{\mu(x)}\left[\int \mu(x)b(x)\,dx + C_1\right]$$

that is,

$$z(x) = \frac{1}{\mu(x)}\,[B(x) + C_1]$$

Substitute the point $x = x_0$ to obtain

$$z(x_0) = y_0 = \frac{1}{\mu(x_0)}\,[B(x_0) + C_1]$$

and therefore

$$C_1 = y_0\mu(x_0) - B(x_0)$$

But we have seen that

$$C_0 = y_0\mu(x_0) - B(x_0)$$

Therefore $C_1 = C_0$, and hence the function $z(x)$ is identical to the function $y_p(x)$. Thus, $y_p(x)$ is the unique solution of the initial-value problem (2.5). ∎

- Existence and uniqueness theorems also hold for several other types of equations that we will encounter later. Their importance lies, first, in guaranteeing that a given problem has a solution, and, second, in guaranteeing the uniqueness of that solution. In fields such as physics, biology, and weather forecasting, mathematical models are often constructed to predict the future behavior of various systems. Uniqueness of the solution is fundamental to the possibility of making such predictions.

- A precise understanding of what the existence and uniqueness theorem states is important in order to avoid errors and confusion. We present several examples.

- **Example 3:** Consider the initial-value problem

$$\begin{cases} xy' = 2y \\ y(0) = 0 \end{cases}$$

It is easy to verify that it has infinitely many distinct solutions of the following form:

$$y(x) = \begin{cases} c_1x^2, & -\infty < x \leq 0 \\ c_2x^2, & 0 \leq x < \infty \end{cases}$$

Note the following points:

A. All the solutions are continuously differentiable on the entire real line!

B. Although this is a first-order equation, it has a two-parameter family of solutions!

C. Each function in this family also satisfies the initial condition $y(0) = 0$, contrary to the assertion of the existence and uniqueness theorem.

D. The initial-value problem

$$\begin{cases} xy' = 2y \\ y(1) = 3 \end{cases}$$

also has infinitely many solutions!

$$y(x) = \begin{cases} c_1x^2, & -\infty < x \leq 0 \\ 3x^2, & 0 \leq x < \infty \end{cases}$$

At first glance, this example seems to invalidate the existence and uniqueness theorem completely, until we observe that the existence and uniqueness theorem applies only to linear equations in normal form! The normal form of our equation is

$$y' - \frac{2}{x}y = 0$$

The function $a(x) = -\frac{2}{x}$ is not continuous on the entire real line, as required by the existence and uniqueness theorem! If we restrict the problem to the interval $(0, \infty)$ or to the interval

$(-\infty, 0)$, then the theorem applies in full. For example, the initial-value problem

$$\begin{cases} xy' = 2y, & (0 < x < \infty) \\ y(1) = 3 \end{cases}$$

has exactly one solution, namely $y_p(x) = 3x^2$, on the interval $(0, \infty)$. We leave the details to the student.

2.3 Newton's Cup of Coffee: Solution

We return to the equation expressing Newton's law of cooling, which we presented in the introductory lecture:

$$T'(t) = k[T(t) - C]$$

In this equation, the function $T(t)$ describes the temperature of a cup of hot coffee at time t in a room whose temperature is C (the assumption is that the temperature C is constant), and k is a physical constant that depends on the type of object being cooled.

The Problem: Newton enters a room whose temperature is $25°$ with a cup of coffee whose initial temperature is $90°$ ($T(0) = 90$). After 10 minutes, the temperature of the coffee was $70°$ Celsius ($T(10) = 70$). When will the temperature of the coffee reach $30°$?

Fig. 2.3: Newton's cup of coffee

Solution: At this stage, we can easily see that this is a first-order linear differential equation with an initial condition:

$$\begin{cases} T'(t) = k[T(t) - 25] \\ T(0) = 90 \end{cases}$$

It is preferable to write the equation in normal form:

$$\begin{cases} T'(t) - kT(t) = -25k \\ T(0) = 90 \end{cases}$$

In ordinary physical problems, the physical constant k is generally known in advance, but in this case we will take the opportunity to show how it can be determined from a single simple measurement: $T(10) = 70$. First, however, we find the general solution: In this case, $a(t) = -k$, $b(t) = -25k$, and therefore our integrating factor is

$$\mu(t) = e^{\int a(t)dt} = e^{-kt}$$

The general solution is

$$\begin{aligned} T(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)b(t)\,dt + C\right] = e^{kt}\left[\int \left(-25ke^{-kt}\right)dt + C\right] \\ &= e^{kt}\left[25e^{-kt} + C\right] = 25 + Ce^{kt} \end{aligned}$$

Thus, the general solution of the differential equation is

$$T(t) = 25 + Ce^{kt}$$

Using the initial condition $T(0) = 90$ and the measurement $T(10) = 70$, we obtain a system of two equations in the two unknowns k, C

$$\begin{aligned} T(0) &= 25 + C = 90 \\ T(10) &= 25 + Ce^{10k} = 70 \end{aligned}$$

The first equation gives $C = 65$, and the second equation gives

$$k = \frac{1}{10} \cdot \ln\frac{45}{65} = -0.03677$$

Therefore, the cooling function of the cup of coffee is

$$T(t) = 25 + 65e^{-0.03677\,t}$$

where t is the time variable measured in minutes. At a glance, we see that the cooling function $T(t)$ tends asymptotically to the room temperature $25°$ (see Figure 2.4).
To determine when the temperature of the coffee reaches $30°$, we solve the equation

$$30 = 65e^{-0.03677\,t} + 25$$

which is equivalent to: $e^{-0.03677\,t} = \frac{5}{65}$, and obtain

$$t = \frac{1}{-0.03677\,t} \cdot \ln\frac{5}{65} = 69.756$$

That is, the cup of coffee will reach a temperature of $30°$ after approximately 70 minutes (enough time to drink the coffee slowly and discover another physical law).

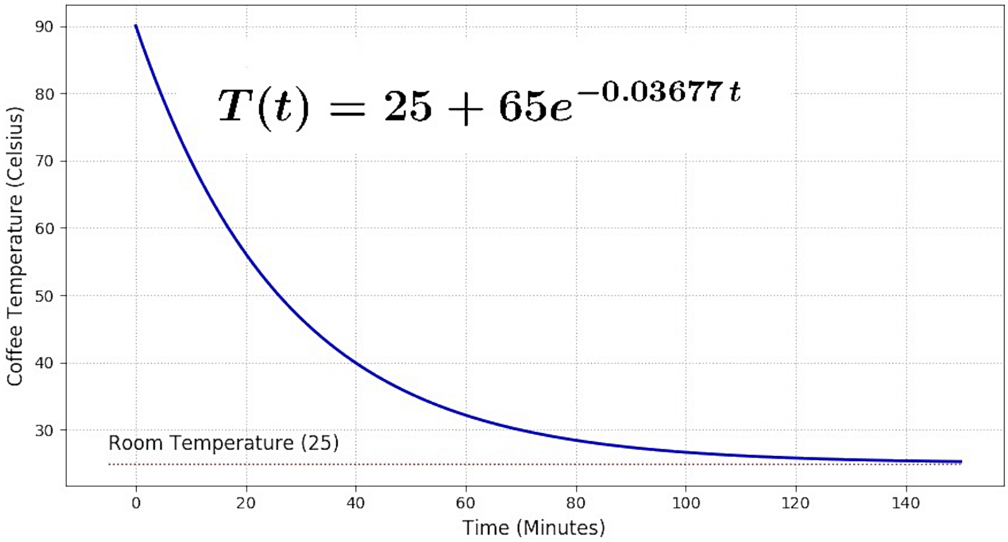

Fig. 2.4: The temperature function of the cup of coffee from $t = 0$ to $t = 150$ (minutes)

2.4 Computing the Fixed Monthly Payment on a Mortgage Loan – Solution

- We return to the mortgage equation presented in the introductory lecture. We restate the problem: Suppose we took out a mortgage of one million NIS at an interest rate of 4.2% for 20 years. What is the fixed monthly payment? And what is the total amount we will repay to the bank by the end of the loan period?

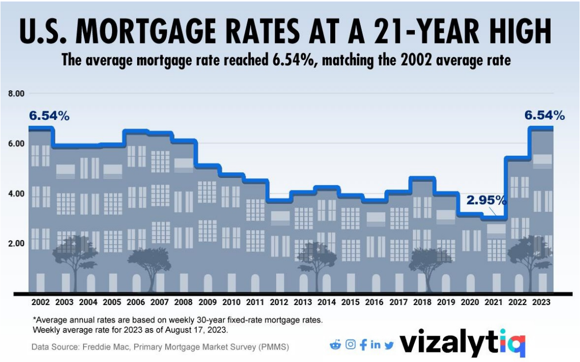

Fig. 2.5: Average interest rate for a 20-year mortgage

- **Notation**
  - t is the time variable (measured in years)
  - $M(t)$ is the size of the loan at time t
  - r is the interest rate as a real number in the interval $[0, 1]$ (in the present example, $r = 0.042$)
  - P is the fixed payment per unit of time (annually, in our case)
  - $years$ is the duration of the mortgage (years)
- The differential equation obtained in the introductory lecture is

$$M'(t) = rM(t) - P$$

This is, of course, a first-order linear differential equation, which we are now able to solve.

First, we write the equation in normal form:

$$M'(t) - rM(t) = -P$$

Therefore

$$a(t) = -r, \quad b(t) = -P$$

Computing the integrating factor:

$$\mu(t) = e^{\int a(t)dt} = e^{-rt}$$

General solution:

$$\begin{aligned} M(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)b(t)\,dt + C\right] = e^{rt}\left[\int \left(-Pe^{-rt}\right)dt + C\right] \\ &= e^{rt}\left[\frac{P}{r}e^{-rt}dt + C\right] = \frac{P}{r} + Ce^{rt} \end{aligned}$$

We have obtained the general solution for computing the mortgage balance at time t (measured in years!)

$$M(t) = Ce^{rt} + \frac{P}{r}$$

To find the particular solution for our problem, we use the data

$$r = 0.042, \quad M(0) = 1000000, \quad \text{years} = 20$$

Substitute $t = 0$ and $t = 20$ into the general solution to obtain a system of two equations in the two unknowns P, C

$$\begin{cases} M(0) = C + \dfrac{P}{r} = 1000000 \\ M(20) = Ce^{20r} + \dfrac{P}{r} = 0 \end{cases}$$

Subtracting the two equalities immediately gives

$$C = \frac{10^6}{1 - e^{20\cdot 0.042}} = -759666.58$$

and substituting C into the second equality gives

$$P = \frac{rM(0)e^{20r}}{e^{20r} - 1} = \frac{0.042 \cdot 10^6 \cdot e^{20\cdot 0.042}}{e^{20\cdot 0.042} - 1} = 73905.996$$

This is the fixed payment for each year. On a monthly basis, this is:

$$\text{Monthly Payment} = \frac{P}{12} = 6158.833$$

The total mortgage repayment after 20 years (excluding indexation and additional costs) is

$$\text{Total return} = 20 * P = 1,478,119.93$$

The mortgage balance at any given time t (continuously from $t = 0$ through the end of the mortgage term at $t = 20$!) is, of course, computed using the particular-solution formula

$$M(t) = Ce^{rt} + \frac{P}{r} = -759666.58e^{0.042t} + 1759666.57$$

The last formula can be used to construct diagrams of the kind shown in Figure 2.6

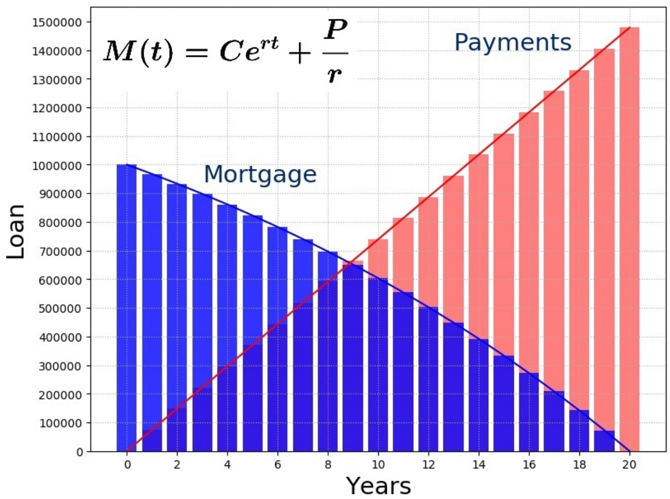

Fig. 2.6: Bar chart of the mortgage balance and repayment amount over the years

Recommended Reading

- http://tutorial.math.lamar.edu/Classes/DE/Linear.aspx
- Paul Dawkins, DE Complete, p. 24

Exercises for chapter 2

Guidance: At the end of the list, there is a link to a Jupyter notebook containing final, complete solutions to most of the exercises. But before clicking the link, it is strongly recommended that you solve them yourself!

Find a general solution of each of the following differential equations.

1. $xy' - y = -x$

2. $y' + y = e^{-x}$

3. $xy' + y = \sin x$

4. $y' - y\cot(x) = \cot(x)$

5. $x^2y' - 2xy = 3$

6. $xy' - xy = (1 + x^2)e^x$

7. $(1 + x^2)y' - 2xy = (1 + x^2)^2$

8. $xy' - ny = e^x x^{n+1}$

9. $y' - \dfrac{y}{\sin x} = \dfrac{\tan x}{2}, \quad 0 < x < \frac{\pi}{2}$

10. $y' + 2xy = 2x^3$

11. $y' - y\sin x = \sin(x)\cos(x)$

12. $xy' - 2y = 2x^4$

13. $(2x + 1)y' = 4x + 2y$

14. $y' + y\tan(x) = \sec(x)$

15. $(xy + e^x)dx - xdy = 0$

16. $x^2y' + xy + 1 = 0$

17. $y = x(y' - x\cos(x))$

18. $(2x + y)dy = ydx + 4\ln y\, dy$

19. $2x(x^2 + y)dx = dy$

20. $(xy' - 1)\ln x = 2y$

21. $(x + y^2)dy = ydx$

22. $(2e^y - x)y' = 1$

23. $\left(\sin^2 y + x\cot y\right) y' = 1$

24. $y' = \dfrac{y}{3x - y^2}$

Link to solutions

Chapter 3

First-Order Nonlinear Differential Equations

- The purpose of this chapter is to formulate sufficient conditions for the existence and uniqueness of solutions of general first-order differential equations, which are not necessarily linear. In addition, we will present various methods for solving nonlinear equations.
- Nonlinear equations are generally more difficult to solve than linear equations, and there is no uniform method for solving them. In most cases, no analytical methods are available for finding a solution of a nonlinear equation. The only way to make progress is therefore to divide them into subfamilies with common characteristics and develop special solution methods for each family.
- At the beginning of the chapter, we will learn to solve several relatively simple families of first-order nonlinear equations: separable equations (separable), equations of homogeneous type, Bernoulli equations, and Riccati equations, exact equations (exact equations), and several other families of equations that can be solved by simple substitutions.
- A substantial portion of the nonlinear equations encountered in physics and other sciences can be solved using uncomplicated mathematical methods (or at least approximated by solvable equations by relaxing the requirement of absolute precision).
- We will conclude the chapter with two formulations of the existence and uniqueness theorem for general first-order equations, and discuss its consequences and various aspects. We will also present an incomplete schematic proof of the theorem (time permitting).

3.1 Solving Separable Differential Equations

Definition 3.1: A first-order differential equation $y' = f(x, y)$ is called a **separable equation (separable)** if $f(x, y)$ can be factored into the product of a function of x and a function of y

$$y' = f(x, y) = a(x)b(y)$$

We begin with several exercises illustrating the technique and then provide a theoretical justification for the technique itself.

Example 1: We solve the equation

$$y' = \frac{x^2}{1 + y^2}$$

We switch to Leibniz notation:

$$\frac{dy}{dx} = \frac{x^2}{1 + y^2}$$

We separate variables:

$$(1 + y^2)dy = x^2 dx$$

We integrate:

$$\int (1 + y^2)dy = \int x^2 dx$$

We obtain

$$y + \frac{y^3}{3} = \frac{x^3}{3} + C$$

We omitted the constant on the left-hand side because it can be absorbed into the constant on the right-hand side. We have obtained a one-parameter family of solutions (in implicit form)

$$y^3 + 3y - x^3 = C$$

Example 2: We solve the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{3x^2 + 4x + 2}{2(y - 1)} \\ y(0) = -1 \end{cases}$$

Separation of variables:

$$2(y - 1)dy = (3x^2 + 4x + 2)dx$$

Integration:

$$(y - 1)^2 = x^3 + 2x^2 + 2x + C$$

In this case, a one-parameter family of solutions can be obtained in explicit form:

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$

Initial condition:

$$y(0) = 1 \pm \sqrt{C} = -1$$

We have obtained an explicit particular solution:

$$y_p(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

Exercise 3.1: Solve the following initial-value problem:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{2x}{y + x^2 y} \\ y(0) = -2 \end{cases}$$

Solution: Very easy. The particular solution should be:

$$y_p(x) = -\sqrt{2\ln(1 + x^2) + 4}$$

3.1.1 Theoretical Basis of the Separation-of-Variables Method

Consider the separable equation

$$\frac{dy}{dx} = f(x, y) = a(x)b(y)$$

Dividing both sides by the expression $b(y)$ yields the following form:

$$b(y)\frac{dy}{dx} = a(x) \tag{3.1}$$

Let

$$B(y) = \int b(y)\, dy$$

Recall that the variable y is actually a function of x, and that this is shorthand for $y(x)$. Therefore, equality (3.1) is equivalent to

$$b(y(x))y'(x) = a(x)$$

The expression $b(y(x))y'(x)$ naturally reminds us of the chain rule for differentiating the composite function $B(y(x))$ with respect to x

$$\frac{d}{dx}B(y) = \frac{d}{dy}B(y) \cdot \frac{dy}{dx} = B'(y) \cdot y'(x) = b(y) \cdot y'(x)$$

Therefore, equality (3.1) is equivalent to

$$\frac{d}{dx}[B(y)] = a(x)$$

Integrating both sides (with respect to x), we obtain

$$B(y) = \int a(x)\, dx$$

that is,

$$\int b(y)\, dy = \int a(x)\, dx$$

The last equality is the basis of the technique for solving a separable equation. After evaluating the integrals, we obtain a general solution (usually in implicit form). We emphasize again that there is no guarantee that this family includes every solution of the equation!

3.2 Substitution Methods

A considerable number of the equations encountered in physics and other sciences can be solved by simple substitutions. We will study several of them in this section.

3.2.1 Homogeneous Differential Equations

- Equations of homogeneous type are equations that can be transformed into separable equations by the substitution $v = \frac{y}{x}$. We emphasize that v is a function of x and that the substitution written in full is:

$$v(x) = \frac{y(x)}{x}$$

- **Example 3:** Find a general solution of the equation

$$y' = \frac{y - 4x}{x - y}$$

Solution: We use the substitution $v = \frac{y}{x}$ to transform the equation into a separable equation

$$\frac{y - 4x}{x - y} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}} = \frac{v - 4}{1 - v}$$

We express y' in terms of v

$$y = xv \implies y' = v + xv'$$

Therefore

$$v + xv' = \frac{v-4}{1-v}$$

and therefore

$$x\frac{dv}{dx} = \frac{v-4}{1-v} - v = \frac{v^2-4}{1-v}$$

Separation of variables:

$$\frac{(1-v)dv}{v^2-4} = \frac{dx}{x}$$

Using partial-fraction decomposition (see Calculus 1, Section 9.3), it is easy to obtain

$$\frac{(1-v)dv}{v^2-4} = \frac{-\frac{1}{4}}{v-2} + \frac{-\frac{3}{4}}{v+2}$$

Integration:

$$\int \left(\frac{-\frac{1}{4}}{v-2} + \frac{-\frac{3}{4}}{v+2} \right) dv = \int \frac{dx}{x}$$

and therefore

$$-\frac{1}{4}\ln(v-2) - \frac{3}{4}\ln(v+2) = \ln x + C$$

Simplification:

$$\ln\left[\frac{1}{(v-2)^{\frac{1}{4}}(v+2)^{\frac{3}{4}}}\right] = \ln x + C$$

Therefore

$$\frac{1}{(v-2)^{\frac{1}{4}}(v+2)^{\frac{3}{4}}} = Ce^x$$

We have obtained a general solution:

$$e^x\left[(v-2)^{\frac{1}{4}}(v+2)^{\frac{3}{4}}\right] = C$$

Substituting $v = \frac{y}{x}$, we obtain a general solution of the original equation

$$\frac{(y-2x)^{\frac{1}{4}}(y+2x)^{\frac{3}{4}}}{x}\, e^x = C$$

Figure 3.1 shows the direction field of the equation together with 50 integral curves (solutions).

Exercise 3.2: Additional examples recommended for practice:

(a) $\frac{dy}{dx} = \dfrac{y^2+2xy}{x^2}$ **(b)** $\frac{dy}{dx} = \dfrac{x^2+3xy+y^2}{x^2}$

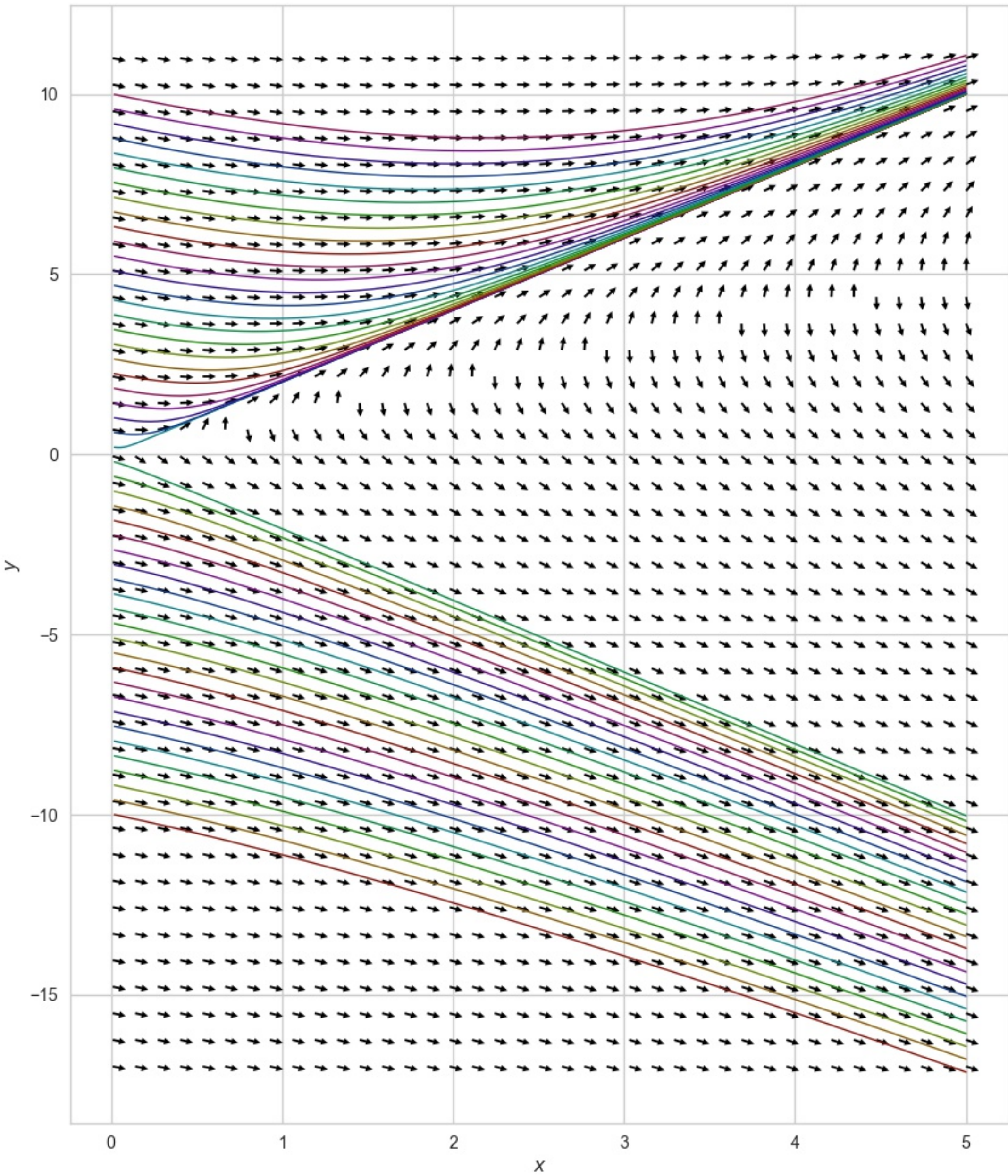

Fig. 3.1: The direction field of the equation $y' = \dfrac{y-4x}{x-y}$

3.2.2 The Bernoulli Equation (Jacob Bernoulli 1655-1705)

- The general form of the Bernoulli equation is

$$y' + p(x)y = q(x)y^n$$

It arises in physics in connection with mathematical models of particle motion through a resistive medium, and provides a generalization of such models. In physics, the common cases are $n = 2$ or $n = 3$, but the method for solving Bernoulli equations also handles noninteger values of n. More information is available at the following links:

Wikipedia page: Bernoulli equation

Bernoulli Equation - HyperPhysics Concepts

- The equation is solved by considering several cases.
- If $n = 0$, we obtain a first-order linear equation

$$y' + p(x)y = q(x)$$

which we learned to solve in the preceding chapter.

- If $n = 1$, we obtain

$$y' + p(x)y = q(x)y$$

and therefore

$$y' + [p(x) - q(x)]y = 0$$

Once again, we have obtained a linear (homogeneous) equation, which is even separable.

- If $n \neq 0, 1$, we perform a **change of variable** that transforms the problem into a linear one:

$$v(x) = y(x)^{1-n}$$

Therefore, we may also write

$$y(x) = v(x)^{\frac{1}{1-n}}$$

Differentiate both sides:

$$y'(x) = \frac{1}{1-n} \cdot v(x)^{\frac{1}{1-n}-1} \cdot v'(x) = \frac{1}{1-n} \cdot v(x)^{\frac{n}{1-n}} \cdot v'(x)$$

Substitute into the equation:

$$\frac{1}{1-n} v(x)^{\frac{n}{1-n}} v'(x) + p(x)v(x)^{\frac{1}{1-n}} = q(x)v(x)^{\frac{n}{1-n}}$$

Divide both sides by the expression $\frac{1}{1-n} v(x)^{\frac{n}{1-n}}$, to obtain the linear equation

$$v'(x) + (1-n)\,p(x)\,v(x) = (1-n)q(x) \tag{3.2}$$

Example 4: Find a general solution of the equation

$$x^2y' + 2xy - y^3 = 0$$

Solution: It is easy to see that this is a Bernoulli equation with $n = 3$

$$y' + \frac{2}{x}y = \frac{1}{x^2}y^3$$

Therefore, the required substitution is

$$v = y^{1-n} = \frac{1}{y^2}, \quad p(x) = \frac{2}{x}, \quad q(x) = \frac{1}{x^2}$$

Substitution into formula (3.2) gives a linear equation

$$v'(x) - \frac{4}{x}v(x) = -\frac{2}{x^2}$$

Integrating factor:

$$\mu(x) = e^{\int -\frac{4}{x}} = e^{-4\ln x} = \frac{1}{x^4}$$

General solution:

$$\begin{aligned} v(x) &= \frac{1}{\mu(x)}\left[\int \mu(x)b(x)\,dx + C\right] \\ &= x^4 \int -\frac{1}{x^4}\cdot\frac{2}{x^2}\,dx = -2x^4\int x^{-6}\,dx \\ &= -2x^4\cdot\left(\frac{x^{-5}}{-5} + C\right) = \frac{2}{5x} - Cx^4 \end{aligned}$$

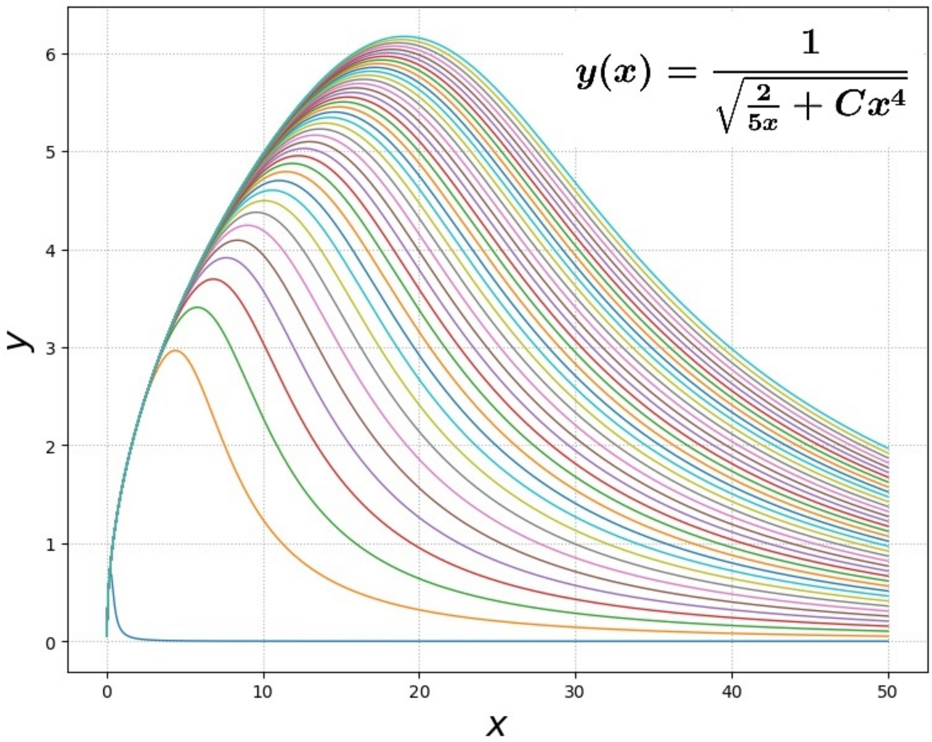

Fig. 3.2: The family of solutions $\frac{1}{\sqrt{\frac{2}{5x}+Cx^4}}$

The general solution of the linear equation is therefore

$$v(x) = \frac{2}{5x} + Cx^4$$

The general solution of the original (Bernoulli) equation is

$$y(x) = [v(x)]^{-\frac{1}{2}} = \frac{1}{\sqrt{\frac{2}{5x} + Cx^4}}$$

We note that this example shows that in some cases the parameter C is not completely free; here, it is subject to the constraint that it must be positive. Figure 3.2 shows the graphs of 40 functions from the family of solutions.

3.2.3 Ratios of Linear Expressions

- The next family of equations for which the substitution method works consists of equations in which the function $f(x, y)$ is a ratio of linear expressions

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

- When $c_1 = c_2 = 0$, we obtain an equation of homogeneous type, for which the substitution $v = \frac{y}{x}$ solves the problem. In other cases, different substitutions are required.
- **Example 5:** The following equation is a simple example of a ratio of two linear expressions:

$$y' = \frac{-x + 2y + 1}{2x - y - 2}$$

It is neither linear nor homogeneous nor separable. However, the simple substitution $\hat{x} = x - 1$ transforms it into a homogeneous equation

$$\frac{dy}{d\hat{x}} = \frac{-\hat{x} + 2y}{2\hat{x} - y}$$

- The preceding substitution will not work for the equation

$$y' = \frac{-x + 2y + 1}{2x - y - 3} \tag{3.3}$$

In this case, a more general substitution is required:

$$\begin{cases} \hat{x} = x + \alpha \\ \hat{y} = y + \beta \end{cases}$$

But a small amount of effort is needed to choose suitable values of α, β.

- Our goal is to eliminate the free constants:

$$\begin{aligned} -\hat{x} + 2\hat{y} &= -x + 2y + 1 \\ 2\hat{x} - \hat{y} &= 2x - y - 3 \end{aligned}$$

Therefore

$$\begin{aligned} -x - \alpha + 2y + 2\beta &= -x + 2y + 1 \\ 2x + 2\alpha - y - \beta &= 2x - y - 3 \end{aligned}$$

We have obtained a system of two equations in two unknowns:

$$\begin{aligned} -\alpha + 2\beta &= 1 \\ 2\alpha - \beta &= -3 \end{aligned}$$

whose solution is

$$\begin{aligned} \alpha &= -\tfrac{5}{3} \\ \beta &= -\tfrac{1}{3} \end{aligned}$$

Thus, the substitution needed to solve differential equation (3.3) is

$$\begin{cases} \hat{x} = x - \frac{5}{3} \\ \hat{y} = y - \frac{1}{3} \end{cases}$$

under which equation (3.3) becomes homogeneous

$$\frac{d\hat{y}}{d\hat{x}} = \frac{-\hat{x} + 2\hat{y}}{2\hat{x} - \hat{y}} \tag{3.4}$$

- In general, every equation of the form

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

can be transformed into a homogeneous equation

$$\frac{d\hat{y}}{d\hat{x}} = \frac{a_1 \hat{x} + b_1 \hat{y}}{a_2 \hat{x} + b_2 \hat{y}}$$

provided that we can solve the system

$$\begin{aligned} a_1 \alpha + b_1 \beta &= c_1 \\ a_2 \alpha + b_2 \beta &= c_2 \end{aligned}$$

This system is solvable if and only if $a_1b_2 - a_2b_1 \neq 0$ or when the equations are dependent: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

- The method fails, for example, in the following case:

$$\frac{dy}{dx} = \frac{5x - 2y + 1}{5x - 2y + 2}$$

because here the resulting system is inconsistent:

$$\begin{aligned} 5\alpha - 2\beta &= 1 \\ 5\alpha - 2\beta &= 2 \end{aligned}$$

In such cases, substituting one of the linear expressions solves the problem

$$v = 5x - 2y + 2$$

We express y'

$$\frac{dv}{dx} = 5 - 2\frac{dy}{dx}$$

and therefore

$$\frac{y}{dx} = \frac{5}{2} - \frac{1}{2}\frac{dv}{dx}$$

Substitution into the equation gives a separable equation

$$\frac{5}{2} - \frac{1}{2}\frac{dv}{dx} = \frac{v-1}{v}$$

- **Ratio of Parallel Linear Expressions:** In general, an equation of the form

$$\frac{dy}{dx} = \frac{\lambda ax + \lambda by + c_1}{ax + by + c_2}, \quad (c_1 \neq \lambda c_2)$$

can be solved using the substitution

$$v = ax + by + c_2$$

The resulting equation will be separable:

$$\boxed{\frac{dv}{dx} = \frac{(a+b)v + b(c_1 - c_2)}{v}}$$

We leave the proof of the formula to the reader. We merely note that it is more important to remember the substitution.

- **Exercise 3.3:** Solve the differential equation

$$\frac{dy}{dx} = \frac{4x - y + 1}{8x - 2y + 1}$$

3.2.4 Differential Equations of the Form $y' = F(ax + by + c)$

- Similar families of differential equations of the form

$$y' = F(ax + by + c)$$

- **Example 6:**

(a) $y' = (x + y + 1)^2$

(b) $y' = \tan^2(x + y + 1)$

(c) $y' = 2 + \sqrt{y - 2x + 3}$

(d) $y' = (x + y)[1 + \ln(x + y)] - 1$

(e) $y' = -1 + \sqrt{e^{2x+2y}}$

(f) $y' = \sin(2x + 2y) - \sin^2(x + y)$

- can be solved in the same way. Here too, the substitution

$$v = ax + by + c$$

transforms the equation into a separable one

- The substitution methods presented in this section are only examples of the broad general idea of substitution. There are infinitely many additional techniques of this kind, and students will sometimes be required to find the appropriate substitutions for a problem even if they have not encountered them before.

3.3 The Riccati Equation (Jacopo Riccati 1676-1754)

- We conclude the section by mentioning the classical Riccati equation. Since it is not included in the course syllabus, we leave the details to the interested student (they are not difficult).
- The general form of the equation is

$$\frac{dy}{dx} = a(x) + b(x)y + c(x)y^2$$

- Its characteristic property is that if one particular solution $y_1(x)$ is known, a general solution (not necessarily the most general one) can be found using the substitution

$$y(x) = y_1(x) + \frac{1}{v(x)}$$

which leads to a first-order linear equation in the function $v(x)$

$$v' + [b(x) + 2c(x)y_1(x)]v = c(x)$$

We leave the derivation of this formula to the student as an exercise.

- **Exercise 3.4:** Find a general solution of the differential equation

$$y' = \frac{1}{x^2} - \frac{y}{x} + y^2$$

given the particular solution $y_1(x) = \frac{1}{x}$.

3.4 Exact Equations

- One creative way to make progress in mathematics is to proceed in reverse, from a solution to a problem (Reverse Engineering), and then change direction from a problem to a solution. The subject of exact equations is an instructive example of this approach.

- An equation of the form

$$F(x, y) = C$$

under suitable conditions on the function, implicitly defines a function $y = y(x)$

$$F(x, y(x)) = C$$

Implicit differentiation of the last equality with respect to x yields the first-order differential equation

$$F_x + F_y \cdot y'(x) = 0$$

The following assertion is easy to prove. (Both directions of the assertion must be checked).

- **Proposition:** The general solution of the differential equation

$$F_x + F_y \cdot y'(x) = 0$$

is $F(x, y) = C$.

- We may infer from this proposition that if we are given a differential equation of the form

$$A(x,y)+B(x,y)\frac{dy}{dx}=0 \tag{3.5}$$

and if there exists a function $F(x,y)$ such that

$$F_x=A(x,y), \quad F_y=B(x,y)$$

then the general solution of (3.5) is

$$F(x,y)=C$$

A differential equation of this type is called an **exact equation**.

- **Example 7:** The following equation

$$(y^2+2xy)+(2xy+x^2)y'=0$$

is exact, because for $F(x,y)=xy^2+x^2y$ we have

$$F_x=y^2+2xy, \quad F_y=2xy+y^2$$

and therefore its general solution is

$$xy^2+x^2y=C$$

- The difficulty is that it is not always easy to guess the function $F(x,y)$, so a clear criterion is needed.

Theorem 3.1: Let $A(x,y)$ and $B(x,y)$ be two continuously differentiable functions on a simply connected domain $\mathscr{D}$ (with respect to both variables). Then the equation

$$A(x,y)+B(x,y)\frac{dy}{dx}$$

is exact if and only if $A_y=B_x$.

Proof: Both directions must be proved.
Direction 1: If the equation is exact, then by definition there exists a function $F(x,y)$ such that

$$F_x=A(x,y), \quad F_y=B(x,y)$$

Therefore

$$A_y=F_{xy}=F_{yx}=B_x$$

The equality $F_{xy} = F_{yx}$ follows from the continuity of the second derivatives.
Direction 2: Assume that $A_y = B_x$. We prove the existence of a function $F(x,y)$ such that

$$(1)\ F_x = A(x,y), \quad (2)\ F_y = B(x,y)$$

Equality (1) implies

$$F(x,y) = \int A(x,y)\,dx + h(y)$$

and equality (2) implies

$$\begin{aligned} F_y(x,y) &= \tfrac{\partial}{\partial y}\left[\int A(x,y)\,dx + h(y)\right] \\ &= \int A_y(x,y)\,dx + h'(y) \\ &= B(x,y) \end{aligned}$$

Differentiation under the integral sign is possible because the partial derivatives of $A(x,y)$ are continuous. Finally, we must find a function $h(y)$ satisfying

$$h'(y) = B(x,y) - \int A_y(x,y)\,dx$$

The difficulty that may arise with the last equality is that its left-hand side is a function of y alone, whereas the right-hand side contains two functions that also depend on x. We prove that the function on the right-hand side is independent of x, because its derivative with respect to x vanishes:

$$\frac{\partial}{\partial x}\left[B(x,y) - \int A_y dx\right] = B_x - A_y = 0$$

Therefore, the function $h(y)$ exists, and its existence proves that the equation is exact. ∎

- The proof of the theorem also outlines the method for computing $F(x,y)$ when the equation is exact.

- **Example 8:** Prove that the equation

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y + 2)y' = 0$$

is exact, and find its general solution.
Solution: In this example,

$$A(x,y) = y\cos x + 2xe^y, \quad B(x,y) = \sin x + x^2e^y + 2$$

Therefore

$$A_y = \cos x + 2xe^y, \quad B_x = \cos x + 2xe^y$$

We have obtained $A_y = B_x$, so the theorem implies that the equation is exact.
We compute $F(x, y)$

$$\begin{aligned} F(x,y) &= \int A(x,y)\,dx \\ &= \int (y \sin x + x^2 e^y) dx \\ &= y \sin x + x^2 e^y + h(y) \end{aligned}$$

It remains to compute $h(y)$

$$F_y = \sin x + x^2 e^y + h'(y) = B(x,y) = \sin x + x^2 e^y + 2$$

Thus $h'(y) = 2$, and therefore $h(y) = 2y$ (the constant is unnecessary because it is already included in the general solution). We have therefore obtained the general solution of the equation

$$y \sin x + x^2 e^y + 2y = C$$

The chances of isolating an explicit solution for $y(x)$ from this equation are probably slim, but numerical methods are available for computing the solutions and plotting them using advanced mathematical software (such as Matlab, Mathematica, or Python/Sympy)

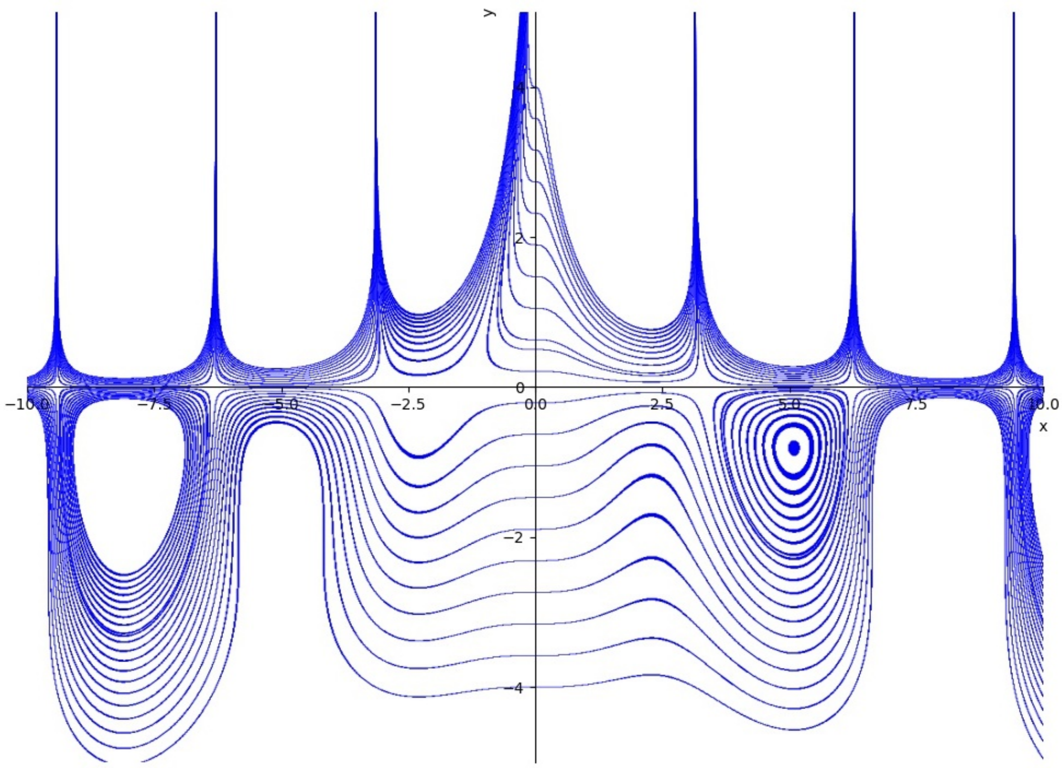

Fig. 3.3: Integral curves of $y \sin x + x^2 e^y + 2y = C$ on the interval $[-10, 10]$ (Python/Sympy tools)

- **Exercise 3.5:** Determine whether the equation

$$(2x^3 - 5y^4)y' = -3x^2 + 1$$

is exact and, if so, find its general solution.
Solution: The equation is exact. The general solution is

$$x^3y^2 + x - y^5 = C$$

3.4.1 Integrating Factors for Exact Equations

- In the preceding discussion, we worked backward from a general solution to derive the family of exact equations. We now move forward again in an attempt to enlarge the family of equations that can be solved in a similar way.
- In general, most equations of the form

$$A(x,y) + B(x,y)\frac{dy}{dx} = 0$$

are not exact. The question is: Can they be made exact by multiplying them by some factor?
- The idea is to find a function $\mu(x,y)$ such that the equation

$$\mu(x,y)A(x,y) + \mu(x,y)B(x,y)\frac{dy}{dx} = 0$$

is exact. If such a function exists, it is called an **integrating factor** of our differential equation.
- An integrating factor must satisfy the equality

$$(\mu A)_y = (\mu B)_x$$

Applying the product rule gives

$$\mu_y A + \mu A_y = \mu_x B + \mu B_x$$

or, equivalently,

$$(A\mu_y - B\mu_x) + (A_y - B_x)\mu = 0 \tag{3.6}$$

- The last equation is a **partial differential equation** (there is a separate course on such equations), which is generally more difficult to solve than the original equation. We will therefore be able to solve it only in special cases.
- Two such special cases occur when the function $\mu(x,y)$ depends on only one of the variables, in

which case equation (3.6) becomes an ordinary differential equation. We begin, however, with a simple example in which μ depends on both variables.

Example 9: The equation

$$(y^2 + 6x^2 y) + (yx\ln x - 2xy)y' = 0 \tag{3.7}$$

is not exact because

$$A_y = 2y + 6x, \quad B_x = y(\ln x + 1) - 2y$$

and it is perfectly clear that $A_y \neq B_x$. If we multiply the equation by the integrating factor

$$\mu(x, y) = \frac{1}{xy}$$

we obtain the exact equation

$$\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0$$

whose solution is

$$\begin{aligned} F(x, y) &= \int \left(\tfrac{y}{x} + 6x\right) dx = y\ln x + 3x^2 + h(y) \\ F_y &= \ln x + h'(y) = \ln x - 2 \end{aligned}$$

Therefore $h(y) = -2y$. The general solution is

$$y\ln x + 3x^2 + 2y = C$$

This is a rare case in which an explicit solution can be isolated:

$$y(x) = \frac{C - 3x^2}{2 + \ln x}$$

The more common cases in which an integrating factor is not difficult to find are those in which μ depends on a single variable. For example, if $\mu = \mu(x)$, then

$$\begin{aligned} (\mu A)_y &= \mu A_y \\ (\mu B)_x &= \mu B_x + B\mu'(x) \end{aligned}$$

Therefore, the equality $(\mu A)_y = (\mu B)_x$ required of an integrating factor implies

$$\mu A_y = \mu B_x + B\mu'(x)$$

We have obtained the equation

$$\frac{\mu'(x)}{\mu(x)} = \frac{A_y - B_x}{B}$$

The condition for the existence of $\mu(x)$ is that the expression $\frac{A_y - B_x}{B}$ depend only on x! If this condition holds, we may write

$$g(x) = \frac{A_y - B_x}{B}$$

and the solution of the equation

$$\frac{\mu'(x)}{\mu(x)} = g(x)$$

is

$$\mu(x) = e^{\int g(x)\,dx} = e^{\int \frac{A_y - B_x}{B}\,dx}$$

If the condition does not hold (as is quite likely), we may try to find an integrating factor of the form $\mu(y)$. Neither form need exist, in which case finding an integrating factor becomes more difficult.

Example 10: Find an integrating factor $\mu(x)$ for the differential equation

$$\frac{y}{2} + (x + \sqrt{x}\sin y)y' = 0$$

It is easy to check that the equation is not exact. However,

$$\begin{aligned}
\frac{A_y - B_x}{B} &= \frac{\frac{1}{2} - 1 - \frac{1}{2\sqrt{x}}\sin y}{x + \sqrt{x}\sin y} = \frac{-\frac{1}{2}\left(1 + \frac{1}{\sqrt{x}}\sin y\right)}{x + \sqrt{x}\sin y} \\
&= \frac{-\frac{1}{2x}\left(x + \sqrt{x}\sin y\right)}{x + \sqrt{x}\sin y} = -\frac{1}{2x}
\end{aligned}$$

To determine $\mu(x)$, we must solve the equation

$$\frac{\mu'(x)}{\mu(x)} = -\frac{1}{2x}$$

As in the case of a first-order linear ODE, here too we assume that $\mu(x) > 0$, and therefore, after integrating both sides, we obtain

$$\ln\mu(x) = -\frac{1}{2}\ln x + C = \ln\frac{1}{\sqrt{x}} + C$$

and therefore

$$\mu(x) = Ce^{\ln\frac{1}{\sqrt{x}}} = \frac{C}{\sqrt{x}}$$

We take the solution $\mu(x) = \frac{1}{\sqrt{x}}$. After multiplying the equation by this factor, we obtain an exact equation

$$\frac{y}{2\sqrt{x}} + \left(\sqrt{x} + \sin y\right) y' = 0$$

whose solution is not difficult to find.

- Similarly, a necessary condition for the existence of an integrating factor $\mu(y)$ is

$$\mu_y A + \mu A_y = \mu B_x$$

Therefore

$$\frac{\mu'(y)}{\mu(y)} = \frac{B_x - A_y}{A} = g(y)$$

The condition for the existence of $\mu(y)$ is that the expression $\frac{B_x - A_y}{A}$ depend only on the variable y; we denote it by $g(y)$. If this condition holds, the solution of the equation

$$\frac{\mu'(x)}{\mu(x)} = g(y)$$

is

$$\mu(y) = e^{\int g(y)\,dy} = e^{\int \frac{B_x - A_y}{A} dy}$$

- If no integrating factor of the form $\mu(x)$ or $\mu(y)$ is found, one may guess integrating factors of various forms, such as $\mu(x,y) = x^m y^n$, $\mu(x,y) = \frac{x^m}{y^n}$, and so forth.

- **Exercise 3.6:** Verify that $\mu(x,y) = (xy)^2$ is an integrating factor of the equation

$$\left(3 + \frac{2y^3}{x}\right) + \left(\frac{2x}{y} + 5y^2\right)\frac{dy}{dx} = 0$$

- **Exercise 3.7:** Find a condition for the existence of an integrating factor of the form $\mu(x,y) = x^m y^n$, where m and n are integers, and then solve the equation

$$(x^3y^2 - y) + (x^2y^4 - x)\frac{dy}{dx} = 0$$

using an integrating factor of this type.

Solution: We must check the condition for the equality

$$(x^m y^n A)_y = (x^m y^n B)_x$$

After differentiating, we obtain

$$n x^m y^{n-1} A + x^m y^n A_y = m x^{m-1} y^n B + x^m y^n B_x$$

We factor out common terms:

$$x^m y^n \left(\frac{n}{y}A + A_y\right) = x^m y^n \left(\frac{m}{x}B + B_x\right)$$

Therefore, the condition for the existence of an integrating factor of the form $x^m y^n$ is the existence of integers m, n for which the following equality holds:

$$\frac{nA}{y} + A_y = \frac{mB}{x} + B_x$$

We check whether our equation has such an integrating factor:

$$(x^3y^2 - y) + (x^2y^4 - x)\frac{dy}{dx} = 0$$

In this case,

$$\begin{aligned} A &= x^3y^2 - y, \qquad & B &= x^2y^4 - x \\ A_y &= 2x^3y - 1, \qquad & B_x &= 2xy^4 - 1 \end{aligned}$$

Therefore

$$\begin{aligned} \frac{nA}{y} + A_y &= n(x^3y - 1) + 2x^3y - 1 = (n+2)x^3y - n - 1 \\ \frac{mB}{x} + B_x &= m(xy^4 - 1) + 2xy^4 - 1 = (m+2)xy^4 - m - 1 \end{aligned}$$

Thus, the following equality must hold for all values of x, y

$$(n+2)x^3y - n - 1 = (m+2)xy^4 - m - 1$$

and this can occur only when $m = n = -2$! Therefore, our integrating factor is $\mu = x^m y^n = x^{-2}y^{-2}$. Multiplying the equation by the integrating factor, we obtain

$$\left(x - \frac{1}{x^2y}\right) + \left(y^2 - \frac{1}{xy^2}\right)\frac{dy}{dx} = 0$$

A simple check proves that this is indeed an exact equation, and the rest of its solution is routine.

Exercise 3.8: Using the same method, prove that the following equation also has an integrating factor of the form $x^m y^n$

$$x^2y^3 + (x + xy^2)\frac{dy}{dx} = 0$$

Find the integrating factor and solve the equation.

Exercise 3.9: Consider the differential equation

$$A(x,y)+B(x,y)y'=0$$

Prove that if the expression

$$\frac{B_x-A_y}{xA-yB}$$

is a function of xy, then the equation has an integrating factor of the form

$$\mu(x,y)=g(xy)$$

Find an integrating factor of this type for the equation

$$(y^2+6x^2y)+(yx\ln x-2xy)y'=0$$

and find a general solution.

Solution: We note that the expression $g(xy)$ is shorthand for a composite function $g(u)$, $u=xy$ (that is, g is actually a function of a single variable).

After multiplying the equation by the integrating factor, we obtain

$$g(xy)A(x,y)+g(xy)B(x,y)y'=0$$

According to the theorem above, the condition for the equation to be exact is

$$\frac{\partial}{\partial y}[g(xy)A(x,y)]=\frac{\partial}{\partial x}[g(xy)B(x,y)]$$

We use the differentiation rules for a composite function:

$$(\mu A)_y=g'(xy)xA+g(xy)A_y(\mu B)_x=g'(xy)yB+g(xy)B_x$$

Thus, for the equation to be exact, the following equality must hold:

$$g'(xy)xA+g(xy)A_y=g'(xy)yB+g(xy)B_x$$

Therefore

$$g'(xy)xA-g'(xy)yB=g(xy)B_x-g(xy)A_y$$

and therefore

$$\frac{B_x-A_y}{xA-yB}=\frac{g'(xy)}{g(xy)} \tag{3.8}$$

We check whether the following equation has such an integrating factor:

$$(y^2 + 6x^2y) + (yx\ln x - 2xy)y' = 0$$

First, we write the expressions we need:

$$\begin{aligned} A &= y^2 + 6x^2y, & B &= xy\ln x - 2xy \\ A_y &= 2y + 6x^2, & B_x &= y + y\ln x - 2y \end{aligned}$$

Substitution into equality (3.8) gives

$$\begin{aligned} \frac{B_x - A_y}{xA - yB} &= \frac{(y + y\ln x - 2y) - (2y + 6x^2)}{x(y^2 + 6x^2y) - y(xy\ln x - 2xy)} \\ &= \frac{y + y\ln x - 2y - 2y - 6x^2}{6x^3y - xy^2\ln x + 3xy^2} \\ &= \frac{y\ln x - 3y - 6x^2}{-xy(y\ln x - 3y - 6x^2)} \\ &= -\frac{1}{xy} \end{aligned}$$

According to equality (3.8), we must find a function g satisfying

$$\frac{g'(xy)}{g(xy)} = -\frac{1}{xy}$$

Substituting $u = xy$, we obtain an ordinary differential equation

$$\frac{g'(u)}{g(u)} = -\frac{1}{u}$$

whose solution is

$$\begin{aligned} \ln g(u) &= -\ln u = \ln\frac{1}{u} \\ g(u) &= \frac{1}{u} \end{aligned}$$

Therefore, our integrating factor is $\mu(x, y) = \frac{1}{xy}$. We already checked and solved the equation above (see equation (3.7) above).

Exercise 3.10: Using the preceding formula, verify that the following equation has an integrating factor of the form $\mu(x, y) = g(xy)$, find the integrating factor, and find a general solution of the equation

$$\left(3 + \frac{2y^3}{x}\right) + \left(\frac{2x}{y} + 5y^2\right)\frac{dy}{dx} = 0$$

3.5 Orthogonal Families of Curves

Definition 3.2:

A. Two curves intersecting at a point (x_0, y_0) are called **orthogonal at the point** (x_0, y_0) (perpendicular) if they intersect at the point (x_0, y_0) and the tangent lines to the two curves at the point (x_0, y_0) are perpendicular to each other (see Figure 3.4).

B. Two curves are called **orthogonal** if they are orthogonal at all their points of intersection.

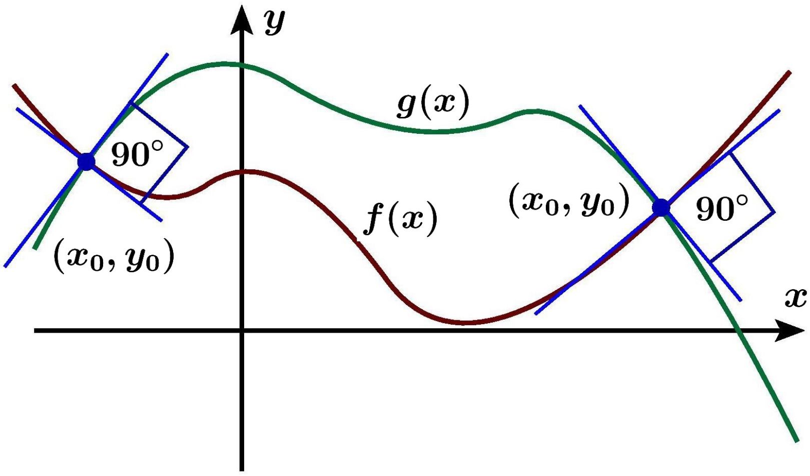

Fig. 3.4: Two orthogonal curves

- The definition implies that both curves have a slope (derivative) at the point (x_0, y_0).
- The limiting case in which one curve has infinite slope requires the existence of an infinite derivative on both sides of the point.
- Two curves may be orthogonal at several points of intersection.
- **Proposition 3.2:** The graphs of the functions $y = f(x)$ and $y = g(x)$ are orthogonal at the point (x_0, y_0) if and only if

$$f'(x_0)g'(x_0) = -1$$

  If $f'(x_0) = 0$ or $g'(x_0) = 0$, the derivative of the other function must be $\pm\infty$.
- **Requirement:** Through every point in the plane, at most one curve from each family may pass. A point through which several curves of the same family pass is called **singular**.

Definition 3.3: **Orthogonal families of curves** are two families of curves such that every curve in the first family is orthogonal to every curve in the second family.

- Orthogonal families of curves are common in physical, geographical, and other contexts.

- On topographic maps (contour maps), for example, the orthogonal family represents optimal climbing or descending routes. Water flowing from the summit of a mountain to its base follows a curve orthogonal to the mountain's contour lines

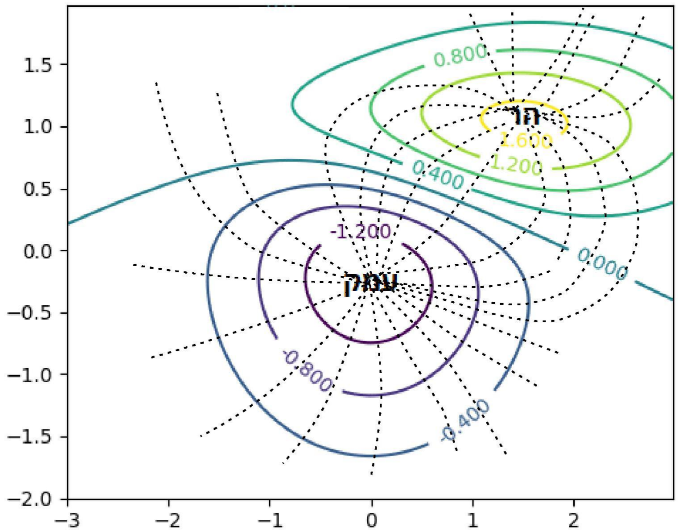

Fig. 3.5: A topographic map (contour lines) and paths of least effort

The mountain summit and the lowest point of the valley are singular points.

- **Geometric Problem:** Consider the family of parabolas $y = kx^2$, where k is an arbitrary real number (for $k = 0$, we obtain a straight line). Find the orthogonal family.

Solution:

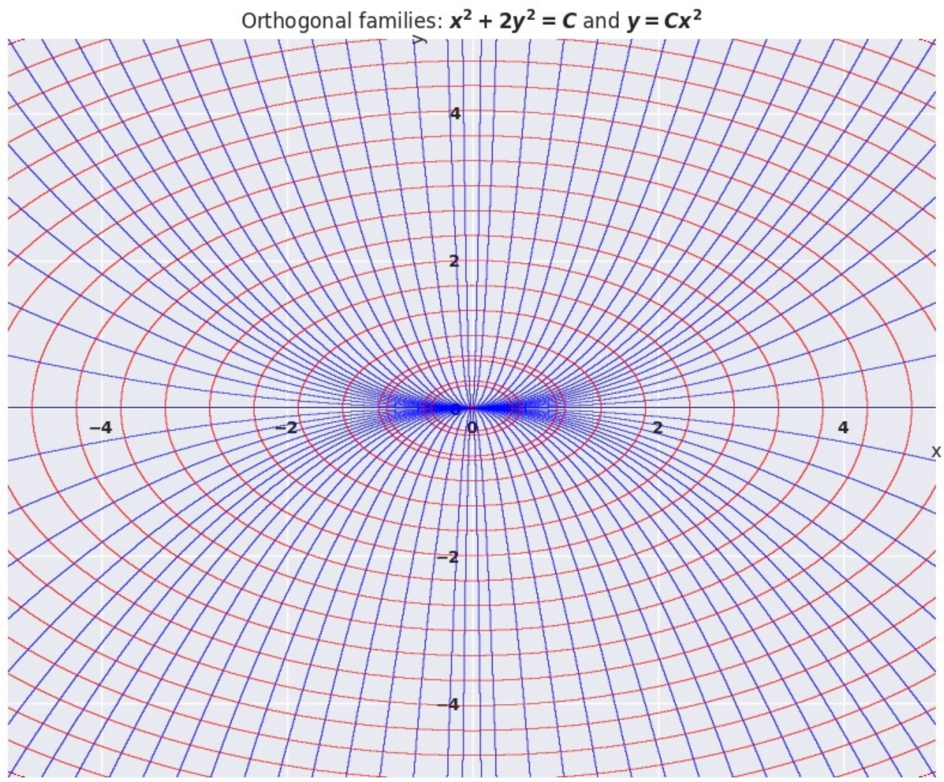

Fig. 3.6: The family of ellipses $x^2 + 2y^2 = C$ is orthogonal to the family of parabolas $y = kx^2$

Let (x, y) be a point in the plane through which a parabola $y = kx^2$ passes. Then $k = \frac{y}{x^2}$, and the slope of the parabola at this point is $2kx = \frac{2y}{x}$. Therefore, the slope of the orthogonal curve at the point (x, y) is $-\frac{x}{2y}$. The orthogonal family is therefore the general solution of the separable equation

$$y' = -\frac{x}{2y}$$

Its general solution is easily obtained:

$$x^2 + 2y^2 = C$$

This is a family of ellipses, as shown in Figure 3.6.

Example 11: Consider the family of circles $(x-k)^2 + y^2 = k^2$. Find the orthogonal family.

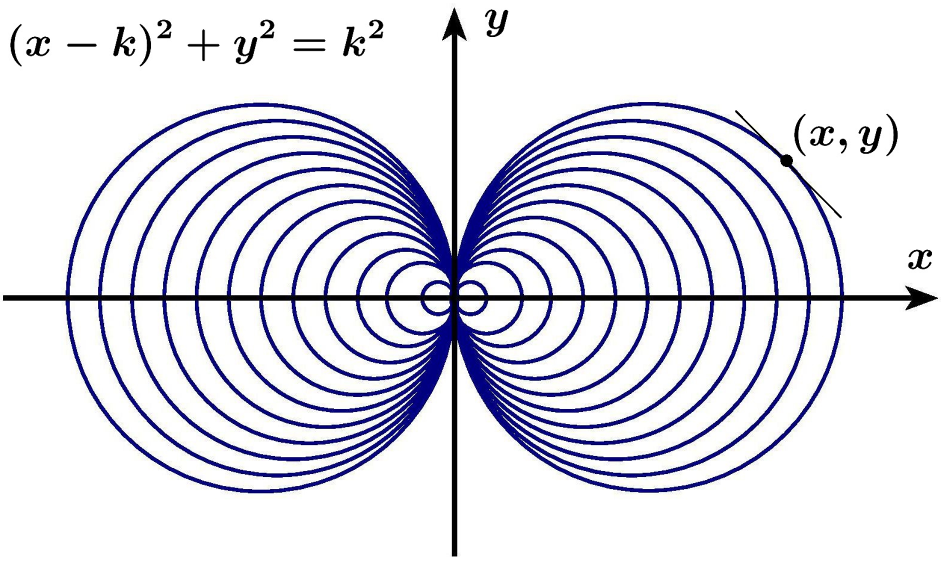

Fig. 3.7: The family of circles $(x-k)^2 + y^2 = k^2$

Solution: Let (x, y) be any point belonging to the family. That is, there exists k such that

$$(x-k)^2 + y^2 = k^2$$

We must express k in terms of x, y

$$x^2 - 2kx + k^2 + y^2 = k^2$$

Therefore

$$k = \frac{x^2 + y^2}{2x}$$

To obtain the slope, we differentiate implicitly:

$$2(x-k) + 2yy' = 0$$

Therefore
$$y' = \frac{k-x}{y} = \frac{\frac{x^2+y^2}{2x} - x}{y} = \frac{x^2+y^2-2x^2}{2xy} = \frac{y^2-x^2}{2xy}$$

The differential equation for the orthogonal family is

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

This is an equation of homogeneous type. The substitution $v = \frac{y}{x}$ gives

$$x\frac{dv}{dx} + v = \frac{2v}{1-v^2}$$

$$\frac{1-v^2}{v^3+v}\,dv = \frac{dx}{x}$$

$$\frac{1-v^2}{v^3+v} = \frac{1+v^2}{v^3+v} - \frac{2v^2}{v^3+v} = \frac{1}{v} - \frac{2v}{v^2+1}$$

Therefore
$$\int \left[\frac{1}{v} - \frac{2v}{v^2+1}\right] dv = \ln\frac{v}{v^2+1} = \ln x + C$$

General solution:
$$\frac{v}{v^2+1} = Cx$$

Substituting $v = \frac{y}{x}$, we obtain the orthogonal family

$$\frac{xy}{x^2+y^2} = Cx$$

or, more simply,
$$x^2 + y^2 - \frac{y}{C} = 0$$

If we replace $\frac{1}{C}$ by the expression $2c$, we obtain the equivalent description

$$x^2 + (y-c)^2 = c^2$$

This is, of course, a family of circles whose centers lie on the y-axis and that are tangent to the x-axis.

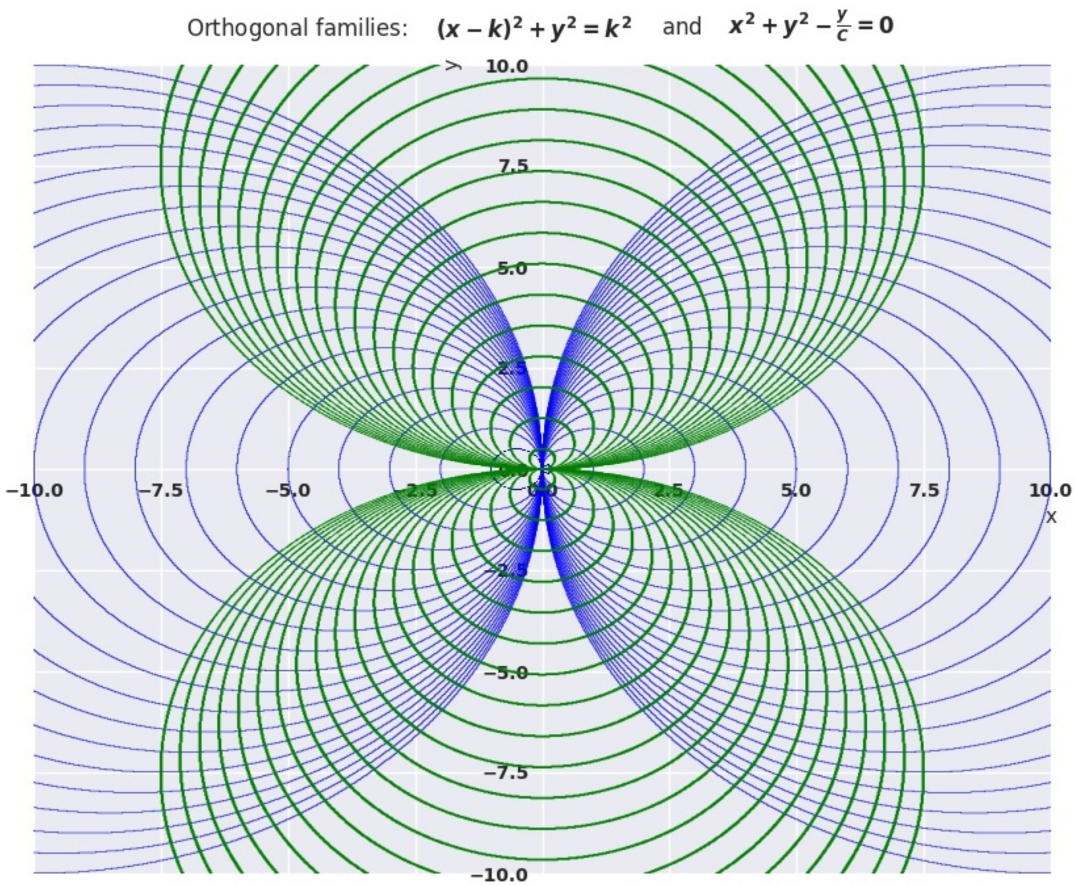

Fig. 3.8: The family of circles $x^2+(y-c)^2=c^2$ is orthogonal to the family of circles $(x-k)^2+y^2=k^2$

Exercise 3.11: Consider the family of circles $x^2+y^2=k^2$. Find a family of curves that intersects the preceding family at an angle of $45°$.

Guidance: If the slopes of two lines are m_1, m_2, then the angle θ between them satisfies

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

This follows from the formula

$$\tan(\beta-\alpha)=\frac{\tan\beta-\tan\alpha}{1+\tan\beta\tan\alpha}$$

where α, β are the angles that the lines make with the x-axis.

After processing the data, we obtain an equation of homogeneous type for the intersecting family

$$y'=\frac{x+y}{x-y}$$

whose solution yields the “orthogonal” family shown in Figure 3.9.

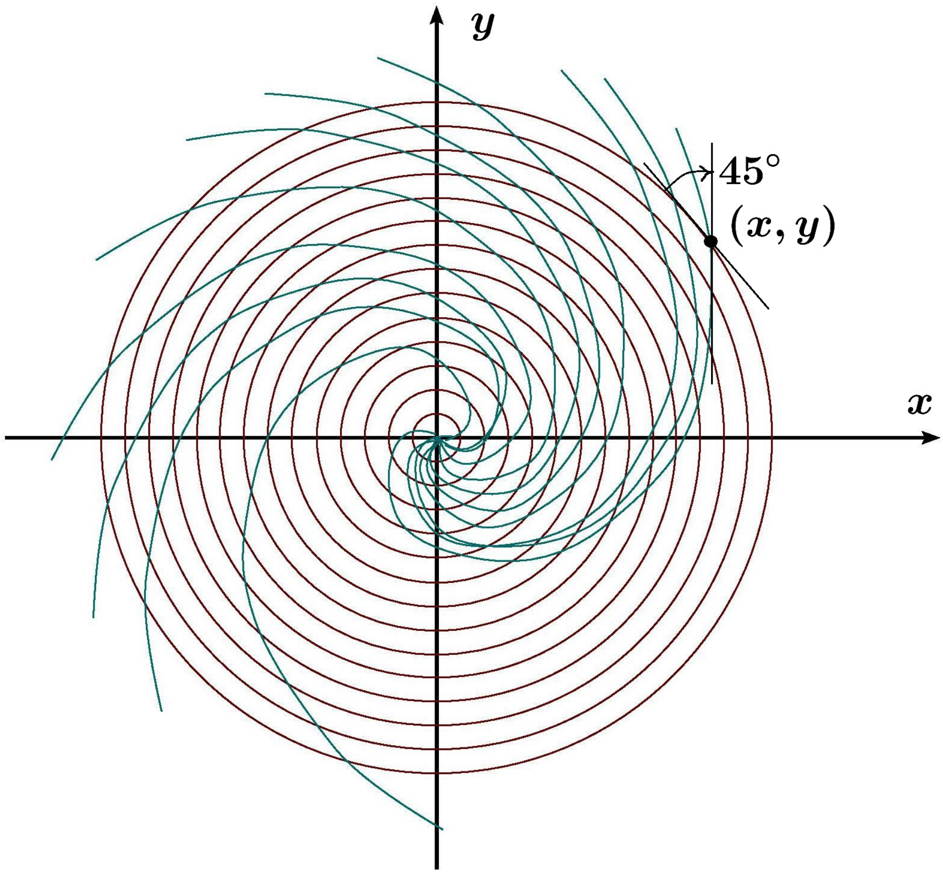

Fig. 3.9: The family of circles $x^2 + y^2 = k^2$ and the family of curves intersecting it at an angle of $45°$

3.6 The Existence and Uniqueness Theorem for First-Order Equations

Example 12: The differential equation

$$(1 + y')^2 + y^2 = -5$$

has no solution, because the left-hand side is always positive regardless of the function $y(x)$.

Example 13: The differential equation

$$(y')^2 + y^2 = 0$$

has exactly one solution: $y(x) \equiv 0$ (the zero function). This contrasts with the common situation in which, without an initial condition, there are usually infinitely many solutions.

Example 14: The differential equation

$$(y')^2 + y^2(y - 1)^2 = 0$$

has exactly two distinct solutions: $y = 0$, $y = 1$.

- As noted, the study of equations in the general implicit form $F(x, y, y') = 0$ is beyond the scope of this course. We restrict ourselves to normal form.
- The general normal form of a first-order initial-value problem is

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

where no algebraic restriction is imposed on $f(x, y)$, which may be a very complicated function.

Theorem 3.3: Consider a first-order initial-value problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{3.9}$$

If the functions $f(x, y)$ and $f_y(x, y)$ are continuous inside a given rectangle in the plane:

$$\alpha < x < \beta, \quad \gamma < y < \delta$$

containing the point (x_0, y_0), then there exists a subinterval $(x_0 - h, x_0 + h)$ of the interval (α, β) on which a solution $\phi(x)$ of our initial-value problem exists and is **unique**.
That is,

$$\begin{cases} \phi'(x) = f(x, \phi(x)), \quad (x_0 - h < x < x_0 + h) \\ \phi(x_0) = y_0 \end{cases}$$

- Recall that the expression $f_y(x, y)$ denotes the **partial derivative** of $f(x, y)$ with respect to the variable y, and is sometimes also denoted by $\frac{\partial f}{\partial y}(x, y)$.
- In general, the quantity h may be smaller than $\min\{x_0 - \alpha, \beta - x_0\}$, but in other cases it may even be ∞. Methods for estimating h will be presented later.
- A partial proof of the theorem emphasizing the general idea will be presented later (if time permits). The full proof is beyond the scope of the course and can be found in the extensive literature on the subject.
- It is important not to forget that the theorem guarantees the existence and uniqueness of a solution in a neighborhood of x_0, but provides no method for finding it (not even a hint!). Indeed, in most cases there are no analytical methods for finding a solution. We will content ourselves with studying several known methods for solving problems commonly encountered in scientific fields.

Example 15: The initial-value problem

$$\begin{cases} x^2y' = y^2 \\ y(0) = 0 \end{cases}$$

has infinitely many distinct solutions:

$$y(x) = \frac{x}{1+Cx} \tag{3.10}$$

This is easily verified by direct substitution. This does not contradict the existence and uniqueness theorem, because the equation in normal form

$$y' = f(x,y) = -\frac{y^2}{x^2}$$

does not satisfy the hypotheses of the theorem near the point $x_0 = 0$.

Example 16: The initial-value problem

$$\begin{cases} x^2y' = y^2 \\ y(1) = 0 \end{cases}$$

has an easily guessed trivial solution: $y(x) \equiv 0$. Prove that this initial-value problem has no other solution.

Note that in the preceding problem, the solution $y(x) \equiv 0$ is not included in its general solution (3.10). Thus, finding an infinite family of solutions of a differential equation does not guarantee that it includes all the solutions of the problem! Nevertheless, a brief look at the graph of the general solution reveals that the function $y(x) \equiv 0$ is a limiting function of the family as $C \to \infty$.

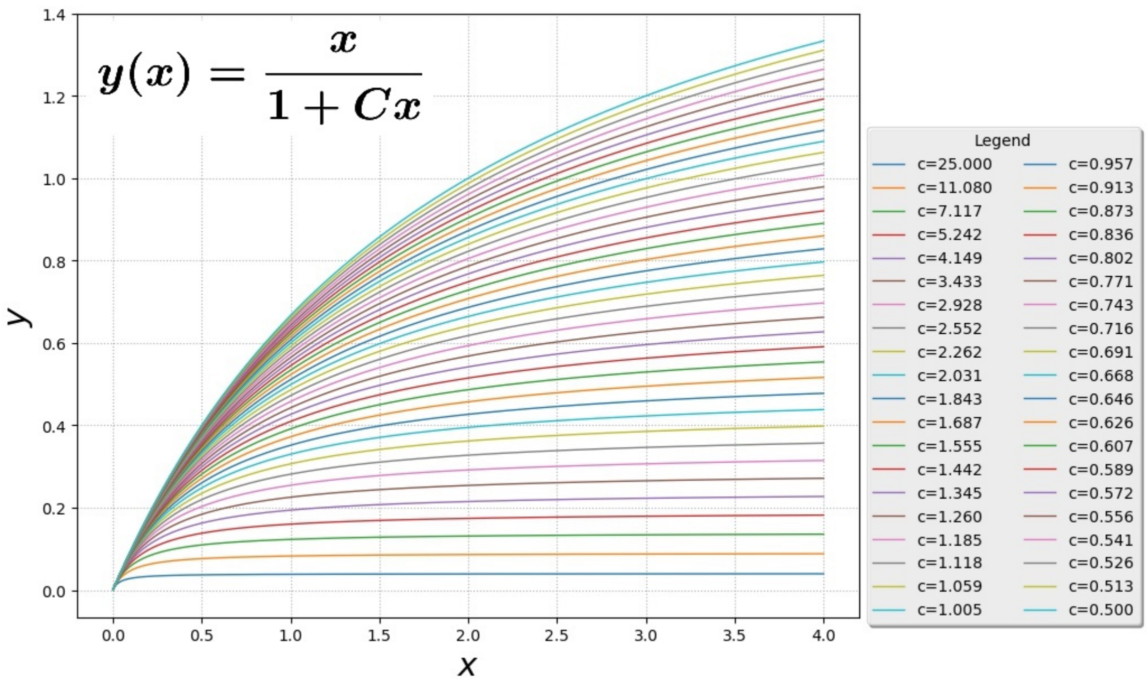

Fig. 3.10: The family of solutions of the equation $x^2y' = y^2$

- The additional solution not included in the family is sometimes called an "envelope". However, there is no guarantee that all solutions not included in the family are of this type.

- **Example 17:** The initial-value problem

$$\begin{cases} y' = y^{\frac{1}{3}} \\ y(0) = 0 \end{cases}$$

has two distinct solutions on the ray $[0, \infty)$:

$$\begin{cases} \phi_1(x) = \left(\dfrac{2}{3}x\right)^{\frac{3}{2}} \\ \phi_2(x) = 0 \end{cases}$$

This does not contradict the existence and uniqueness theorem, because the function $f(x, y) = y^{\frac{1}{3}}$ has no partial derivative with respect to y at the point $(0, 0)$ (in fact, f_y does not exist at this point).

- There is a second formulation of the existence and uniqueness theorem that makes it possible to estimate the parameter h.

- **Example 18:** A problem given on a midterm examination in the spring semester of the Hebrew year 5744 (30.3.1984)

A. Find all the solutions of the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x\sqrt{y}}{e^{\sqrt{y}}} \\ y(0) = 0 \end{cases}$$

B. Explain in detail why the nonuniqueness of the solution does not contradict the existence and uniqueness theorem.

Solution: This is a separable equation with a routine solution:

$$\frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = x\,dx$$

General solution:

$$2e^{\sqrt{y}} = \frac{x^2}{2} + C$$

From the initial condition $y(0) = 0$, we obtain $C = 2$, and therefore

$$e^{\sqrt{y}} = \frac{x^2}{4} + 1$$

Thus

$$\sqrt{y} = \ln\left(\frac{x^2}{4} + 1\right)$$

We have obtained the first solution

$$y = \ln^2\left(\frac{x^2}{4} + 1\right)$$

The second solution "disappeared" when we divided both sides by $\sqrt{y}$. Such division assumes that $y \neq 0$, so the function $y = 0$ must be checked separately. The check shows that $y = 0$ is also a solution of the initial-value problem, and thus we have obtained two solutions of the initial-value problem, contrary to the conclusion of the existence and uniqueness theorem. Reading the theorem carefully again, we see that the function

$$f(x, y) = \frac{x\sqrt{y}}{e^{\sqrt{y}}}$$

does not satisfy the hypotheses of the theorem at the point $(0, 0)$, because it is not defined to the left of the point $y = 0$ (the theorem requires the function to exist in a rectangular neighborhood centered at the point $(0, 0)$).

- Even if $f(x, y)$ satisfies the hypotheses of the theorem and also has wonderful properties (such as infinitely many continuous derivatives, differentiability, and so forth), it is not always possible to find the explicit solution guaranteed by the existence and uniqueness theorem. In many cases, numerical methods must be used to approximate the solution to various levels of accuracy (these will be studied in the fifth chapter of the course, together with many other methods taught in numerical analysis courses).

Theorem 3.4: (The Existence and Uniqueness Theorem, Version 2)

Consider a first-order initial-value problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

If the functions $f(x,y)$ and $f_y(x,y)$ are continuous inside a symmetric rectangle around the point (x_0, y_0):

$$\begin{cases} x_0 - a < x < x_0 + a \\ y_0 - b < y < y_0 + b \end{cases}$$

and, in addition, there exists a real constant M such that $|f(x,y)| \le M$ throughout the rectangle, then there exists a **unique** solution on the subinterval $(x_0 - h, x_0 + h)$, where

$$h = \min\left\{a, \frac{b}{M}\right\}$$

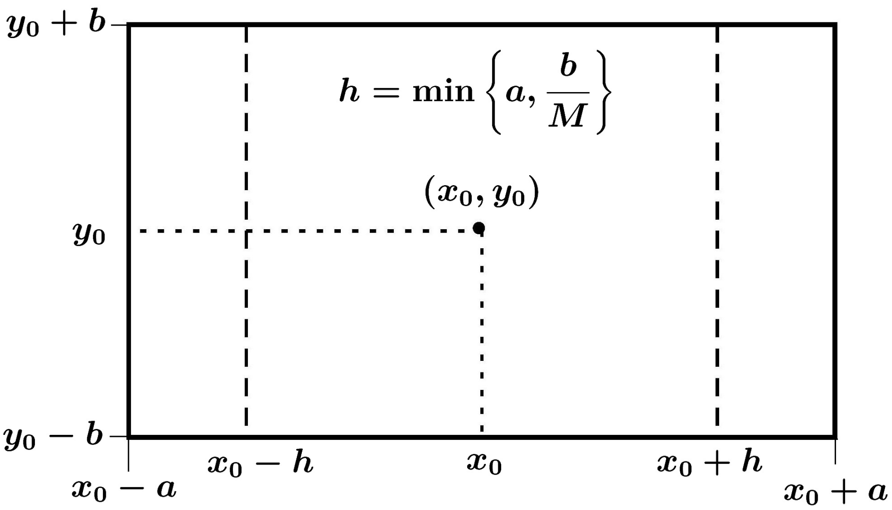

Fig. 3.11: The domain of definition of the unique solution

- The symmetric rectangle may be an "infinite strip" if $a = \infty$ or $b = \infty$. If both $a = \infty$ and $b = \infty$, then the rectangle is the entire real plane $\mathbb{R}^2$. See Figure 3.11.
- In practice, in many cases the value of h may be greater than the theorem guarantees.
- If both $a = \infty$ and $b = \infty$, then the value of h is $h = \infty$, and therefore this formulation of

the existence and uniqueness theorem allows us to conclude that a unique solution exists on the entire real line (which was not possible before).

Example 19: The initial-value problem

$$\begin{cases} y' = \dfrac{xy}{1+x^2+y^2} \\ y(0) = 1 \end{cases}$$

has a unique solution on the entire real line, because

A. The function $f(x,y) = \frac{xy}{1+x^2+y^2}$ is defined and continuous on the entire real plane $\mathbb{R}^2$.

B. The derivative f_y is defined and continuous on the entire plane
Therefore, we may take $a = b = \infty$

C. For every point $(x,y) \in \mathbb{R}^2$,

$$|f(x,y)| = \left|\frac{xy}{1+x^2+y^2}\right| < M = 1$$

D. Therefore

$$h = \min\left\{a, \frac{b}{M}\right\} = \infty$$

Exercise 3.12: Does there exist a solution of the equation

$$y' = \frac{xy}{1+x^2+y^2}$$

that satisfies both $y(0) = 0$ and $y(2) = 2$?

Exercise 3.13: Prove that the initial-value problem

$$\begin{cases} y' = \sqrt{1-y^2} \\ y(0) = 8 \end{cases}$$

has no solution. Does this fact contradict the existence and uniqueness theorem?

Exercise 3.14: Prove that the initial-value problem

$$\begin{cases} y' = \sqrt{1-y^2} \\ y(0) = -1 \end{cases}$$

has two distinct solutions defined on the entire real line:

$$y_1(x) = -1, \qquad y_2(x) = -\cos x$$

Does this fact contradict the existence and uniqueness theorem?

Remark: The general solution of the equation is $y = \sin(x + c)$, but there are two "envelope" solutions, $y = -1$, $y = 1$.

- **Exercise 3.15:** Use the existence and uniqueness theorem to prove that the initial-value problem

$$\begin{cases} y' = \sqrt{1 - y^2} \\ y(0) = \frac{1}{2} \end{cases}$$

has a unique solution on the interval $(-\frac{1}{2}, \frac{1}{2})$.

- Before proceeding to the partial proof of the existence and uniqueness theorem, we introduce the concept of a direction field, which is very helpful in understanding the behavior of solutions of first-order differential equations.

3.7 The Direction Field of a Function $f(x, y)$ in the Plane

- One of the practical concepts that aids the understanding and analysis of solutions of first-order differential equations (there is no analogous concept for higher-order equations) is the concept of a **direction field (directions field)**.
- Given a general first-order differential equation in normal form,

$$y' = f(x, y)$$

it can be interpreted geometrically as describing the slope (or direction) of a solution at the point (x, y). The expression $y'(x)$ denotes the geometric slope of the solution at the point (x, y). In other words: The expression $y' = f(x, y)$ can be interpreted as a rule telling us "in which direction to move in order to remain on a graph that solves the equation".

- A direction vector can be drawn at every point (x, y) in the plane (at which the function f is defined), yielding a "direction map" or a **direction field**, as it is more commonly called, which indicates the direction in which to proceed at each point in order to follow a solution of the equation.
- For convenience, the vector has a fixed length (to make the drawing clear and legible), but its slope equals $f(x, y)$. The angle of the vector, $\theta = \arctan f(x, y)$, always lies in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, and the arrow always points toward the positive direction of the x-axis. Thus, the direction of flow is always from left to right.
- **An integral curve (integral line)** is obtained by choosing a point (x_0, y_0) in the xy-plane and flowing along the vectors in the direction field. It is easy to see that constructing an integral curve is equivalent to finding a solution of an initial-value problem.

Example 20: The normal form of the differential equation solved earlier,

$$xy' + y = x^3$$

is

$$y' = -\frac{y}{x} + x^2$$

There are very simple numerical methods for drawing the direction field of

$$f(x,y) = -\frac{y}{x} + x^2$$

which provide a glimpse of the geometric form of the solutions, as can be seen in Figure 3.12.

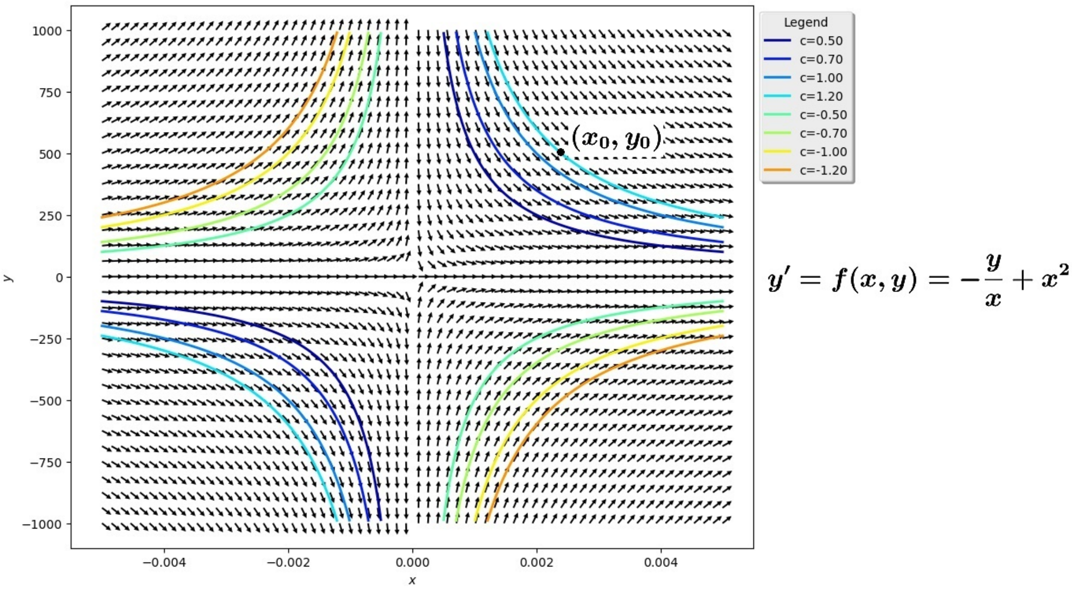

Fig. 3.12: The direction field of the equation $y' = -\frac{y}{x} + x^2$

Figuratively speaking: To find the unique solution passing through the point (x_0, y_0), we stand at that point and "flow in the direction" of the vectors in the field. The one–parameter family of solutions of the equation looks as follows:

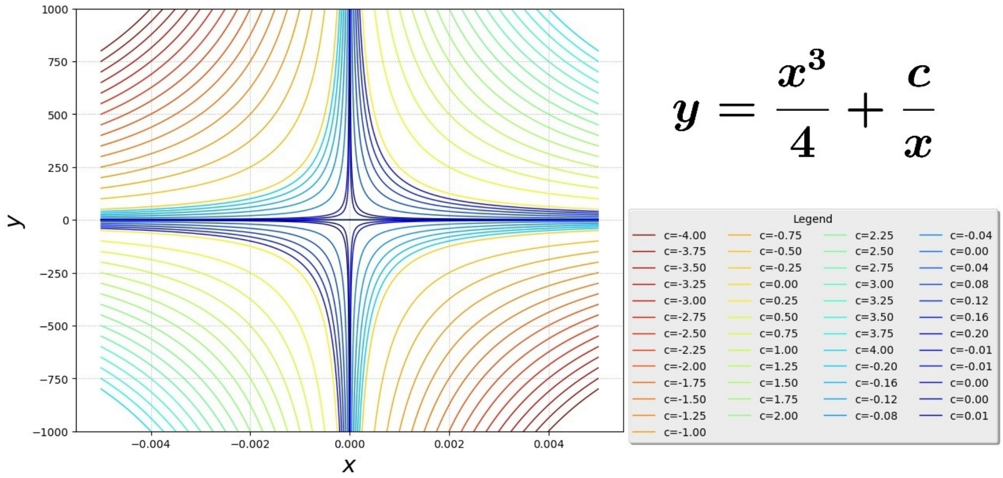

Fig. 3.13: The family of solutions of the equation $xy' + y = x^3$

Example 21: The normal form of the differential equation

$$xy' + 2y = \sin x$$

is

$$y' = -\frac{2y}{x} + \frac{\sin x}{x}$$

The direction field of $f(x, y) = -\frac{2y}{x} + \frac{\sin x}{x}$ looks as follows:

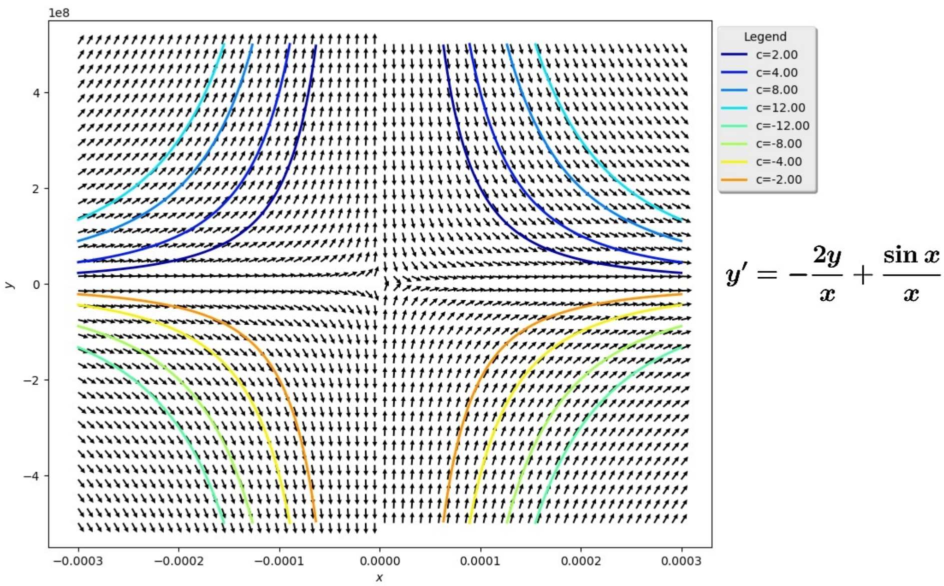

Fig. 3.14: The direction field of the equation $y' = -\frac{2y}{x} + \frac{\sin x}{x}$

The one–parameter family of solutions of the equation looks as follows:

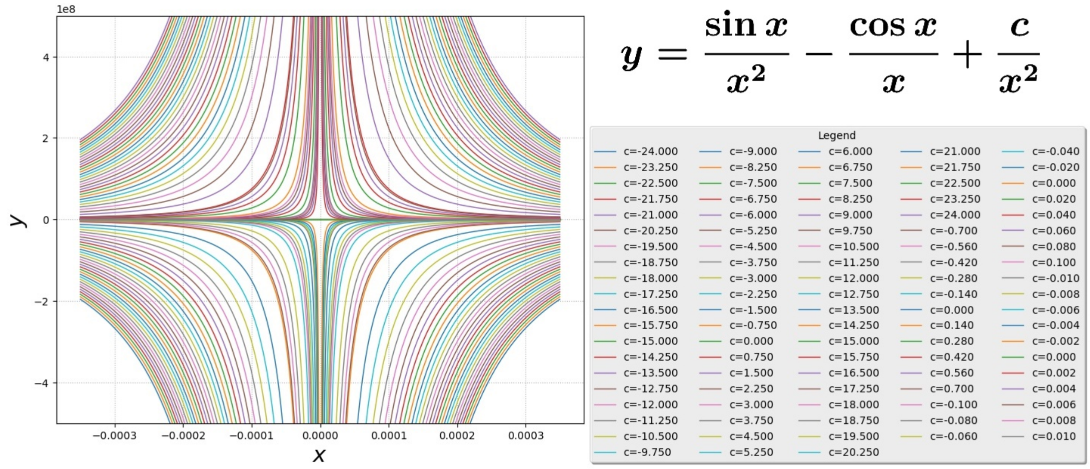

Fig. 3.15: The family of solutions of the equation $xy' + 2y = \sin x$

In the preceding two examples, note the following two facts:

A. No two solutions have a common point. All the solutions are disjoint.

B. If (x, y) is any point in the plane ($x \neq 0$), then exactly one solution passes through it.

Example 22: Find a solution of the initial-value problem, and find the largest domain of definition that the existence and uniqueness theorem can guarantee for the solution you found

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

- This is an easy separable equation. It has the unique solution $y(x) = \frac{1}{1-x}$
- The point $x = 1$ is singular, even though there is no sign or indication of this in the problem itself, and therefore the solution does not exist on the entire real line, even though the function $f(x, y) = y^2$ and its partial derivative with respect to y exist and are continuous on the entire plane $\mathbb{R}^2$!?
- This does not contradict the existence and uniqueness theorem, because the function $f(x, y)$ is not bounded on $\mathbb{R}^2$ as required by the theorem.
- The source of the difficulty becomes apparent when we try to apply the existence and uniqueness theorem to obtain a unique solution on the largest possible domain. The maximum value of the function $f(x, y) = y^2$ on the rectangle $(-a, a) \times (1 - b, 1 + b)$ is $M = (1 + b)^2$. Thus, as we increase b, we decrease h:

$$h = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{b}{(1+b)^2}\right\}$$

We may choose a as large as we wish, but as b increases, the value of $\frac{b}{(1+b)^2}$ tends to zero and drastically reduces h. It is easy to check that the maximum value of the expression

$\frac{b}{(1+b)^2}$ is $\frac{1}{4}$, and therefore the largest value of h we can obtain is $h = \frac{1}{4}$.

- The largest domain of existence guaranteed by the existence and uniqueness theorem is therefore $(-\frac{1}{4}, \frac{1}{4})$.
- If we slightly change the initial condition

$$\begin{cases} y' = y^2 \\ y(0) = 2 \end{cases}$$

we obtain the solution $y(x) = \dfrac{1}{1-2x}$

- This time the singular point moves to $x = \frac{1}{2}$, and again there is no sign or indication of this in the problem itself.

Example 23: We return once more to the preceding initial-value problem

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

We saw that it has a unique solution $y_p(x) = \frac{1}{1-x}$ on the domain of existence $(-\frac{1}{4}, \frac{1}{4})$, which is the largest one that the existence and uniqueness theorem could provide. Nevertheless, the function $y_p(x)$ solves the equation on the entire ray $(-\infty, 1)$, and it is difficult to imagine that there is another solution on the ray $(-\infty, 1)$ satisfying $y(0) = 1$. Is there a way to prove this conjecture?

Guidance: Assume, to the contrary, that there is another solution $z(x)$ of the initial-value problem, distinct from $y_p(x) = \frac{1}{1-x}$, defined on the interval $(a, 1)$, $a < 0$. Let

$$x_1 = \sup\left\{ x \,\middle|\, a < x < 0,\ z(x) \neq \frac{1}{1-x} \right\}$$

By the definition of the supremum, there are points x_2, x_3 in the interval $(a, 1)$ such that

$$\begin{cases} x_2 \leq x_1 \leq x_3 \\ |x_2 - x_3| < \dfrac{1}{4} \\ z(x_2) \neq \dfrac{1}{1-x} \\ z(x_3) = \dfrac{1}{1-x} \end{cases}$$

Use the existence and uniqueness theorem to prove that $\frac{1}{1-x}$ is the unique solution on the interval $(x_3 - \frac{1}{4}, x_3 + \frac{1}{4})$ and obtain a contradiction to $z(x_2) \neq \frac{1}{1-x}$.

3.8 Implicit Solutions

- As we have seen in preceding examples, methods for solving differential equations do not always lead to an explicit solution, and in many cases we must settle for implicit solutions from which we have no way to isolate an explicit solution.

- **Example 24:** The family of solutions of the equation

$$y' = -\frac{2xy}{(1+2y^2)}$$

is given by

$$ye^{x^2+y^2} = C$$

This is easily checked by implicit differentiation of the equality.

- Although we do not have an explicit solution, we can learn quite a lot about it and its derivatives by implicit differentiation (including higher-order differentiation) and suitable mathematical theorems. Numerical methods based on the equation's direction field can be used to compute desired values of the solutions to very high levels of accuracy.

- The direction field of our equation was drawn using a Python program of fewer than 20 lines of code. It gives us some idea of the form of the solutions by allowing us to observe the directions of flow.

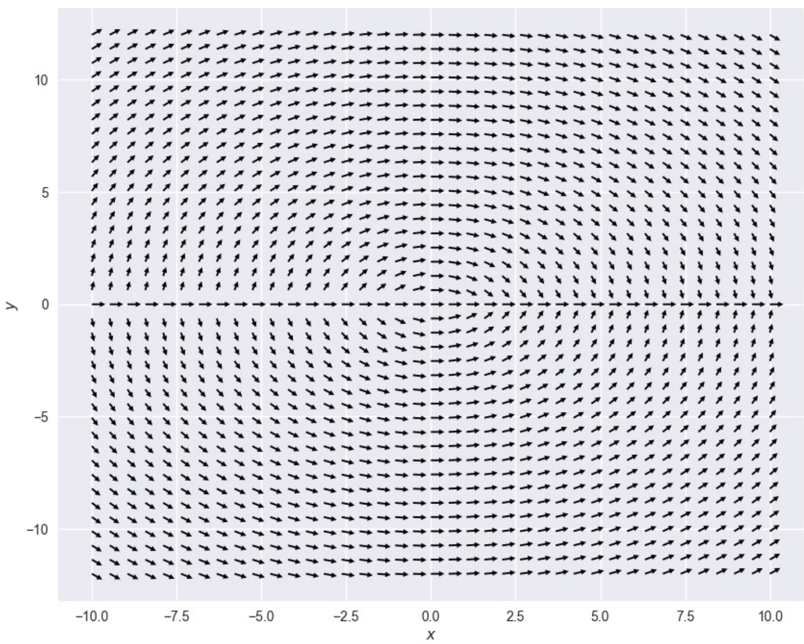

Fig. 3.16: The direction field of $\dfrac{-2xy}{1+2y^2}$

- Moreover, it is now very easy to write and run code to draw integral curves and thereby compute any desired value of a solution to any required level of accuracy.

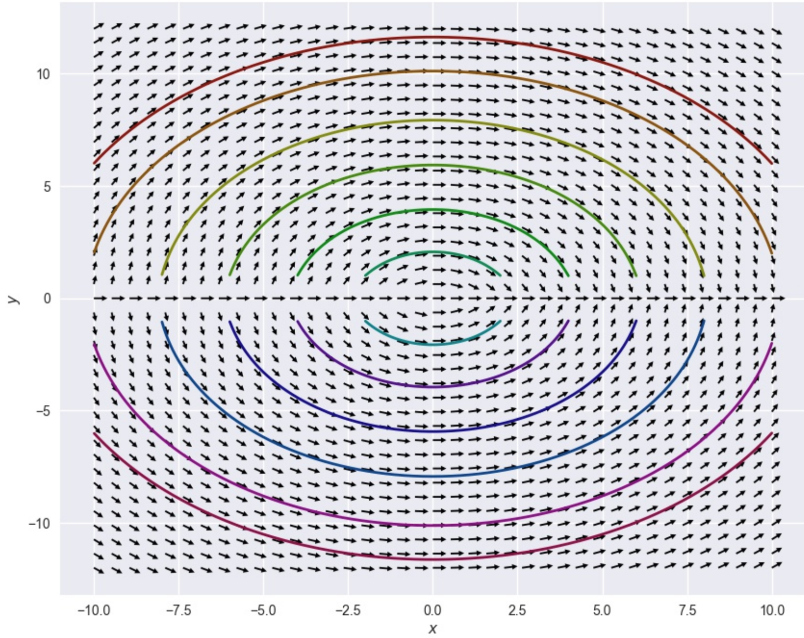

Fig. 3.17: Integral curves in the direction field of $\dfrac{-2xy}{1+2y^2}$

3.9 A Schematic Discussion of the Proof of the Existence and Uniqueness Theorem Picard's Method of Successive Approximations

We recall the theorem:

Theorem 3.3: (The Existence and Uniqueness Theorem, Version 2)

Consider a first-order initial-value problem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

If the functions $f(x,y)$ and $f_y(x,y)$ are continuous inside a symmetric rectangle around the point (x_0, y_0) :

$$\begin{cases} x_0 - a < x < x_0 + a \\ y_0 - b < y < y_0 + b \end{cases}$$

and, in addition, there exists a real constant M such that $|f(x,y)| \leq M$ throughout the rectangle, then there exists a **unique** solution on the subinterval $(x_0 - h, x_0 + h)$, where

$$h = \min\left\{a, \frac{b}{M}\right\}$$

We present a schematic description of the method of successive approximations developed by the mathematician Émile Picard, who used it to prove the existence and uniqueness theorem.

3.9.1 Step 1: Normalizing the Problem

For simplicity and convenience, assume that our initial-value problem is centered at the point (x_0, y_0)

$$\begin{cases} \frac{dy}{dx} = f(x, y) \\ y(0) = 0 \end{cases}$$

This can be achieved by translating the variables:

$$\begin{cases} \hat{x} = x - x_0 \\ \hat{y} = y - y_0 \end{cases}$$

We may therefore restrict ourselves to proving the existence and uniqueness theorem at the point $(0, 0)$, as follows:

The Existence and Uniqueness Theorem Restricted to the Point $(0, 0)$:

Consider a first-order initial-value problem

$$\begin{cases} y' = f(x, y) \\ y(0) = 0 \end{cases}$$

If the functions $f(x, y)$ and $f_y(x, y)$ are continuous inside a symmetric rectangle:

$$\begin{cases} -a < x < a \\ -b < y < b \end{cases}$$

and, in addition, there exists a real constant M such that $|f(x, y)| \leq M$ throughout the rectangle, then there exists a **unique** solution on the subinterval $(-h, h)$, where

$$h = \min\left\{a, \frac{b}{M}\right\}$$

3.9.2 Step 2: Conversion to an Integral Equation

In the second step, Émile Picard transformed the equation into an integral equation after realizing that it would be much more convenient to handle than the differential equation. Assume that there exists a function $\phi(t)$ that solves the problem on some interval $(-h, h)$, that is,

$$\begin{cases} \phi'(t) = f(t, \phi(t)), \ -h < t < h \\ \phi(0) = 0 \end{cases}$$

Fig. 3.18: Charles Émile Picard 1856-1941

- The composite function $f(t, \phi(t))$ depends only on the variable t, and therefore we may write

$$\int_0^x \phi'(t)dt = \int_0^x f(t, \phi(t))dt$$

But by the fundamental theorem of calculus,

$$\int_0^x \phi'(t)dt = \phi(x) - \phi(0) = \phi(x) - 0 = \phi(x)$$

and we obtain an integral equation

$$\phi(x) = \int_0^x f(t, \phi(t))dt$$

Clearly, every solution of this equation is also a solution of our original differential equation.

- It turns out to be simpler to prove the existence of a unique solution of the integral equation than of the differential equation.

3.9.3 Step 3: Constructing a Sequence of Approximations to the Solution

- We construct a sequence of functions $\{\phi_n(x)\}_{n=1}^{\infty}$ that is intended to converge to the solution $\phi(x)$.
- The first function in the sequence is the zero function:

$$\phi_0(x) = 0, \quad -h < x < h$$

The function $\phi_0(x)$ satisfies the initial condition $\phi_0(0) = 0$ but does not necessarily satisfy the integral equation (otherwise the process ends successfully!).

- If $\phi_0(x)$ does not solve the equation, define the next function in the sequence by

$$\phi_1(x) = \int_0^x f(t, \phi_0(t))dt$$

  Again, the initial condition $\phi_1(0) = 0$ clearly holds, and if $\phi_1(x)$ solves the equation, we are done.

- Induction step: If the process terminates successfully for some n ($\phi_n = \phi_{n-1}$), we are done. Otherwise, define the next term in the sequence, $\phi_{n+1}(x)$, by

$$\phi_{n+1}(x) = \int_0^x f(t, \phi_n(t))dt$$

- At each stage, note the following five issues:
  A. Does the function $\phi_n(x)$ exist at all? After all, there is no guarantee that the integral defining it exists!
  B. Does the sequence of functions $\{\phi_n(x)\}_{n=1}^{\infty}$ converge?
  C. If the sequence converges to a limiting function $\phi(x)$, does it solve the problem?
  D. If the limit $\phi(x)$ is indeed a solution of the problem, is it the unique solution?
  E. What is the solution's domain of definition? It may shrink during the process.

- We will try to examine the above points using a relatively simple specific example that also illustrates how to design an algorithm for finding the solution. The complete proof of the existence and uniqueness theorem can be found in Uri Elias's book (Sections 3.3, 3.4) or in the book by Boyce/Diprima.

3.9.4 An Example of Picard's Method of Successive Approximations

- We solve the following initial-value problem using Picard's method of successive approximations:

$$\begin{cases} y' = f(x,y) = 3x^2(y+1) \\ y(0) = 0 \end{cases}$$

- As noted, the first approximation in the sequence is the zero function:

$$\phi(x) = 0, \quad -h < x < h$$

  This approximation is not a solution of the problem, so we must proceed to the next approximation.

- The second approximation in the sequence is

$$\phi_1(x) = \int_0^x f(t,\phi_0(t))dt = \int_0^x f(t,0)dt = \int_0^x 3t^2dt = x^3$$

We obtain

$$\phi_1(x) = x^3$$

This still does not solve the equation. We proceed to the next approximation.

- The third approximation in the sequence is

$$\phi_2(x) = \int_0^x f(t,\phi_1(t))dt = \int_0^x f(t,t^3)dt = \int_0^x 3t^2(t^3+1)dt = x^3 + \frac{1}{2}x^6$$

We obtain

$$\phi_2(x) = x^3 + \frac{1}{2}x^6$$

Again, we do not obtain a solution.

- The fourth approximation in the sequence is

$$\begin{aligned}\phi_3(x) &= \int_0^x f(t,\phi_2(t))dt = \int_0^x f(t,t^3+\tfrac{1}{2}t^6)dt \\ &= \int_0^x 3t^2(1+t^3+\tfrac{1}{2}t^6)dt = \int_0^x (3t^2+3t^5+\tfrac{3}{2}t^8)dt \\ &= x^3+\tfrac{1}{2}x^6+\tfrac{1}{6}x^9\end{aligned}$$

We obtain

$$\phi_3(x) = x^3 + \frac{1}{2}x^6 + \frac{1}{6}x^9$$

which, as expected, does not solve the problem.

- Using induction, it can be proved that for every natural number n,

$$\phi_n(x) = \frac{x^3}{1!} + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots + \frac{x^{3n}}{n!} + \cdots$$

The ratio test for power series easily shows that the series converges, and its radius of convergence is the entire real line. Denote its limiting function by $\phi(x)$

$$\phi(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

- From the familiar power series of the function $y = e^x$, it is easy to prove that

$$\phi(x) = e^{x^3} - 1$$

A simple check proves that $\phi(x)$ is indeed a solution of the problem.

3.9.5 Domain of Existence and Uniqueness of the Solution

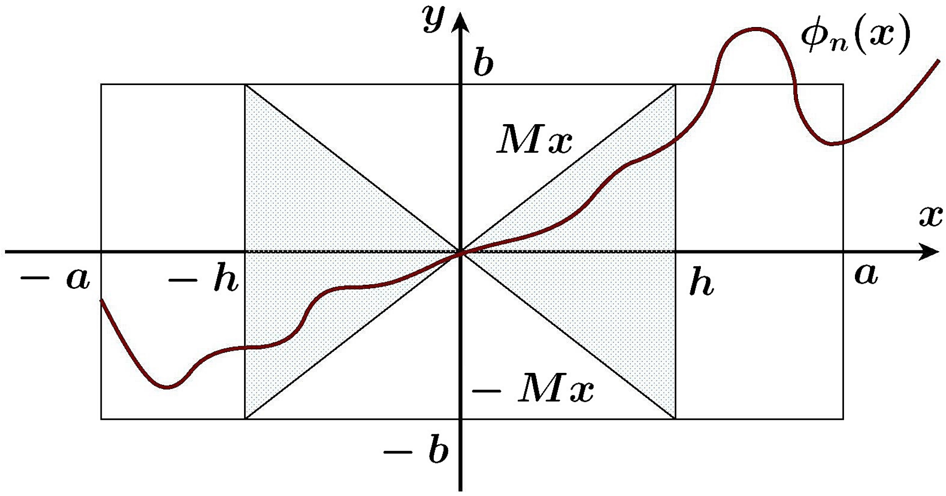

Fig. 3.19: The domain of definition of $\phi_n(t)$ versus the domain of definition of $f(x,y)$

- In the preceding discussion, we did not really address the solution's domain of definition or why the solution is unique. Without going into all the details (which can be read in the textbooks mentioned earlier), we will try to explain the general ideas.

- We begin by addressing item A: The worst-case scenario that may occur at some stage of the process is that the value of $\phi_n(t)$ exceeds the bounds of the rectangle (which is actually the domain of definition of $f(x,y)$); for example, if for some t in the interval $(-a,a)$ we have $\phi_n(t) > b$, as shown in Figure 3.19. If this happens, the expression $f(t,\phi_n(t))$ is undefined, and therefore the expression $\int_0^x f(t,\phi_n(t))$ is also undefined! The process will then become stuck, and we will be unable to proceed to the next approximation in the sequence.

- To prevent such a scenario, we must restrict ourselves to a smaller interval, $(-h,h)$, on which $f(x,\phi_n(x))$ does not exceed the bounds of the rectangle (see Figure 3.19).

- For this purpose, we use the assumption that the function $f(x,y)$ is bounded by M on the rectangle. Therefore, at every stage n of the approximation process, $\phi_n(x)$ satisfies

$$|\phi'_{n+1}(x)| = |f(x,\phi_n(x))| \leq M$$

That is, the slope of the function $\phi_n(x)$ is bounded by M, and therefore the function is bounded by the lines $y = Mx$ and $y = -Mx$, as shown in Figure 3.19.

$$|\phi_n(x)| \leq Mx$$

For this reason, if we restrict ourselves to the interval $(-h, h)$ with $h = \frac{b}{M}$, then $|f(x, \phi_n(x))| \leq b$ for every x in the interval $(-h, h)$. If $a < \frac{b}{M}$, then of course we must take $h = a$ in order not to exceed the bounds of the rectangle. Therefore, the precise definition of h is

$$h = \min\left\{a, \frac{b}{M}\right\}$$

- Convergence of the sequence of approximations on the interval $(-h, h)$ is proved by replacing the sequence of functions $\{\phi_n(x)\}_{n=1}^{\infty}$ with the series of functions

$$\sum_{n=0}^{\infty}[\phi_{n+1}(x) - \phi_n(x)]$$

and applying the Weierstrass M-test, which guarantees **uniform convergence**.

- Convergence of the series is established by proving the inequality

$$|\phi_{n+1}(x) - \phi_n(x)| \leq 2bM^n\frac{x^n}{n!} \leq \frac{M^n h^n}{n!}$$

Since the series $\sum\limits_{n=0}^{\infty}\frac{M^n h^n}{n!}$ converges, the Weierstrass M-test implies that our series of functions also converges. Complete details can be found in the books by Uri Elias and Boyce/Diprima.

- **Uniqueness of the Solution:** If $\phi(x)$ and $\psi(x)$ are two solutions of the integral equation

$$y(t) = \int_0^x f(t, y(t))dt$$

that is, for every x in the interval $(-h, h)$,

$$\begin{aligned}\phi(x) &= \int_0^x f(t, \phi(t))dt \\ \psi(x) &= \int_0^x f(t, \psi(t))dt\end{aligned}$$

then

$$\phi(x) - \psi(x) = \int_0^x [f(t, \phi(t)) - f(t, \psi(t))]dt$$

By the intermediate value theorem (Calculus 1), there exists an intermediate value $\hat{y}(t)$ between

$\phi(t)$ and $\psi(t)$ such that

$$f(t,\phi(t)) - f(t,\psi(t)) = f_y(t,\hat{y}(t)) \cdot [\phi(t) - \psi(t)]$$

Therefore

$$\phi(x) - \psi(x) = \int_0^x f_y(t,\hat{y}(t)) \cdot [\phi(t) - \psi(t)]dt$$

Therefore

$$0 \le |\phi(x) - \psi(x)| \le \int_0^x |f_y(t,\hat{y}(t))| \cdot |\phi(t) - \psi(t)|dt$$

The continuity of f_y implies that f_y is bounded by some K, and therefore

$$0 \le |\phi(x) - \psi(x)| \le K\int_0^x |\phi(t) - \psi(t)|dt$$

Let

$$g(x) = \int_0^x |\phi(t) - \psi(t)|dt$$

Then $g'(x) = |\phi(x) - \psi(x)|$, and in addition $g(0) = 0$. Our inequality becomes

$$0 \le g'(x) \le Kg(x)$$

That is, $g(x)$ is a solution of the following **differential inequality** problem:

$$\begin{cases} g'(x) - Kg(x) \le 0, \quad -h < x < h \\ g(x) \ge 0 \\ g(0) = 0 \end{cases}$$

Multiply the inequality by the integrating factor e^{-Kx}

$$\left[e^{-Kx}g(x)\right]' = e^{-Kx}[g'(x) - Kg(x)] \le 0$$

and integrate both sides of the inequality over the interval $[0,x]$:

$$\int_0^x e^{-Kt}[g'(t) - Kg(t)] = e^{-Kx}g(x) \le 0$$

We have obtained $g(x) \le 0$. But we also know that $g(x) \ge 0$. Therefore, necessarily $g(x) = 0$ for every x in the interval $(-h,h)$. That is, $\phi(x) = \psi(x)$ for every x in the interval $(-h,h)$, and therefore the solution of our original differential equation is unique.

3.10 Appendix: The Existence and Uniqueness Theorem for a Differential Equation of Order n

To complete the picture, we conclude the chapter by stating the existence and uniqueness theorem for equations of arbitrary order.

Theorem 3.5: Consider a normalized differential equation of order n, together with n initial conditions:

$$\begin{cases} y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}) \\ y(x_0) = \alpha_0 \\ y'(x_0) = \alpha_1 \\ y''(x_0) = \alpha_2 \\ \quad \vdots \\ y^{(n-1)}(x_0) = \alpha_{n-1} \end{cases}$$

where $f(x, z_0, z_1, z_2, \ldots, z_{n-1})$ is a continuous function of $n+1$ variables, with n continuous partial derivatives f_{z_0}, f_{z_1}, $\ldots$, $f_{z_{n-1}}$, in a neighborhood of the point $(x_0, \alpha_0, \alpha_1, \ldots, \alpha_{n-1})$. Then there exists a neighborhood $(x_0 - h, x_0 + h)$ on which the equation with all the initial conditions has a unique solution $y(x)$, such that all the derivatives $y'(x)$, $y''(x)$, $\ldots$, $y^{(n)}(x)$ exist and are continuous on the neighborhood $(x_0 - h, x_0 + h)$.

Exercises for chapter 3

Guidance: At the end of the list, there is a link to a Jupyter notebook containing final and/or complete solutions to a small number of the exercises. But before clicking the link, it is strongly recommended that you solve them yourself! Additional solutions will be added over time. We would be very pleased to receive additional solutions from students for exercises that have not yet been solved!

1. Find a general solution.

(a) $xy' - y = y^3$

(b) $xyy' = 1 - x^2$

(c) $y' \tan x = y$

(d) $y' - xy^2 = 2xy$

(e) $e^{-s}\left(1 + \frac{ds}{dt}\right) = 1$

(f) $y' = 10^{x+y}$

(g) $(x+1)dx + (y-1)dy = 0$

(h) $\cos\sqrt{x}\, dx - \sqrt{x}\, dy = 0$

(i) $\sqrt{1-x^2}dy - \sqrt{1-y^2}dx = 0$

(j) $e^x(1+e^y)dx + e^y(1+e^x)dy = 0$

(k) $xy^3y' = x^4 + y^4$

2. Solve the following initial-value problems.

(l) $\begin{cases} (x^2-1)y' + 2xy^2 = 0 \\ y(2) = 1 \end{cases}$

(m) $\begin{cases} y' = 3\sqrt[3]{y^2} \\ y(2) = 0 \end{cases}$

(n) $\begin{cases} (xy^2+x)dx + (x^2y-y)dy = 0 \\ y(0) = 1 \end{cases}$

(o) $\begin{cases} y'\sin x + y\ln y = 0 \\ y(\frac{\pi}{2}) = 1 \end{cases}$

(p) $\begin{cases} 2xyy' + x^2 - y^2 = 0 \\ y(1) = 0 \end{cases}$

(q) $\begin{cases} xy' + y - e^x = 0 \\ y(a) = b \end{cases}$

(r) $\begin{cases} y' - 2y = -x^2 \\ y(0) = \frac{1}{4} \end{cases}$

(s) $\begin{cases} xy' + y = 2x + 1, \quad x > 0 \\ y(1) = 1 \end{cases}$

(t) $\begin{cases} (x-2)dx + (y-1)dy = 0 \\ y(2) = 1 \end{cases}$

(u) $\begin{cases} (1+x^2)dy - 2x(y+3)dx = 0 \\ y(0) = -1 \end{cases}$

(v) $\begin{cases} y'\sqrt{1-x^2} = x \\ y(1) = 0 \end{cases}$

(x) $\begin{cases} y'\tan x = 1 + y \\ y(\frac{\pi}{6}) = -0.5 \end{cases}$

3. Find a general solution.

(a) $y' = \dfrac{2xy}{x^2+y^2}$

(b) $xy' - y = x\tan\frac{y}{x}$

(c) $xy' - y = (x+y)\ln\frac{x+y}{x}$

(d) $\left(x - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}\,dy = 0$

(e) $y' - \dfrac{y}{x} = x$

(f) $x(y' - y) = e^x$

(g) $xy' + (x+1)y = 3x^2e^{-x}$

(h) $xy' - 2y = x^3 + x$

(i) $y'\sin x - y\cos x = 1$

(j) $\dfrac{dy}{dx} = \dfrac{2y - x + 5}{2x - y - 4}$

(k) $(2x + y - 3)dx + (x + y - 1)dy = 0$

4. Find a general solution.

(a) $xy^2y' = x^2 + y^2$

(b) $xy\,dy = (y^2 + x)dx$

(c) $x^2y' = y(x + y)$

(d) $y' + y = xy^3$

(e) $y' + x^3\sqrt{y} = 3y$

(f) $y' + \frac{y}{x} = x^2y^2$

(g) $(2 - 9xy^2)xdx + (4y^2 - 6x^3)ydy = 0$

(h) $2(1 + \sqrt{x^2 - y})xdx - \sqrt{x^2 - y}dy = 0$

(i) $(1 + y^2\sin 2x)dx - 2y\cos^2 x\,dy = 0$

(j) $\dfrac{2xdx}{y^3} + \dfrac{y^2 - 3x^2}{y^4}dy = 0$

(k) $xdx + ydy = \dfrac{xdy - ydx}{x^2 + y^2}$

(l) $\left(y + \dfrac{1}{1 + x^2}\right)dx + \left(x - \dfrac{1}{1 + y^2}\right)dy = 0$

(m) $xy' - 2x^2\sqrt{y} = 4y$

(n) $xy^2y' = x^2 + y^3$

(o) $(x + 1)(yy' - 1) = y^2$

5. Find a general solution of the following equations by finding an integrating factor of the form $\mu = \mu(x)$ or $\mu = \mu(y)$.

(a) $(x + y^2)dx - 2xydy = 0$

(b) $(x^2 + y^2 + x)dx + ydy = 0$

(c) $y(1 + xy)dx - xdy = 0$

(d) $(x^2y - 1)dx + (x^3 + x^2\cos y)dy = 0$

(e) $(2xy\ln y + y^2\cos x)dx + (x^2 + y\sin x)dy = 0$

6. Solve the following equation using an integrating factor of the form $\mu(x, y) = f(xy)$

$$\left(\frac{2}{x} + \frac{2}{x^2y}\right)dx + \left(\frac{2}{y} + \frac{2}{xy^2} + \frac{1}{x}\right)dy = 0$$

7. Find a general solution.

(1) $(x+y)y' = y$

(2) $(x-2)y' = y^2$

(3) $y' = 2xy + x$

(4) $(x^2 + 2xy^3)dx + (y^2 + 3x^2y^2)dy = 0$

(5) $y' + y\cos x = \sin x \cos x$

(6) $(2e^x + y^4)dy - ye^x dx = 0$

(7) $\tan x \cdot \frac{dy}{dx} - y = a$

(8) $(3x^2 + 2xy - y^2)dx + (x^2 - 2xy - 3y^2)dy = 0$

(9) $\dfrac{y}{x}dx + (y^3 + \ln x)dy = 0\,, \quad 0 < x < \infty$

(10) $x(y^4 - x^2)dy + y(y^4 + x^2)dx = 0$

(11) $(3x^2y + 2xy + y^3)dx + (x^2 + y^2)dy = 0$

(12) $y' = \dfrac{3x^2 - 2xy}{x^2 - 4y^3}$

(13) $y' = \sqrt{4x + 2y - 1}$

(14) $\dfrac{dx}{dy} = \cos(y - x)$

(15) $2x(1 + \sqrt{x^2 - y})dx - \sqrt{x^2 - y}dy = 0$

(16) $3x^2(1 + \ln y)dx = (2y - \frac{x^3}{y})dy$

(17) $\left(y - \frac{1}{x}\right)dx - \dfrac{dy}{y} = 0$

(18) $y^2dx - (xy + x^3)dy = 0$

(19) $y^2dx + (xy + \tan(xy))dy = 0$

(20) $\frac{2}{3}xyy' = \sqrt{x^6 - y^4} + y^2$

(21) $x^3dy - y(x^2 + y^2)dx = 0$

(22) $xy' - y = \sqrt{x^2 + y^2}$

(23) $(y + \sqrt{xy})dx = xdy$

(24) $xy' = \sqrt{x^2 - y^2} + y$

(25) $(2x-4y+6)dx+(x+y-3)dy=0$

(26) $(2x+y+1)dx-(4x+2y-3)dy=0$

(27) $x-y-1+(y-x+2)y'=0$

(28) $(y+2)dx=(2x+y-4)dy$

(29) $y'=2\left(\dfrac{y+2}{x+y-1}\right)^2$

(30) $y'=\dfrac{y+2}{x+1}+\tan\dfrac{y-2x}{x+1}$

(31) $2xy'+y=y^2\sqrt{x-x^2y^2}$

(32) $2y'+x=4\sqrt{y}$

(33) $y+y'\ln^2 y=(x+2\ln y)y'$

(34) $y'=\dfrac{1}{x-y^2}$

(35) $x-\dfrac{y}{y'}=\dfrac{2}{y}$

(36) $(x+y)^2y'=1$

(37) $2x^3yy'+3x^2y^2+7=0$

(38) $\frac{dx}{x}=\left(\frac{1}{y}-2x\right)dy$

(39) $xy'=e^y+2y'$

(40) $dy+(xy-xy^3)dx=0$

(41) $\dfrac{y-xy'}{x+yy'}=2$

(42) $(xy^4-x)dx+(y+yx)dy=0$

(43) $(y+\sin x)dy+(y\cos x-x^2)dx=0$

(44) $yy'+y^2\cot x=\cos x$

(45) $(e^y+2xy)dx+(e^y+x)xdy=0$

(46) $y'+x\sqrt[3]{y}=3y$

(47) $y'\sqrt{x}=\sqrt{y-x}+\sqrt{x}$

(48) $(x\cos y+\sin 2y)y'=1$

(49) $(4xy-3)y'+y^2=1$

(50) $(2x-2y-1)dx-(y-x-1)dy=0$

(51) $y'=\dfrac{1}{x\tan(y)+e^{5\sin y}}$

(52) $(x^4-x^4y+x)y'=3\,,\quad 0<x<\infty$

8. Solve the following initial-value problems.

(1) $\begin{cases} y'=\frac{y^2-x^2}{2xy} \\ y(2)=-2 \end{cases}$ **(2)** $\begin{cases} y'=2+\sqrt{9-(y-2x-1)^2} \\ y(0)=4 \end{cases}$

(3) $\begin{cases} (1+x^2)y'-xy=2x \\ y(0)=0 \end{cases}$ **(4)** $\begin{cases} xy'-3y=x^4e^x \\ y(1)=e \end{cases}$

(5) $\begin{cases} y'\sin x-y\cos x=1 \\ y(\pi)=1 \end{cases}$ **(6)** $\begin{cases} xy'-2y=x^3e^x \\ y(1)=0 \end{cases}$

(7) $\begin{cases} y'-y\tan x=\sec x \\ y(0)=0 \end{cases}$ **(8)** $\begin{cases} xy'-4y=x^2\sqrt{y} \\ y(1)=1 \end{cases}$

(9) $\begin{cases} y'-y\tan x=y^4\cos x \\ y(\pi)=-1 \end{cases}$ **(10)** $\begin{cases} (2x-1)dy=(y+1)dx \\ y(5)=0 \end{cases}$

(11) $\begin{cases} \frac{dy}{\sqrt{y}}+dx=\frac{dx}{\sqrt{x}} \\ y(0)=1 \end{cases}$ **(12)** $\begin{cases} xy+y^2-(2x^2+xy)y'=0 \\ y(1)=1 \end{cases}$

9. Does the solution of the initial-value problem

$$\begin{cases} y'=\sin x(y-1) \\ y(2)=1 \end{cases}$$

intersect the x-axis? Explain.

Hint: Try to guess a constant solution of the equation.

10. Consider the following initial-value problem:

$$\begin{cases} y'=\sin xy(y-2) \\ y(5)=1 \end{cases}$$

A. Prove that the solution of the problem is positive throughout its domain of definition.

B. Prove that the solution of the problem is bounded by $M = 2$ throughout its domain of definition.

Hint: Guess two constant solutions of the equation.

11. Consider the family of curves $\cos y = ke^{x^2}$.
Find a family orthogonal to the given family.

12. Consider the family of curves $y^3 = -3\ln(kx)$.
Find a family orthogonal to the given family.

13. Consider the family of curves $y^2 = ke^x + x + 1$.
Find a family orthogonal to the given family.

14. Consider the family of curves $x^2 + y^2 = ky$, $-\infty < k < \infty$

(a) Find a family orthogonal to the given family.

(b) Find the curves from the two families that intersect at the point $(1, 2)$.

15. Consider the family of curves $x^2 + (y + k)^2 = k^2 - 1$, $|k| > 1$

A. Find a family orthogonal to the given family.

B. Find the family orthogonal to the family:

$$\begin{cases} x^2 + y^2 = Cy \\ -\infty < C < \infty \end{cases}$$

C. Find the curves from the two orthogonal families that intersect at the point $(1, 2)$.

16. In the family orthogonal to the family of curves

$$x^2 + y^2 = 18\ln|y| + k$$

there is exactly one circle. Find its equation and explain the solution method.

17. Find a plane curve passing through the point $(1, 1)$ and perpendicular to every curve in the family

$$(x + k)y + kx - 1 = 0, \quad -\infty < k < \infty$$

18. Consider the family of canonical ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

whose width (in the x-direction) is three times their height (in the y-direction). Find the equation of the curve that is orthogonal to this family and passes through the point $(2, 1)$.

19. Solve the initial-value problem

$$\begin{cases} y' + \dfrac{\ln^2 x}{\sin^2 x} = 0 \\ y(4) = 0 \end{cases}$$

What is the domain of definition of the solution you found?

20. Consider the initial-value problem

$$\begin{cases} y' = \dfrac{y^2 - 1}{y} \cdot \sin(x^3) \\ y(0) = -\frac{1}{2} \end{cases}$$

(a) Prove that the solution you found is bounded above on its domain of definition.

(b) On which intervals is the solution increasing and decreasing?

Guidance: Note that both $y = -1$ and $y = 1$ are solutions of the equation. Use the nonintersection property of the family of solutions (under the hypotheses of the existence and uniqueness theorem, of course).

21. Consider the initial-value problem

$$\begin{cases} y' - 2x = \sqrt[3]{(y - x^2 - 2)^2} \\ y(1) = 3 \end{cases}$$

(a) Find two distinct solutions of the problem.

(b) Explain why this result does not contradict the existence and uniqueness theorem.

Guidance: Use the substitution $v(x) = y(x) - x^2 - 2$. Check for singular solutions.

22. Consider the initial-value problem

$$\begin{cases} y' + y(4y - 8)e^{4xy} = 7x^5 \sin(\pi y) \\ y(1) = 1 \end{cases}$$

Prove that the solution $y(x)$ of the problem is bounded (above and below) throughout its domain of definition.

Guidance: Find two constant solutions of the equation.

23. Consider the differential equation

$$y' = \frac{x}{y} + \ln\left[1 + (y - x)^2\right] \cdot \ln\left[1 + \left(y - \sqrt{x^2 + 3}\right)^2\right]$$

Without solving the equation, answer the following two questions:

(a) Guess two distinct solutions $y_1(x)$, $y_2(x)$ of the equation.

(b) Let $\phi(x)$ be a solution of the equation on the ray $(0,\infty)$ satisfying $\phi(1)=\frac{3}{2}$. Prove that

$$\lim_{x\to\infty}\frac{\phi(x)}{x}=1$$

Guidance: Determine when the function $\ln$ vanishes.

24. Consider the initial-value problem

$$\begin{cases} y' = (2-\sin^2 x)(y^2-y^5) \\ y(0)=0.25 \end{cases}$$

(a) Does the problem have a solution? If so, is it unique?

(b) If a solution exists, does $\lim\limits_{x\to\infty} y(x)$ exist?

25. Consider the following differential equation together with an initial condition:

$$\begin{cases} (16-y^2)\cos(y)dx + (y^2+5y-15)dy = 0 \\ y(0)=3 \end{cases}$$

Prove that its solution is bounded throughout its domain of definition.

Guidance: Try to guess two constant solutions. Use the existence and uniqueness theorem.

26. Consider the differential equation $y' = (x^3+y^3)(2\cos y - 1)$.
Prove that every solution converges to a finite limit as x tends to infinity.

Guidance: Try to guess infinitely many constant solutions ...

27. Let $y(x)$ be a solution of the initial-value problem

$$\begin{cases} y' = \left(y-\sqrt{2x^2+1}\right)^2\left(y-\sqrt{2x^2+5}\right)^2 + \dfrac{2x}{y} \\ y(0)=2 \end{cases}$$

Compute the limit $\lim\limits_{x\to\infty}\dfrac{y(x)}{x}$.

Guidance: Use the existence and uniqueness theorem. Check for singular solutions.

28. Consider the equation $y' = 2\sqrt{y-2x-1}\cdot e^{-\sqrt{y-2x-1}} + 2$.
Find two explicit solutions (that is, of the form $y=f(x)$) of the equation that satisfy the initial condition $y(3)=7$, and explain why this does not contradict the existence and uniqueness theorem.

29. Consider the differential equation

$$y' = \frac{\sqrt{1-y^2}}{x}, \qquad 0 < x < \infty$$

Find two solutions of the equation satisfying $y(1) = 1$, and explain why this does not contradict the existence and uniqueness theorem.

30. Let $y(x)$ be a solution of the initial-value problem

$$\begin{cases} y' = \dfrac{x}{4y} + \left(2y - \sqrt{x^2+1}\right)(2y - x) \\ y(0) = \alpha \end{cases}$$

It is known that $0 < \alpha < \frac{1}{2}$. Compute the limit $\lim\limits_{x \to \infty} \dfrac{y(x)}{x}$.

31. For the equation

$$y dx - (x^2 + y^2 + x) dy = 0$$

find an integrating factor of the form $f(x^2 + y^2)$. Solve the equation and express the solution in the form $x = x(y)$.

32. For the equation

$$x dx + (y + 4y^3x^2 + 4y^5) dy = 0$$

find an integrating factor of the form $f(x^2 + y^2)$. Find its general solution.

33. Consider the differential equation

$$x dy + y dx - 3x^3y^2 dy = 0$$

It is known that the equation has an integrating factor of the form $\mu(x, y) = x^{\alpha}y^{\beta}$. Find the integrating factor.

34. Consider the differential equation

$$x^3y^3(2y dx + x dy) - (5y dx + 7x dy) = 0$$

It is known that the equation has an integrating factor of the form $\mu(x, y) = x^{\alpha}y^{\beta}$. Find α, β, and find a particular solution passing through the point $(1, 1)$.

35. Consider the differential equation

$$(y^4 - 2x^3y) dx + (x^4 - 2xy^3) dy = 0$$

It is known that the equation has an integrating factor of the form $\mu(x,y) = x^\alpha y^\beta$. Find α, β.

36. Consider the differential equation

$$(-x+y-2)dx + (-5x-3y-6)dy = 0$$

It is known that the equation has an integrating factor of the form $\mu(x,y) = \mu(x+3y)$. Find the integrating factor.

37. Let $y_1(x)$ and $y_2(x)$ be two solutions of the differential equation

$$y' = F(x,y)$$

on the interval (α,β), where $F(x,y)$ and $F_y(x,y)$ are defined and continuous on the entire plane. Prove that if there is a point x_0 in the interval (α,β) for which $y_1(x_0) < y_2(x_0)$, then $y_1(x) < y_2(x)$ for every point x in the interval.

38. An air-filled balloon with initial radius R loses air at a rate of $-3V(t)$ cubic centimeters per second, where $V(t)$ is the volume of the balloon at time t. After how many seconds will the balloon's volume reach one-eighth of its initial volume?

39. Two kilograms of salt are uniformly dissolved in a water tank containing 500 liters of water. Salt water with a concentration of 0.1 kilograms of salt per liter flows into the tank at a rate of 25 liters per minute, and at the same time, the same quantity of liquid is drained from the tank (through another opening). Calculate the salt concentration after 10 minutes.

40. The growth rate of the world's population during the years 1975–2015 was $r = 0.02$ per year. It is known that in 1980 the world's population was 4 billion. What was the population in 2010? If this growth rate remains valid in the future, what will the world's population be in 2050?

41. Find the equation of the plane curve whose normal line at every point also passes through the point $(1,1)$.

42. A raindrop falls from a stationary cloud at an altitude of 2 kilometers above sea level. Only two forces act on the raindrop: Earth's gravitational force and air resistance opposing the direction of the fall, whose magnitude is kv^2, where k is the drag coefficient and $v(t)$ is the velocity of the raindrop t seconds after it begins falling. Find the formula for the velocity of the raindrop, and determine when it reaches sea level.

43. Find the family of plane curves $F(x,y) = 0$, $x > 0$, such that, for every tangent to a curve, the x-axis bisects the segment between the point of tangency P and the point where the tangent intersects the y-axis.

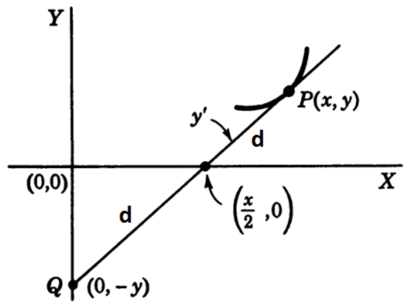

Fig. 3.20: The x-axis bisects every tangent segment between the point of tangency P and its intersection with the y-axis

44. Find the family of plane curves $F(x, y) = 0$ such that for every tangent to a curve, the distance d between the point of tangency and the intersection with the x-axis is constant.

45. A 100-liter water tank contains 30 kilograms of dissolved salt, mistakenly added instead of 20 kilograms. To correct the mistake, 3 liters of liquid per minute are drained from the tank through a lower opening at a constant rate. At the same time, 3 liters of pure water per minute flow into the tank. Assuming that the salt concentration in the tank remains uniform throughout the process (by continuous mixing), how long will it take to reach the desired concentration?

46. A thermometer stored at a temperature of T_0 degrees is placed in a hot-water bath maintained at a constant temperature of T degrees at time $t = 0$.

A. Write a differential equation for the temperature function $y(t)$ of the thermometer as a function of time t (in minutes).

B. Find an explicit formula for $y(t)$ if it is known that after t_1 minutes the thermometer indicated the temperature T_1. Express $y(t)$ in terms of T, T_0, T_1, t_1, t.

47. Find a first-order differential equation whose general solution is

$$y = (x^2 + C)^3 + 1$$

48. Find all functions $q(x)$ for which the equation

$$(x^2 + y)dx + q(x)dy = 0$$

has the integrating factor $\mu(x) = x$.

Link to solutions

Chapter 4

Linear Differential Equations of Order $n > 1$

- After an extensive study of first-order differential equations, we now turn to differential equations of higher order ($n > 1$). We begin by considering several familiar examples.
  - Second-order linear equations with constant coefficients: $y'' - 5y' + y = e^x$
  - The Bessel equations: $x^2y'' + xy' + (x^2 - n^2)y = y$
  - Legendre equations: $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1) = 0$
  - The spring equation: $my'' + cy' + ky = F(x)$
  - Free fall with air resistance: $my''(t) = mg - ky'(t)$
- For example, let us solve the following simple equation:

$$y'' = 6x$$

This is a second-order linear equation. Two simple integrations yield its general solution:

$$\begin{aligned} y' &= 3x^2 + C_1 \\ y'' &= x^3 + C_1x + C_2 \end{aligned}$$

This is indeed a two-parameter family of solutions that contains <u>all</u> the solutions of the equation. It involves two constants, and therefore an initial-value problem requires two initial conditions: $y(x_0) = y_0$, $y'(x_0) = y'_0$.

- **Example 1:** The solution of the initial-value problem

$$\begin{cases} y'' = 6x \\ y(0) = -1 \\ y'(0) = 5 \end{cases}$$

is $y(x) = x^3 + 5x - 1$.

- In general, the general solution of a differential equation of order n contains n constants.

- **Example 2:** Using reduction of order, solve the equation for free fall with air resistance

$$my''(t) = mg - ky'(t)$$

(introduced in the opening lecture of the course - see Chapter 1), where
m – the mass of the falling body
g – the gravitational constant
k – the drag coefficient

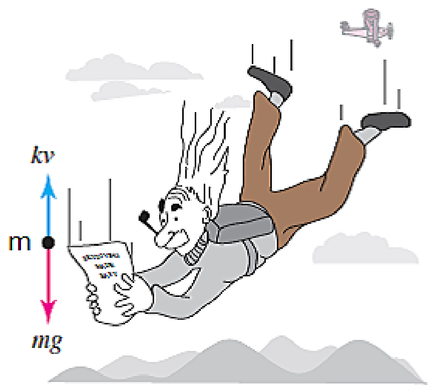

Fig. 4.1: Free fall with air resistance

Solution: If we set $v(t) = y'(t)$ (its physical meaning is velocity), we obtain a first-order linear equation with constant coefficients:

$$v'(t) + \frac{k}{m}v(t) = g$$

whose general solution is

$$v(t) = \frac{mg}{k} + c_1 e^{-\frac{kt}{m}}$$

and therefore

$$y(t) = \int v(t) = \frac{mgt}{k} + c_1 e^{-\frac{kt}{m}} + c_2$$

- **Example 3:** Solve the initial-value problem using reduction of order:

$$\begin{cases} y''' - y'' = 1 \\ y(0) = 2 \\ y'(0) = 8 \\ y''(0) = 2 \end{cases}$$

- **Example 4:** Find a general solution of the equation

$$x^2y'' + 2xy' - 1 = 0, \quad 0 < x < \infty$$

4.1 Theoretical Discussion

- The general form of a linear differential equation of order n is

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where all the functions $a_i(x)$ are continuous on an open interval (α, β). The possibilities $\alpha = -\infty$ and $\beta = \infty$ are of course included.

- When $g(x) \equiv 0$, the equation is called **homogeneous**.
- We will also assume that $a_n(x)$ is not the zero function (otherwise the order of the equation is $n-1$), and hence we may divide the equation by $a_n(x)$ to obtain the normal form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = q(x)$$

- The general solution of such an equation contains n constants c_1, c_2, $\ldots$, c_n, and n initial conditions are required to determine a unique solution.

$$\begin{cases} y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = q(x) \\ y(x_0) = \alpha_0 \\ y'(x_0) = \alpha_1 \\ y''(x_0) = \alpha_2 \\ \qquad \vdots \\ y^{(n-1)}(x_0) = \alpha_{n-1} \end{cases}$$

4.2 The Existence and Uniqueness Theorem for Linear Equations of Order n

Theorem 4.1: Consider a linear differential equation of order n with n initial conditions:

$$\begin{cases} y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = q(x) \\ y(x_0) = \alpha_0 \\ y'(x_0) = \alpha_1 \\ y''(x_0) = \alpha_2 \\ \quad\vdots \\ y^{(n-1)}(x_0) = \alpha_{n-1} \end{cases} \tag{4.1}$$

where

(a) $p_i(x)$ and $q(x)$ are continuous functions on the interval (α, β), $i = 0, 1, 2, \ldots, n-1$;

(b) the point x_0 belongs to the interval: $\alpha < x_0 < \beta$;

(c) the values α_0, α_1, α_2, $\ldots$, α_{n-1} are arbitrary real numbers.

Then there exists a unique solution $\phi(x)$ of the equation on the entire interval (α, β) that satisfies all the initial conditions.

- The existence and uniqueness theorem for a general differential equation of order n guarantees the existence of a solution in some neighborhood $(x_0 - h, x_0 + h)$, which may be smaller than the domain (α, β) of the coefficients of the equation. The theorem above, however, makes a stronger assertion: the domain of the solution is identical to the domain of the coefficients of the equation. The proof of this theorem is beyond the scope of the course and will not be covered.

- **Exercise 4.1:** On which intervals are solutions of the following equation guaranteed to exist?

$$x(x-1)y'' + e^x y' + x^2 y = \cos x$$

- **Exercise 4.2:** What is the unique solution of the following initial-value problem?

$$\begin{cases} y'' + x^3 y + (x^2 - 1)y = 0 \\ y(5) = 0 \\ y'(5) = 0 \end{cases}$$

Hint: This is a **homogeneous** linear equation.

Exercise 4.3: Suppose that $y_1(x)$ is a solution of the equation

$$y'' + x^3 y + (x^2 - 1)y = 0 \tag{4.2}$$

that satisfies

$$\begin{cases} y_1(0) = 0 \\ y_1'(0) = 1 \end{cases}$$

and suppose that $y_2(x)$ is a second solution of (4.2) that satisfies

$$\begin{cases} y_2(0) = 1 \\ y_2'(0) = 0 \end{cases}$$

Prove: if $y_3(x)$ is a solution of equation (4.2) that satisfies

$$\begin{cases} y_3(0) = 1 \\ y_3'(0) = 1 \end{cases}$$

then necessarily $y_3(x) = y_1(x) + y_2(x)$ for every real x!

Exercise 4.4: Continuing the preceding exercise, what can you say about a solution $y_4(x)$ of equation (4.2) that satisfies the initial conditions

$$\begin{cases} y_4(0) = 15 \\ y_4'(0) = -32 \end{cases}$$

Proposition 4.2: If $y(x)$ is a solution of equation (4.2) that satisfies the initial conditions

$$\begin{cases} y(0) = a \\ y'(0) = b \end{cases}$$

then necessarily $y(x) = ay_1(x) + by_2(x)$ for every real x.

4.3 Homogeneous Equations and the Superposition Principle

As noted above, when the right-hand side of a linear differential equation is zero, the equation is called a **homogeneous equation**.

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

Homogeneous equations have special properties that make them easier to handle.

- We begin by studying homogeneous linear equations and later proceed to nonhomogeneous linear equations.

Theorem 4.3: If

$$y_1(x), \quad y_2(x), \quad \ldots, \quad y_k(x)$$

is a set of k solutions of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

and c_1, c_2, $\ldots$, c_k are k arbitrary real numbers, then the function

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution of the homogeneous equation.

Proof: We prove the theorem for a homogeneous equation of order 2:

$$y'' + p_1(x)y' + p_0(x)y = 0$$

The proof of the general case is identical. By assumption,

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

Therefore,

$$\begin{aligned}
\phi'(x) &= c_1 y_1'(x) + c_2 y_2'(x) + \cdots + c_k y_k'(x) \\
\phi''(x) &= c_1 y_1''(x) + c_2 y_2''(x) + \cdots + c_k y_k''(x)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\phi''(x) + p_1(x)\phi'(x) + p_0\phi(x) \\
&\quad = \left[c_1 y_1''(x) + \cdots + c_k y_k''(x)\right] + p_1(x)\left[c_1 y_1'(x) + \cdots + c_k y_k'(x)\right] \\
&\qquad + p_0(x)\left[c_1 y_1(x) + \cdots + c_k y_k(x)\right] \\
&\quad = c_1\left[y_1''(x) + p_1(x)y_1'(x) + p_0 y_1(x)\right] + \cdots + c_k\left[y_k''(x) + p_1(x)y_k'(x) + p_0 y_k(x)\right] \\
&\quad = 0 + \cdots + 0 = 0
\end{aligned}$$

We have shown that $\phi(x)$ is also a solution of the homogeneous linear equation. ∎

- Every expression of the form

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is called a **linear combination**, also known simply as a **combination**.

- More explicitly, we say that the function $\phi(x)$ is a **linear combination of** $\{y_1(x),\ y_2(x),\ \ldots,\ y_k(x)\}$

- **Example 5:** It is easy to verify that the functions $\sin x$ and $\cos x$ are solutions of the equation

$$y'' + y = 0$$

By the preceding theorem, we may conclude that for any two real numbers c_1, c_2, $c_1 \sin x + c_2 \cos x$ is also a solution of the equation. Is

$$y(x) = c_1 \sin x + c_2 \cos x$$

the general solution of the equation?
We will prove later that the answer is affirmative.

4.3.1 Differential Operators

- In the proof of the preceding theorem, we had to restrict ourselves to linear equations of order 2 because the width of a standard page is insufficient for very long linear combinations; the trees would obscure the forest. Such situations compel us to develop notation that saves many words and displays the logical structure more compactly.

- For this purpose, the theory of **linear operators**, together with a suitable algebra, was developed as part of the foundations of functional analysis. These are mappings between function spaces.

- The basic idea is to regard the symbol $\frac{d}{dx}$ as a mapping that takes a function $f(x)$ as input and produces a new function $f'(x)$.

$$\frac{d}{dx}[f(x)] = f'(x)$$

or, more simply,

$$\frac{d}{dx}[y] = y'$$

- The operators $\dfrac{d^n}{dx^n}$, $n = 1, 2, 3, \ldots$, are defined similarly.

$$\frac{d^n}{dx^n}[y] = y^{(n)}$$

These operators are sometimes also denoted by D, D^n, D_x, and D_x^n. Consequently, differential equations such as

$$y'' - 3y' + 2y = 0$$

may sometimes also be written in the form

$$\left(\frac{d^2}{dx^2} - 3\frac{d}{dx} + 2\right)[y] = 0$$

or

$$(D^2 - 3D + 2)[y] = 0$$

- We may add any two given operators to obtain a new operator L. For example:

$$L = \frac{d^3}{dx^3} + \frac{d}{dx}$$

We may apply the new operator L as follows:

$$L[f(x)] = \left(\frac{d^3}{dx^3} + \frac{d}{dx}\right)[f(x)] = \frac{d^3}{dx^3}[f(x)] + \frac{d}{dx}[f(x)] = f'''(x) + f'(x)$$

We may multiply any operator by an expression $p(x)$ (including constants) to obtain a very wide variety of operators, such as

$$\begin{aligned} L_1 &= \frac{d^3}{dx^3} + 5\frac{d^2}{dx^2} - 2\frac{d}{dx} + 7 \\ L_2 &= \frac{d^3}{dx^3} - 2e^x\frac{d^2}{dx^2} + 2\frac{d}{dx} \\ L_3 &= \frac{d^4}{dx^4} - 5x\frac{d^3}{dx^3} - 3\frac{d^2}{dx^2} + x^2\frac{d}{dx} - 4 \end{aligned}$$

or, in a more readable form,

$$\begin{aligned} L_1 &= D^3 + 5D^2 - 2D + 7 \\ L_2 &= D^3 - 2e^x D^2 + 2D \\ L_3 &= D^4 - 5xD^3 - 3D^2 + x^2 D - 4 \end{aligned}$$

- The operator L may also be viewed as a kind of algorithm, or as a sequence of instructions for performing a series of operations on a given function.
- For example, the operator L_2 in the preceding list is equivalent to the following sequence of instructions:

  A. Differentiate the function three times.

  B. Add to the result the product of the second derivative and $-2e^x$.

  C. Add to the result the product of the first derivative and 2.

- For example, let us apply the operator L_3 to the function $y = \sin x$:

$$\begin{aligned} L_3[y] &= (D^4 - 5xD^3 - 3D^2 + x^2D - 4)\,[y] \\ &= y'''' - 5xy''' - 3y'' + x^2y' - 4y \\ &= \sin x - 5x(-\cos x) - 3(-\sin x) + x^2\cos x - 4\sin x \\ &= \sin x + 5x\cos x + 3\sin x + x^2\cos x - 4\sin x \\ &= (x^2 + 5x)\cos x \end{aligned}$$

- Note that the constant 7 in the operator L_1 translates into the operation $7y$.

Definition 4.1: An operator L is called a **linear operator** if it has the following two properties:

A. For any two functions $f(x)$, $g(x)$,

$$L[f+g] = L[f] + L[g]$$

B. For every function $f(x)$ and every real constant c,

$$L[cf] = cL[f]$$

- The two requirements may be combined into one: L is a linear operator if and only if for every pair of functions $f(x)$, $g(x)$ and every pair of real numbers c_1, c_2,

$$L[c_1f + c_2g] = c_1L[f] + c_2L[g]$$

The last property is sometimes called the **superposition principle**.

Definition 4.2: An operator L is called a **differential operator** if it is a finite sum of operators of the form $p(x)\frac{d^k}{dx^k}$.

- **Example 6:** We have already encountered the following three examples of differential operators:

$$\begin{aligned} L_1 &= D^3 + 5D^2 - 2D + 7 \\ L_2 &= D^3 - 2e^xD^2 + 2D \\ L_3 &= D^4 - 5xD^3 - 3D^2 + x^2D - 4 \end{aligned}$$

- As noted above, an operator is a mapping between two function spaces (in fact, vector spaces as studied in linear algebra). A differential operator is said to be **of order n** if the highest-order derivative that appears in it has order n. The vector space on which it acts is usually denoted by $\mathscr{C}^n(\alpha, \beta)$, which is the space of all functions that are continuously differentiable n times on the interval (α, β). The range of the operator is usually the space of piecewise continuous functions on the interval (α, β).

Theorem 4.4: Every differential operator is also a linear operator.

Proof: The theorem follows from the fact that a sum of linear operators is a linear operator, and that every differential operator is a sum of operators of the form $p(x)\frac{d^n}{dx^n}$. It is therefore enough to prove that the operator $p(x)D^k$ is linear. For this purpose, it is enough to prove that

$$\left(p(x)D^k\right)[f+g] = p(x)D^k[f] + p(x)D^k[g]$$

$$\left(p(x)D^k\right)[cf] = c \cdot p(x)D^k[f]$$

Both statements follow immediately from the properties of the derivative. ∎

We can now give a complete proof of the theorem proved earlier for equations of order 2. We state it again:

Theorem 4.5: If

$$y_1(x),\ y_2(x),\quad \ldots,\ y_k(x)$$

is a set of k solutions of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

and c_1, c_2, $\ldots$, c_k are k arbitrary real numbers, then the function

$$\phi(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is also a solution of the homogeneous equation.

Proof: This time, we begin by defining the linear operator

$$L = D^n + p_{n-1}(x)D^{n-1} + \cdots + p_1(x)D + p_0(x)$$

With this notation, our homogeneous equation becomes much shorter:

$$L[y] = 0$$

The solution set of the equation is precisely the **kernel** of the operator L (recall that the kernel of a linear mapping consists of all vectors that the mapping sends to zero).

The fact that the functions $y_1(x)$, $y_2(x)$, $\ldots$, $y_k(x)$ are solutions of the equation may now be written as

$$L[y1] = 0, \quad L[y_2] = 0, \quad \ldots, \quad L[y_k] = 0$$

To prove that the linear combination

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution, it is enough to prove that $L[\phi] = 0$. But this follows immediately from the fact that L is a linear operator:

$$\begin{aligned} L[\phi] &= L[c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)] \\ &= c_1 L[y_1(x)] + c_2 L[y_2(x)] + \cdots + c_k L[y_k(x)] \\ &= c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_k \cdot 0 = 0 \end{aligned}$$

- In the language of linear algebra, the assertion of the theorem is equivalent to the familiar fact that the kernel of a linear operator is a vector subspace of the space $\mathscr{C}^n(\alpha, \beta)$.
- It turns out that there is an interesting relationship between the dimension of the solution space and the order of the equation: they are equal.

4.4 A Fundamental Set of Solutions for Homogeneous Equations

Definition 4.3: A set of solutions

$$\{y_1(x),\ y_2(x),\quad \ldots,\ y_n(x)\}$$

is called a **fundamental set of solutions** of the homogeneous equation

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = 0$$

if every other solution of the equation is a linear combination of its members.

- **Example 7:** The set $\{\cos x,\ \sin x\}$ is a fundamental set of solutions of the homogeneous equation

$$y'' + y = 0$$

Proof: Let $y(x)$ be any solution of the homogeneous equation. Set

$$\begin{aligned} c_1 &= y(0) \\ c_2 &= y'(0) \end{aligned}$$

Define the function

$$z(x) = c_1 \cos x + c_2 \sin x$$

It is easy to verify that $z(x)$ is a solution of the initial-value problem

$$\begin{cases} y'' + y = 0 \\ y(0) = c_1 \\ y'(0) = c_2 \end{cases}$$

Therefore, by the existence and uniqueness theorem for linear equations of order n,

$$z(x) \equiv y(x)$$

that is, $y(x) = c_1 \cos x + c_2 \sin x$.

In conclusion, every solution of the homogeneous equation is a linear combination of the set $\{\cos x,\ \sin x\}$, and hence this is a fundamental set of solutions. ∎

- In the language of linear algebra, we say that the set $\{\cos x, \sin x\}$ is a **basis** for the solution space of the homogeneous equation. This is not yet completely precise, however, unless we prove that it is also a **linearly independent set of solutions**. In the present case this is clear, and hence it is indeed a basis.

- The conclusion from the preceding example is that the solution space of the homogeneous equation $y'' + y = 0$ is a vector space spanned by the basis $\{\cos x, \sin x\}$, and therefore its dimension is 2, exactly the order of the equation. Is this a coincidence?

- **Question:** How can we determine whether a given set of functions is a fundamental set of solutions of a given homogeneous equation?

  - To avoid being overwhelmed by notation, we focus on a homogeneous equation of order 2:

$$y'' + p_1(x)y' + p_0(x)y = 0, \qquad \alpha < x < \beta$$

  - Let $y_1(x)$ and $y_2(x)$ be two solutions of the equation on the interval (α, β). The condition for the set $\{y_1(x),\ y_2(x)\}$ to be a fundamental set of solutions is that, for every solution $\phi(x)$, we can find constants c_1, c_2 such that

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x)$$

  - We follow the idea used in the preceding example. Choose a point x_0 in the interval (α, β) and define

$$\begin{cases} \alpha_0 = \phi(x_0) \\ \alpha_1 = \phi'(x_0) \end{cases}$$

    and seek a solution of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for the initial-value problem

$$\begin{cases} y'' + p_1(x)y' + p_0(x)y = 0, & \alpha < x < \beta \\ y(x_0) = \alpha_0 \\ y'(x_0) = \alpha_1 \end{cases}$$

If we can prove the existence of constants c_1, c_2 for which $y(x)$ solves this initial-value problem, then, by the existence and uniqueness theorem for higher-order linear equations, the solution $y(x)$ must be identical to the solution $\phi(x)$, and our task is complete.

- Substituting $x = x_0$ into the initial conditions gives a system of two equations in the two unknowns c_1, c_2:

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) & = & \phi(x_0) \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) & = & \phi'(x_0) \end{cases}$$

From linear algebra, we know that the system has a unique solution if and only if its coefficient determinant is nonzero:

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) \neq 0$$

- Since the point x_0 was chosen arbitrarily, we may conclude that the set $\{y_1(x),\ y_2(x)\}$ is a fundamental set of solutions if and only if the determinant is nonzero at every point x in the domain:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \neq 0, \qquad \alpha < x < \beta$$

- The function $W(x)$ is called the **Wronskian** of the set $\{y_1(x),\ y_2(x)\}$, named after the Polish mathematician (Józef Maria Hoene-Wroński, 1776-1853) who first discovered it. It is sometimes denoted by the expression $W[y_1, y_2]$.

$$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

➜ We have thus proved the following theorem.

Theorem 4.6: Consider a homogeneous linear equation

$$y'' + p_1(x)y' + p_0(x)y = 0, \qquad \alpha < x < \beta$$

where $p_0(x)$ and $p_1(x)$ are continuous functions on the interval (α, β). A set of solutions $\{y_1(x),\ y_2(x)\}$ is a fundamental set if and only if its Wronskian $W[y_1, y_2]$ is nonzero at every point of the interval (α, β).

- **Corollary:** If $\{y_1(x),\ y_2(x)\}$ is not a fundamental set, then its Wronskian vanishes identically throughout the interval (α, β).

$$W[y_1, y_2](x) = 0, \quad \alpha < x < \beta$$

- **Example 8:** The Wronskian of the set $\{\cos x,\ \sin x\}$ is

$$W[\cos, \sin](x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Therefore, $\{\cos x,\ \sin x\}$ is a fundamental set of solutions of the homogeneous equation $y'' + y = 0$.

- **Example 9:** The two functions $\{e^x,\ e^{-x}\}$ are solutions of the homogeneous equation

$$y'' - y = 0$$

Prove that the two-parameter family

$$y(x) = c_1 e^x + c_2 e^{-x}$$

is the general solution of the equation on the entire real line $(-\infty, \infty)$.
Proof: The Wronskian of the set $\{e^x,\ e^{-x}\}$ is

$$W[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$

For every real x, we have found that the value of the Wronskian is -2, and therefore, by the preceding theorem, $\{e^x,\ e^{-x}\}$ is a fundamental set of solutions. Thus, every solution of the equation $y'' - y = 0$ is a linear combination of $\{e^x,\ e^{-x}\}$,

$$y(x) = c_1 e^x + c_2 e^{-x}$$

and hence this is the form of the general solution. ∎

- **Exercise 4.5:** Prove that $y(x) = c_1 x + c_2 x e^x$ is a general solution of the following homogeneous equation on the ray $(0, \infty)$:

$$y'' - \frac{x+2}{x} y' + \frac{x+2}{x^2} y = 0$$

- The second natural question is: Does every normalized homogeneous equation have a fundamental set of solutions?

  The preceding theorem provides a method for determining whether a set of solutions is fundamental, but it does not guarantee that such a set exists! The following theorem answers this question affirmatively (for the time being, for equations of order 2; later we will encounter a more general theorem).

Theorem 4.7: Every normalized homogeneous linear equation

$$y'' + p_1(x) y' + p_0(x) y = 0, \qquad \alpha < x < \beta$$

has a fundamental set of solutions.

Proof: Let x_0 be any point in the interval (α, β). By the existence and uniqueness theorem for linear equations, there exists a solution $y_1(x)$ of the initial-value problem

$$\begin{cases} y'' + p_1(x) y' + p_0(x) y = 0, & \alpha < x < \beta \\ y(x_0) = 1 \\ y'(x_0) = 0 \end{cases}$$

and there exists a solution $y_2(x)$ of the initial-value problem

$$\begin{cases} y'' + p_1(x) y' + p_0(x) y = 0, & \alpha < x < \beta \\ y(x_0) = 0 \\ y'(x_0) = 1 \end{cases}$$

We prove that the set of solutions $\{y_1(x),\ y_2(x)\}$ is fundamental using the Wronskian test. For this purpose, it is enough to evaluate the Wronskian only at the point x_0:

$$W[y_1, y_2](x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Since the Wronskian is nonzero at $x = x_0$, it follows that $\{y_1(x),\ y_2(x)\}$ is a fundamental set. ∎

- The proof for $n = 2$ makes clear how the proof works for equations of order $n = 3$ and higher.

- In the language of linear algebra, a fundamental set is also called a ***spanning set***, and sometimes a **generating set**. Thus, in different contexts, we may say that the set $\{y_1(x),\ y_2(x)\}$ **spans** all the solutions of the equation, or that the **solution space** of the equation is **spanned** by the set $\{y_1(x),\ y_2(x)\}$.

- The theorem does not address whether there is only one such set or more than one. Clearly, if there is one fundamental set, infinitely many other fundamental sets can be generated from it (how?).

- The theorem also does not address the size of the set. It guarantees the existence of a set of size 2 (for an equation of order 2). It is very easy, however, to construct fundamental sets containing three or more functions: Simply begin with a fundamental set of two functions and add any collection of functions we wish (the first two functions already span all the solutions, and the additional functions may be multiplied by zeros).

- Another question that may arise for second-order equations is: Can a fundamental set of solutions contain only one function? In this case, it is easy to prove that for every solution $y_1(x)$ of a homogeneous linear equation of order 2, one can find a solution $y_2(x)$ that is not a multiple of $y_1(x)$ by ”cooking up“ suitable initial conditions.

- A minimal fundamental set (a minimal spanning set) is called a **basis**. For an equation of order 2, a basis always has size 2, and therefore the dimension of the solution space is 2.

- **Linear Dependence and Independence:** Another important concept from linear algebra that is relevant here is linear dependence. A set of solutions $\{y_1(x),\ y_2(x),\ \ldots,\ y_k(x)\}$ is called **linearly dependent** if there are constants c_i, not all zero, such that

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x) = 0$$

  Equivalently, a set of solutions $\{y_1(x),\ y_2(x),\ \ldots,\ y_k(x)\}$ is linearly dependent if at least one member of the set is a linear combination of the others.

- The preceding discussion readily yields the following theorem. To avoid confusion and errors, note that the theorem concerns only solutions of a homogeneous linear equation!

Theorem 4.8: A set of solutions $\{y_1(x),\ y_2(x),\ \ldots,\ y_n(x)\}$ of a homogeneous linear equation of order n

A. is a basis for the solution space of the equation if and only if its Wronskian is nonzero at some point in the domain.

B. If the Wronskian vanishes at some point, then the set is linearly dependent (and therefore is not a basis for the solution space).

- We have thus obtained a simple test for determining linear dependence or independence for a set of solutions of a homogeneous equation.

$$\{y_1(x), \quad y_2(x), \quad \ldots, \quad y_n(x)\}$$

It is enough to find one point x_0 in the domain of existence at which the Wronskian is nonzero in order to determine that the set is linearly independent. If the Wronskian $W[y_1, y_2, \ldots, y_n]$ vanishes at a point x_0, then the set is linearly dependent.

- We emphasize again that this test works only for sets of solutions of homogeneous equations! For arbitrary functions, the Wronskian is of little use. For example, it is easy to see that the following two functions are linearly independent, although their Wronskian vanishes on the entire real line:

$$y_1(x) = \begin{cases} 0, & x \leq 0 \\ x^2, & x > 0 \end{cases} \qquad y_2(x) = \begin{cases} x^2, & x \leq 0 \\ 0, & x > 0 \end{cases}$$

- At this point, we are assured that every homogeneous linear equation has a fundamental set of solutions, but we still have no methods for finding one. In the following sections, we will learn methods for finding solutions.

4.5 Finding a Solution by Reduction of Order

- Given one solution of a homogeneous linear equation, a second, linearly independent solution can be found by reducing the order of the equation. We begin with an example.

- **Example 10:** It is easy to verify that $y_1(x) = e^{-x}$ is a solution of the equation

$$y'' + 2y' + y = 0$$

Idea: Analogously to the integrating-factor idea encountered in two earlier cases, we try to find a second solution that is the product of the first solution and a function $v(x)$:

$$y_2(x) = v(x)y_1(x)$$

(one can always try; in the worst case, the attempt fails and we move on to the next idea)

$$\begin{aligned} y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

After substituting into the equation,

$$y'' + p_1(x)y' + p_2(x)y = 0$$

we obtain

$$v[y_1'' + p_1(x)y_1' + p_2(x)y_1] + v'[2y_1' + p_1(x)y_1] + v''y_1 = 0$$

The first expression vanishes because y_1 is a solution of the homogeneous equation, leaving

$$v'[2y_1' + p_1(x)y_1] + v''y_1 = 0$$

and after normalization,

$$v'' + \left[p_1(x) + 2\frac{y_1'}{y_1}\right]v' = 0$$

The reduction of order occurs after the substitution $u = v'$:

$$\boxed{u' + \left[p_1(x) + 2\frac{y_1'}{y_1}\right]u = 0}$$

This is a homogeneous linear equation of order 1 in the variable u.

- In our example, $p_1(x) + 2\frac{y_1'}{y_1} = 0$, and we obtain the very simple equation

$$u' = 0$$

It follows easily that $v(x) = x$, and hence $y_2(x) = xe^{-x}$. It is easy to verify that $\{e^{-x}, xe^{-x}\}$ is a fundamental set of solutions, and therefore the general solution is $a_1e^{-x} + a_2xe^{-x}$.

- Of course, the general solution for $v(x)$ is of no interest to us. We need only one instance (preferably as simple as possible).

- **Example 11:** It is given that $y_1(x) = \frac{1}{x}$ is a solution of the equation

$$x^2y'' + 3xy' + y = 0$$

Find the general solution.

4.6 Abel's Formula (Niels Henrik Abel, 1802-1829)

Theorem 4.9: (Abel's formula)

If

$$\{y_1(x),\ y_2(x),\quad \ldots,\ y_n(x)\}$$

are n solutions of the homogeneous linear equation of order n,

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

then their Wronskian satisfies

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix} = Ke^{-\int p_{n-1}(x)dx}$$

for some real constant K. When the set is not fundamental (is linearly dependent), $K = 0$.

Proof: We prove the theorem for order $n = 2$. Let $\{y_1(x),\ y_2(x)\}$ be two solutions of the equation. That is,

$$\begin{aligned} y_1'' + p_1(x)y_1' + p_0(x)y_1 &= 0 \\ y_2'' + p_1(x)y_2' + p_0(x)y_2 &= 0 \end{aligned}$$

Multiply the first equation by y_2 and the second equation by y_1.

$$\begin{aligned} y_2y_1'' + p_1(x)y_2y_1' + p_0(x)y_2y_1 &= 0 \\ y_1y_2'' + p_1(x)y_1y_2' + p_0(x)y_1y_2 &= 0 \end{aligned}$$

Subtracting the second equation from the first gives

$$(y_1y_2'' - y_2y_1'') + p_1(x)(y_1y_2' - y_2y_1') = 0 \tag{4.3}$$

Recall that the Wronskian for $n = 2$ is

$$W(x) = y_1y_2' - y_2y_1'$$

and therefore

$$W'(x) = y_1y_2'' - y_2y_1''$$

Looking again at equation (4.3), we see that it is a differential equation for the Wronskian W:

$$W'(x) + p_1(x)W(x) = 0$$

This is in fact a first-order separable equation. Separating variables gives the formula

$$\frac{dW}{W} = -p_1(x)dx$$

and therefore

$$\ln W(x) = -\int p_1(x)dx$$

and therefore

$$W(x) = Ke^{-\int p_1(x)dx} \tag{4.4}$$

The constant K may be computed by substituting any point x_0 in the domain into the vector definition of $W(x)$. ∎

- Abel's formula is sometimes written in the form

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p_{n-1}(t)dt} \tag{4.5}$$

  where x_0 is any point in the domain of existence of the equation. This formula is useful only when $W(x_0)$ can be computed easily from the determinant, or for theoretical purposes.

- For various applications, it is also useful to remember the differential equation preceding Abel's formula in the general case:

$$\frac{W'(x)}{W(x)} = -p_{n-1}(x) \tag{4.6}$$

- Inspection of formula (4.4) provides further confirmation of the fact that either $W(x) = 0$ for every $\alpha < x < \beta$, or $W(x) \neq 0$ for every $\alpha < x < \beta$, where (α, β) is the domain of existence of the equation. The first case occurs when $K = 0$, and the second when $K \neq 0$.

- A second conclusion from Abel's formula is that the Wronskian is essentially the same for every fundamental set, up to multiplication by a constant.

- For $n = 2$, the Wronskian of two solutions $y_1(x)$, $y_2(x)$ recalls the numerator in the derivative formula for the quotient of the functions $\frac{y_2(x)}{y_1(x)}$:

$$\frac{d}{dx}\left[\frac{y_2(x)}{y_1(x)}\right] = \frac{y_1(x)y_2'(x) - y_1'(x)y_2(x)}{y_1(x)^2} = \frac{W(x)}{y_1(x)^2} = \frac{Ke^{-\int p_{n-1}(x)dx}}{y_1(x)^2}$$

Thus, if one solution $y_1(x)$ of the equation is known, we can find the second solution $y_2(x)$ from the equality

$$\left[\frac{y_2(x)}{y_1(x)}\right]' = \frac{e^{-\int p_{n-1}(x)dx}}{y_1(x)^2}$$

that is,

$$\frac{y_2(x)}{y_1(x)} = \int \frac{e^{-\int p_{n-1}(x)dx}}{y_1(x)^2}$$

and therefore

$$y_2(x) = y_1(x)\int \frac{e^{-\int p_{n-1}(x)dx}}{y_1(x)^2} \tag{4.7}$$

- Abel's formula now enables us to proceed and solve homogeneous linear equations with constant coefficients.

4.7 Solving Homogeneous Linear Equations with Constant Coefficients

- In this section, we focus on a homogeneous linear equation of order n with constant coefficients:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{4.8}$$

where a_i, $0 \le i \le n$, are real constants.

- By the existence and uniqueness theorem for linear equations, the domain of existence of every solution of the equation is the entire real line $\mathbb{R}$.
- **Definition:** The polynomial

$$P(r) = a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$$

is called the **characteristic polynomial** of equation (4.8).

- If r_1 is a real root of the characteristic polynomial, then it can be factored into two factors:

$$P(r) = Q(r)(r - r_1)$$

where $Q(r)$ is a real polynomial of degree $n-1$.

- If we consider the operator form of the equation,

$$P(D)[y] = Q(D)(D - r_1)[y] = Q(D)\Big[(D - r_0)[y]\Big] = 0$$

the solution of the linear equation

$$(D - r_1)[y] = y' - r_1 y = 0$$

is $c_1 e^{r_1 x}$ (as is easily verified).

- Thus, the real roots $\{r_1, r_2, \ldots, r_k\}$ of the characteristic polynomial $P(r)$ correspond to the solutions

$$\{e^{r_1 x}, \quad e^{r_2 x}, \quad \ldots, \quad e^{r_k x}\}$$

which a simple Wronskian calculation (at the point $x = 0$) shows to be linearly independent.
Reminder: A common exercise in linear algebra is the ***Vandermonde determinant***:

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_n \\ r_1^2 & r_2^2 & \ldots & r_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \ldots & r_n^{n-1} \end{vmatrix} = \prod_{i<j}(r_j - r_i)$$

The proof is by induction.

- The difficulty is that the number of real roots may be less than n, and we will therefore need to find additional solutions in order to obtain a fundamental set of solutions.

- From our previous familiarity with this type of equation, we know that fundamental solutions usually have forms such as e^{rx}, $\sin rx$, and $\cos rx$.

- A second way to prove the last result is simply to substitute $y(x) = e^{rx}$ into the differential equation:

$$\begin{aligned} y(x) &= e^{rx} \\ y'(x) &= re^{rx} \\ y''(x) &= r^2 e^{rx} \\ &\vdots \\ y^{(n)}(x) &= r^n e^{rx} \end{aligned}$$

and therefore

$$\begin{aligned} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y &= (a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) e^{rx} \\ &= 0 \end{aligned}$$

Example 12: Find a general solution of

$$y'' - 5y' + 6y = 0$$

Solution: The characteristic polynomial here is $r^2 - 5r + 6 = 0$. The roots of the polynomial are $\{2,\ 3\}$, and hence we obtain the fundamental set of solutions $\{e^{2x},\ e^{3x}\}$. Therefore, the general solution of the equation is

$$y(x) = c_1 e^{2x} + c_2 e^{3x}$$

- By the fundamental theorem of algebra, the characteristic polynomial $P(r)$ (which has degree n) has n roots, some of which may be repeated and some complex (the complex roots may also be repeated!). We examine each case separately.

- **Real Roots, All of Multiplicity 1:** If all the roots are distinct and real, $\{r_1,\ r_2,\ \ldots,\ r_n\}$, then we obtain a fundamental set of solutions $\{e^{r_1 x},\ e^{r_2 x},\ \ldots,\ e^{r_n x}\}$ and we are done.

- **A Complex Root of Multiplicity 1:** If one of the roots is complex, $r = s + it$, then its conjugate $r = s - it$ also appears in the list. It can be verified that the functions e^{s+it} and e^{s-it} are also solutions of the equation, but they are complex-valued functions. We can, however, obtain from them two linearly independent real solutions by taking the following linear combinations:

$$\begin{aligned} e^{sx} \cos tx &= \tfrac{1}{2} \left[e^{s+it} + e^{s-it} \right] \\ e^{sx} \sin tx &= \tfrac{1}{2i} \left[e^{s+it} - e^{s-it} \right] \end{aligned}$$

A Wronskian calculation verifies that these two solutions are linearly independent.

- **A Real Root of Multiplicity k Greater Than 1:** If r_1 is a root of multiplicity k, then the characteristic polynomial $P(r)$ can be factored as follows:

$$P(r) = (r - r_1)^k Q(r)$$

where $Q(r)$ is a polynomial of degree $n-k$, but r_1 is not a root of $Q(r)$ (that is, $Q(r_1) \neq 0$). The root r_1 corresponds to k distinct (and linearly independent) solutions:

$$\{e^{r_1 x}, \quad x e^{r_1 x}, \quad \ldots, \quad x^{k-1} e^{r_1 x}\}$$

To prove this assertion, denote our equation by $L[y] = 0$. It is easy to see that for every real r,

$$L[e^{rx}] = e^{rx}P(r) = e^{rx}(r - r_1)^k Q(r)$$

Differentiating both sides with respect to the variable r, we obtain

$$\begin{aligned} L[xe^{rx}] &= xe^{rx}(r - r_1)^k Q(r) + e^{rx}\left[k(r - r_1)^{k-1}Q(r) + (r - r_1)^k Q'(x)\right] \\ &= e^{rx}\left[x(r - r_1)^k Q(r) + k(r - r_1)^{k-1}Q(r) + (r - r_1)^k Q'(x)\right] \end{aligned}$$

Substituting $r = r_1$ gives $L[xe^{rx}] = 0$. Thus, $y(x) = xe^{rx}$ is a solution of the equation. Similarly, if we continue differentiating the last equality with respect to the variable r, we find that all the functions in the set

$$\{e^{r_1 x}, \quad xe^{r_1 x}, \quad \ldots, \quad x^{k-1}e^{r_1 x}\}$$

are solutions of our equation.

- **A Complex Root of Multiplicity k Greater Than 1:** Because the coefficients of the equation are real, every such root $s + it$ is accompanied by its conjugate $s - it$. Exactly as before, we prove that the set

$$\{e^{(s\pm it)x}, \quad xe^{(s\pm it)x}, \quad \ldots, \quad x^{k-1}e^{(s\pm it)x}\}$$

is a fundamental set of $2k$ solutions. From it, we can extract $2k$ real solutions:

$$\begin{array}{ll} e^{sx}\cos tx, & e^{sx}\sin tx, \\ xe^{sx}\cos tx, & xe^{sx}\sin tx, \\ x^2e^{sx}\cos tx, & x^2e^{sx}\sin tx, \\ \vdots & \vdots \\ x^{k-1}e^{sx}\cos tx, & x^{k-1}e^{sx}\sin tx \end{array}$$

- **Example 13:** Find a general solution of

$$y'' - 6y' + 9y =$$

Solution: The characteristic polynomial of the equation is $P(r) = (r - 3)^2$. There is a single root $r = 3$ of multiplicity 2. Therefore, by the result obtained above, $\{e^{3x}, \; xe^{3x}\}$ is a fundamental set of solutions, and hence the general solution is

$$y(x) = c_1e^{3x} + c_2xe^{3x}$$

Example 14: Similarly, we prove that $\{e^x, xe^x, x^2e^x\}$ is a fundamental set of solutions of the equation

$$y''' - 3y'' + 3y' - y = 0$$

Example 15: Find a general solution of

$$y'''' - 6y''' + 13y'' = 0$$

Solution: The equation here is of fourth order. The characteristic polynomial

$$P(r) = r^4 - 6r^3 + 13r^2 = r^2(r^2 - 6r + 13)$$

has a real root $r = 0$ of multiplicity 2 and two complex roots $r = 3 \pm 2i$ of multiplicity 1. Therefore, our fundamental set is

$$\{1, \quad x, \quad e^{3x}\cos 2x, \quad e^{3x}\sin 2x\}$$

and hence the general solution of the equation is

$$y(x) = c_1 + c_2x + c_3e^{3x}\cos 2x + c_4e^{3x}\sin 2x$$

Example 16: Find a general solution of

$$y'''' + 2y'' + y = 0$$

Solution: The characteristic polynomial

$$P(r) = r^4 + 2r^2 + 1 = (r^2 + 1)^2 = (r - i)^2(r + i)^2$$

has two complex roots of multiplicity 2: $r_1 = i$, $r_2 = -i$. Therefore, our fundamental set this time is

$$\{\cos x, \quad x\cos x, \quad \sin x, \quad x\sin x\}$$

Thus, the general solution is

$$y(x) = c_1\cos x + c_2x\cos x + c_3\sin x + c_4x\sin x$$

Example 17: Find a general solution of

$$y'''' + y = 0$$

Solution: The characteristic polynomial

$$P(r) = r^4 + 1$$

has four distinct complex roots, $r^4 = -1 = e^{\pi i}$.

$$r = e^{\frac{\pi i}{4}}, \quad e^{\frac{3\pi i}{4}}, \quad e^{\frac{5\pi i}{4}}, \quad e^{\frac{7\pi i}{4}}$$

Using the formula $e^{\theta i} = \cos\theta + i\sin\theta$, we obtain the following four roots:

$$r_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad r_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad r_3 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad r_4 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

Therefore, our fundamental set is

$$\{e^{\frac{\sqrt{2}}{2}x}\cos\frac{\sqrt{2}}{2}x, \quad e^{\frac{\sqrt{2}}{2}x}\sin\frac{\sqrt{2}}{2}x, \quad e^{-\frac{\sqrt{2}}{2}x}\cos\frac{\sqrt{2}}{2}x, \quad e^{-\frac{\sqrt{2}}{2}x}\sin\frac{\sqrt{2}}{2}x\}$$

and the general solution is, as usual, a linear combination of these four functions.

4.8 Nonhomogeneous Equations

Theorem 4.10: If $y_1(x)$ and $y_2(x)$ are two solutions of a linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = q(x)$$

then the difference $y_h(x) = y_1(x) - y_2(x)$ is a solution of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

Proof: We use operator notation:

$$L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y$$

By the superposition principle,

$$\begin{aligned} L[y_h(x)] &= L[y_1(x) - y_2(x)] \\ &= L[y_1(x)] - L[y_2(x)] \\ &= q(x) - q(x) = 0 \end{aligned}$$

We have therefore shown that the difference $y_h(x) = y_1(x) - y_2(x)$ is a solution of the homogeneous equation. ∎

The following theorem is an immediate consequence of this result.

Theorem 4.11: If a particular solution $y_p(x)$ of the nonhomogeneous equation is known,

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = q(x)$$

and if the general solution of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

is

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

then the general solution of the nonhomogeneous equation is

$$y_p(x) + c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

Proof: We again use operator notation:

$$L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y$$

The nonhomogeneous equation may be written as

$$L[y] = q(x)$$

and the homogeneous equation as

$$L[y] = 0$$

Suppose that a particular solution $y_p(x)$ of the nonhomogeneous equation is known.

$$L[y_p(x)] = q(x)$$

Let $y(x)$ be any solution of the nonhomogeneous equation.

$$L[y(x)] = q(x)$$

By the preceding theorem, $y(x) - y_p(x)$ is a solution of the homogeneous equation, and hence there are constants c_1, c_2, $\ldots$, c_n such that

$$y(x) - y_p(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

and therefore

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

∎

- The meaning of the last theorem is that, in order to solve a nonhomogeneous differential equation, we need one particular solution and the general solution of the homogeneous equation.

- **Example 18:** Find the general solution of the nonhomogeneous equation

$$y'' + y = 3x$$

by guessing a particular solution.

Solution: It is easy to guess that $y_p(x) = 3x$ is a particular solution of the equation. From the preceding sections, we know that the general solution of the homogeneous equation

$$y'' + y = 0$$

is

$$c_1 \cos x + c_2 \sin x$$

Therefore, the general solution of the nonhomogeneous equation is

$$y(x) = 3x + c_1 \cos x + c_2 \sin x$$

- Another tool that can help solve nonhomogeneous equations is the superposition principle: If the nonhomogeneous term $q(x)$ can be decomposed as

$$q(x) = q_1(x) + q_2(x) + \cdots + q_k(x)$$

and if y_{p_i} is a particular solution of

$$L[y] = q_i(x)$$

then the sum

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

is a solution of

$$L[y] = q(x)$$

- **Example 19:** Find a general solution of the nonhomogeneous equation

$$y'' + y = 1 + x + \frac{3}{4} \sin \frac{x}{2} \tag{4.9}$$

Solution: By the superposition principle, it is enough to find three particular solutions of the following equations:

(1) $$y'' + y = 1$$

(2) $$y'' + y = x$$

(3) $$y'' + y = \frac{3}{4} \sin \frac{x}{2}$$

It is easy to guess three suitable particular solutions:

(1) $$y_{p_1}(x) = 1$$

(2) $$y_{p_2}(x) = x$$

(3) $$y_{p_3}(x) = \sin \frac{x}{2}$$

Therefore, a particular solution of the nonhomogeneous equation (4.9) is

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + y_{p_3}(x) = 1 + x + \sin \frac{x}{2}$$

and hence the general solution of the nonhomogeneous equation (4.9) is

$$y(x) = 1 + x + \sin \frac{x}{2} + c_1 \cos x + c_2 \sin x$$

4.8.1 The Method of Undetermined Coefficients

- It turns out that guessing a particular solution for nonhomogeneous equations with constant coefficients is straightforward in typical cases.
- Let
$$L[y] = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = q(x)$$

The following table gives the form of the appropriate particular-solution ansatz for the term $q(x)$. Of course, one must substitute $y_p(x)$ into the nonhomogeneous equation and determine the appropriate coefficients.

	$q(x)$	$y_p(x)$
1	$A_0 + A_1x + A_2x^2 + \cdots + A_mx^m$	$x^k(B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)$
2	$(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{\alpha x}$	$x^k(B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)e^{\alpha x}$
3	$(A_0 + A_1x + \cdots + A_mx^m)e^{\alpha x}\cos\beta x$	$x^k\big[(B_0 + B_1x + \cdots + B_mx^m)e^{\alpha x}\cos\beta x$ $+(C_0 + B_1x + \cdots + C_mx^m)e^{\alpha x}\sin\beta x\big]$
4	$(A_0 + A_1x + \cdots + A_mx^m)e^{\alpha x}\sin\beta x$	$x^k\big[(B_0 + B_1x + \cdots + B_mx^m)e^{\alpha x}\cos\beta x$ $+(C_0 + C_1x + \cdots + C_mx^m)e^{\alpha x}\sin\beta x\big]$

Table 4.1: Form of the particular-solution ansatz

- Associate a pair of constants α, β with each row of the table. The constant α corresponds to the factor $e^{\alpha x}$, and the constant β corresponds to the factor $\cos\beta x$ or $\sin\beta x$.
- In row 1 of the table, $\alpha = 0$, $\beta = 0$, and therefore row 1 is actually a special case of row 3 (if we substitute $\alpha = 0$, $\beta = 0$ into row 3, we obtain row 1).
- In row 2 of the table, $\beta = 0$, and therefore row 2 is actually a special case of row 3.
- The first two rows could therefore be omitted from the table, but they are customarily included for convenience.
- **The rule for determining k:**
  If $r = \alpha + \beta i$ is a root of the characteristic polynomial, then k is the algebraic multiplicity of r. Otherwise, $k = 0$.
- It is preferable to use rows 3 and 4, but when using row 1, remember that $\alpha = 0$, $\beta = 0$ in that row, and hence $r = 0 + 0i = 0$. Thus, if $r = 0$ is a root of the characteristic polynomial, then k is its algebraic multiplicity; otherwise, $k = 0$.
- When using row 2, remember that $\beta = 0$ in that row, and therefore

$$r = \alpha + 0i = \alpha$$

  Thus, if $r = \alpha$ is a root of the characteristic polynomial, then k is its algebraic multiplicity; otherwise, $k = 0$.
- **Example 20:** Find a general solution of the equation

$$y'' + y = x^2$$

Solution: In this case, clearly $k = 0$, because $r = 0$ is not a root of the characteristic polynomial $r^2 + 1 = 0$. Therefore, the form of the particular solution here is

$$y_p(x) = Ax^2 + Bx + C$$

Substituting $y_p(x)$ into the nonhomogeneous equation gives

$$y_p''(x) + y_p(x) = Ax^2 + Bx + (C + 2A)$$

and therefore
$$\begin{cases} A = 1 \\ B = 0 \\ C + 2A = 0 \end{cases}$$

and therefore
$$\begin{cases} A = 1 \\ B = 0 \\ C = -2 \end{cases}$$

We have obtained the following particular solution of the nonhomogeneous equation:

$$y_p(x) = x^2 - 2$$

It is advisable to verify that this is indeed a particular solution of the nonhomogeneous equation by substituting it into the equation. The general solution of the nonhomogeneous equation is therefore
$$y(x) = x^2 - 2 + c_1 \cos x + c_2 \sin x$$

Example 21: Find a general solution of the equation

$$y'' - y = e^x$$

Solution: We use row 3 of the table: $\alpha = 1$, $\beta = 0$. $r = \alpha + \beta i = 1$ is a root of the characteristic polynomial $r^2 - 1 = 0$, and therefore, according to the table, the form of the particular solution is
$$y_p(x) = Axe^x$$

because $y = e^x$ is a solution of the homogeneous equation.

$$\begin{aligned} y_p'(x) &= Ae^x + Axe^x = A(x+1)e^x \\ y_p''(x) &= A(x+2)e^x \\ y_p''(x) - y_p(x) &= 2Ae^x = e^x \end{aligned}$$

Thus, $A = \frac{1}{2}$, and therefore the particular solution is

$$y_p(x) = \frac{1}{2}xe^x$$

The general solution of the homogeneous equation $y'' - y = 0$ is

$$y(x) = c_1e^x + c_2e^{-x}$$

Hence, the general solution of the nonhomogeneous equation is

$$y(x) = \frac{1}{2}xe^x + c_1e^x + c_2e^{-x}$$

Example 22: Find a general solution of the equation

$$y''' - 4y' = x + 3\cos x + e^{-2x}$$

Solution: To guess a particular solution of the nonhomogeneous equation, it is useful to begin with the general solution of the homogeneous equation.

$$y''' - 4y' = 0$$

The roots of the characteristic polynomial $r^3 - 4r = 0$ are $r = 0, -2, 2$, and therefore the general solution is

$$y(x) = c_1x + c_2e^{-2x} + c_3e^{2x}$$

Next, we guess particular solutions of the following three nonhomogeneous equations:

$$y''' - 4y' = x \tag{1}$$

$$y''' - 4y' = 3\cos x \tag{2}$$

$$y''' - 4y' = e^{-2x} \tag{3}$$

Equation 1:
The appropriate row of the table is row 1. In the present case, we must take $k = 1$ because $x^0 = 1$ is a solution of the homogeneous equation. Therefore, the ansatz is

$$y_p(x) = x^1(Ax + B) = Ax^2 + Bx$$

Substitute into equation (1):

$$\begin{array}{l} y_p'(x) = 2Ax + B \\ y_p''(x) = 2A \\ y_p'''(x) = 0 \\ \hline y_p'''(x) - 4y_p'(x) = -8Ax - 4B = x \end{array}$$

Thus, $A = -\frac{1}{8}$, $B = 0$, and we obtain

$$y_{p_1}(x) = -\frac{1}{8}x^2$$

Equation 2:
The appropriate row of the table is row 3 ($\alpha = 0$, $\beta = 1$). Since

$$r = \alpha + \beta i = i$$

is not a root of the characteristic polynomial, we take $k = 0$, and therefore the form of the particular solution is

$$y_p(x) = A\cos x + B\sin x$$

Substitute into equation (2):

$$\begin{array}{l} y_p'(x) = -A\sin x + B\cos x \\ y_p''(x) = -A\cos x - B\sin x \\ y_p'''(x) = A\sin x - B\cos x \\ \hline y_p'''(x) - 4y_p'(x) = 5A\sin x - 5B\cos x = 3\cos x \end{array}$$

Thus, $A = 0$, $B = -\frac{3}{5}$, and we obtain

$$y_{p_2}(x) = -\frac{3}{5}\sin x$$

Equation 3:
In this case, $r = -2$ is a root of the characteristic polynomial of multiplicity $k = 1$, and therefore the form of the particular solution is

$$y_p(x) = Axe^{-2x}$$

Substitute into equation (3):

$$\begin{aligned}
y_p'(x) &= Ae^{-2x} - 2Axe^{-2x} = A(1-2x)e^{-2x} \\
y_p''(x) &= -2Ae^{-2x} - 2A(1-2x)e^{-2x} = A(-4+4x)e^{-2x} \\
y_p'''(x) &= 4Ae^{-2x} - 2A(-4+4x)e^{-2x} = A(12-8x)e^{-2x} \\
\hline
y_p'''(x) &- 4y_p'(x) = 8Ae^{-2x} = e^{-2x}
\end{aligned}$$

Thus, $A = -\frac{1}{8}$, and we obtain

$$y_{p_3}(x) = -\frac{1}{8}e^{-2x}$$

By the superposition principle, a solution of the nonhomogeneous equation

$$y''' - 4y' = x + 3\cos x + e^{-2x}$$

is

$$\begin{aligned}
y_p(x) &= y_{p_1}(x) + y_{p_2}(x) + y_{p_3}(x) \\
&= -\frac{1}{8}x^2 - \frac{3}{5}\sin x - \frac{1}{8}e^{-2x}
\end{aligned}$$

and hence the general solution is

$$y(x) = -\frac{1}{8}x^2 - \frac{3}{5}\sin x - \frac{1}{8}e^{-2x} + c_1x + c_2e^{-2x} + c_3e^{2x}$$

Example 23: Find a general solution of the equation

$$y^{(4)} - y''' - y'' + y' = x\sin x$$

Solution: We begin by finding the general solution of the homogeneous equation:

$$y^{(4)} - y''' - y'' + y' = 0$$

Characteristic polynomial:

$$\begin{aligned}
r^4 - r^3 - r^2 + r &= r^3(r-1) - r(r-1) \\
&= (r^3 - r)(r-1) \\
&= r(r^2-1)(r-1) \\
&= r(r-1)^2(r+1)
\end{aligned}$$

Roots: $r = 0, -1, 1, 1$
General solution:

$$y(x) = c_1 + c_2e^{-x} + c_3e^x + c_4xe^x$$

We use row 4 of the table. In our case, $q(x) = x\sin x$, and hence $\alpha = 0$, $\beta = 1$, $r = \alpha + \beta i = i$. Since $r = i$ is not a root of the characteristic polynomial, we take $k = 0$, and therefore, according to row 4 of the table, the form of the particular solution of the nonhomogeneous equation is

$$y_p(x) = (Ax + B)\cos x + (Cx + D)\sin x$$

The first four derivatives of $y_p(x)$ must be computed:

$$\begin{aligned}
y_p'(x) &= A\cos x - (Ax + B)\sin x + C\sin x + (Cx + D)\cos x \\
&= (A + D + Cx)\cos x + (C - B - Ax)\sin x \\
y_p''(x) &= C\cos x - (A + D + Cx)\sin x - A\sin x + (C - B - Ax)\cos x \\
&= (2C - B - Ax)\cos x - (2A + D + Cx)\sin x \\
y_p'''(x) &= -A\cos x - (2C - B - Ax)\sin x - C\sin x - (2A + D + Cx)\cos x \\
&= -(3A + D + Cx)\cos x - (3C - B - Ax)\sin x \\
y_p^{(4)}(x) &= -C\cos x + (3A + D + Cx)\sin x + A\sin x - (3C - B - Ax)\cos x \\
&= (Ax + B - 4C)\cos x + (Cx + 4A + D)\sin x
\end{aligned}$$

Substitute $y_p(x)$ into the nonhomogeneous equation to obtain

$$\begin{aligned}
&y_p^{(4)} - y_p''' - y_p'' + y_p' = \\
&\qquad (3A - 6C + D)\cos x + (2A + 2C)x\cos x \\
&\qquad + (6A - 2B + 4C + 2D)\sin x + (-2A + 2C)x\sin x \\
&= x\sin x
\end{aligned}$$

We obtain a linear system of four equations in four unknowns:

$$\begin{cases}
4A + 2B - 6C + 2D = 0 \\
2A + 2C = 0 \\
6A - 2B + 4C + 2D = 0 \\
-2A + 2C = 1
\end{cases}$$

whose solution is

$$A = -\frac{1}{4}, \quad B = \frac{1}{2}, \quad C = \frac{1}{4}, \quad D = \frac{3}{4}$$

and therefore

$$\begin{aligned} y_p(x) &= (Ax+B)\cos x + (Cx+D)\sin x \\ &= \left(-\frac{x}{4}+\frac{1}{2}\right)\cos x + \left(\frac{x}{4}+\frac{3}{4}\right)\sin x \end{aligned}$$

The general solution of the nonhomogeneous equation

$$y^{(4)} - y''' - y'' + y' = x\sin x$$

is

$$y(x) = \left(-\frac{x}{4}+\frac{1}{2}\right)\cos x + \left(\frac{x}{4}+\frac{3}{4}\right)\sin x + c_1 + c_2e^{-x} + c_3e^{x} + c_4xe^{x}$$

- **Programming Corner:** Programmers can easily verify that $y_p(x)$ is a solution of the nonhomogeneous equation using **SymPy** (Python):

http://live.sympy.org

(otherwise, a manual check may consume a great deal of time and ink and is also prone to errors)

```
from sympy import *
   x = Symbol("x")
yp = (-x/4 + 1/2)*cos(x) + (x/4 + 3/4)*sin(x)
yp1 = diff(yp, x)
yp2 = diff(yp1, x)
yp3 = diff(yp2, x)
yp4 = diff(yp3, x)
simplify(yp4 - yp3 - yp2 + yp1)
```

(copy and paste the code into the window in **SymPy**)

- **Exercise 4.6:** Find a general solution of the equation

$$y^{(4)} - y''' - y'' + y' = e^{x}$$

Final Answer: $y(x) = \frac{x^2}{4}e^{x} + c_1 + c_2e^{-x} + c_3e^{x} + c_4xe^{x}$

4.8.2 The Method of Variation of Parameters

- So far, we have studied linear differential equations with constant coefficients. The method of variation of parameters is intended for solving nonhomogeneous linear equations with variable coefficients

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = q(x)$$

when the general solution of the homogeneous equation is known.

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

- Suppose that the general solution of the homogeneous equation is known:

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

- The method of variation of parameters assumes that the nonhomogeneous equation has a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + \cdots + c_n(x)y_n(x)$$

where $c_1(x)$, $c_2(x)$, $\ldots$, $c_n(x)$ are n functions to be determined (hence the name of the method: the parameters c_1, $\ldots$, c_n become variable coefficients).

- We illustrate the method with a second-order equation:

$$L[y] = y'' + p_1(x)y' + p_0(x)y = q(x)$$

Suppose that we have the general solution $y(x) = c_1y_1(x) + c_2y_2(x)$ of the homogeneous equation

$$L[y] = y'' + p_1(x)y' + p_0(x)y = 0$$

We seek a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$$

- The derivative of $y_p(x)$ is

$$y_p'(x) = c_1'(x)y_1(x) + c_1(x)y_1'(x) + c_2'(x)y_2(x) + c_2(x)y_2'(x)$$

- The considerable freedom in choosing $c_1(x)$ and $c_2(x)$ allows us to impose the additional constraint

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$$

and therefore we may write

$$y_p'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x)$$

- The second derivative of $y_p(x)$ is

$$y_p''(x) = c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_1(x)y_1''(x) + c_2(x)y_2''(x)$$

- Substitute y_p, y_p', and y_p'' into the nonhomogeneous equation:

$$\begin{aligned}
&y_p'' + p_1(x)y_p' + p_0(x)y_p \\
&\quad = (c_1'y_1' + c_2'y_2' + c_1y_1'' + c_2y_2'') + p_1(c_1y_1' + c_2y_2') + p_0(c_1y_1 + c_2y_2) \\
&\quad = c_1(y_1'' + p_1y_1' + p_0y_1) + c_2(y_2'' + p_1y_1' + p_0y_2) + c_1'y_1' + c_2'y_2' \\
&\quad = c_1 \cdot 0 + c_2 \cdot 0 + c_1'y_1' + c_2'y_2' \\
&\quad = c_1'y_1' + c_2'y_2' \\
&\quad = q(x)
\end{aligned}$$

- In summary, we have obtained the following system:

$$\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = q(x) \end{cases}$$

By Cramer's rule for solving systems of linear equations,

$$c_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ q(x) & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = \frac{-y_2(x)q(x)}{W(x)}$$

$$c_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & q(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = \frac{y_1(x)q(x)}{W(x)}$$

We have obtained simple formulas for computing the derivatives $c_1'(x)$ and $c_2'(x)$:

$$\begin{aligned} c_1'(x) &= \frac{-y_2(x)q(x)}{W(x)} \\ c_2'(x) &= \frac{y_1(x)q(x)}{W(x)} \end{aligned}$$

- Simple integration, of course, gives $c_1(x)$ and $c_2(x)$.
- Similar formulas are obtained for equations of order 3 and higher.
- Division by the Wronskian $W(x)$ has no effect on the domain of existence of the solutions because we are given that $\{y_1(x), y_2(x)\}$ is a linearly independent set, and hence the Wronskian does not vanish on the domain of existence of the equation (α, β).
- **Example 24:** Find a general solution of the equation

$$y'' + y = \tan x, \quad (0 < x < \frac{\pi}{2})$$

Solution: First, note that although the coefficients of the equation are constant, the equation cannot be solved by the method of undetermined coefficients. Recall that the general solution of the homogeneous equation $y'' + y = 0$ is

$$y(x) = c_1 \cos x + c_2 \sin x$$

Therefore, the form of the particular solution of the nonhomogeneous equation is

$$y_p(x) = c_1(x) \cos x + c_2(x) \sin x$$

We must solve the system

$$\begin{cases} \cos x \cdot c_1'(x) + \sin x \cdot c_2'(x) = 0 \\ -\sin x \cdot c_1'(x) + \cos x \cdot c_2'(x) = \tan x \end{cases}$$

We use Cramer's rule to solve the system of linear equations:

$$c_1'(x) = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \cdot \tan x}{1} = -\sin x \cdot \tan x$$

$$c_2'(x) = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \cdot \tan x}{1} = \sin x$$

Clearly, $c_2(x) = -\cos x$. To obtain $c_1(x)$, we must evaluate the integral

$$\begin{aligned} c_1(x) &= \int -\sin x \cdot \tan x \, dx = \int -\frac{\sin^2 x}{\cos x} dx \\ &= \int \frac{\cos^2 x - 1}{\cos x} dx = \int \left(\cos x - \frac{1}{\cos x} \right) dx \\ &= \sin x - \int \frac{\cos x}{\cos^2 x} dx = \sin x - \int \frac{\cos x \, dx}{1 - \sin^2 x} \\ &= \sin x - \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} \end{aligned}$$

Therefore,

$$\begin{aligned} y_p(x) &= \left(\sin x - \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} \right) \cos x - \cos x \cdot \sin x \\ &= -\frac{1}{2} \cos x \cdot \ln \frac{1 + \sin x}{1 - \sin x} \end{aligned}$$

and hence the general solution of the nonhomogeneous equation is

$$y(x) = -\frac{1}{2} \cos x \cdot \ln \frac{1 + \sin x}{1 - \sin x} + c_1 \cos x + c_2 \sin x$$

Programming Corner: Programmers can easily verify that $y_p(x)$ is a solution of the nonhomogeneous equation using **SymPy** (Python):

http://live.sympy.org

(otherwise, a manual check may consume a great deal of time and ink and is also prone to errors)

```
from sympy import *
   x = Symbol("x")
yp = -0.5*cos(x) * ln((1+sin(x))/(1-sin(x)))
yp1 = diff(yp, x)
yp2 = diff(yp1, x)
simplify(yp + yp2)
```

Copy and paste the code into the window in **SymPy**

- In exactly the same way, we can find a particular solution of a third-order nonhomogeneous linear differential equation:

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = q(x)$$

If the general solution of the homogeneous equation

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

is known,

$$y(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x)$$

then the particular solution of the nonhomogeneous equation has the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + c_3(x)y_3(x)$$

After differentiating and imposing two additional constraints, we obtain the system

$$\begin{cases} c_1'(x)y_1(x) + c_2'(x)y_2(x) + c_3'(x)y_3(x) = 0 \\ c_1'(x)y_1'(x) + c_2'(x)y_2'(x) + c_3'(x)y_3'(x) = 0 \\ c_1'(x)y_1''(x) + c_2'(x)y_2''(x) + c_3'(x)y_3''(x) = q(x) \end{cases}$$

Using Cramer's rule, we obtain appropriate formulas for $c_1'(x)$, $c_2'(x)$, and $c_3'(x)$. For example, the formula for $c_1'(x)$ is

$$c_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) & y_3(x) \\ 0 & y_2'(x) & y_3'(x) \\ q(x) & y_2''(x) & y_3''(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1'(x) & y_2'(x) & y_3'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) \end{vmatrix}} = \frac{q(x)[y_2(x)y_3'(x) - y_3(x)y_2'(x)]}{W(x)}$$

- **Exercise 4.7:** Find a general solution of the equation

$$y''' - y' = x$$

Guidance: The problem can be solved by the method of undetermined coefficients or by a simple reduction of order ($v = y'$). Nevertheless, it is advisable to solve the exercise in order to practice variation of parameters. The final result should be

$$y(x) = -\frac{x^2}{2} + c_1 + c_2e^{-x} + c_2e^{x}$$

Exercise 4.8: Find a general solution of the equation

$$y'' - 4y' + 5y = e^{2x}\sin^2 x$$

Guidance: The final answer is

$$y(x) = 29\sin^2 x - 15\sin 2x + \frac{23}{312}e^{2x}\cos^2 x + c_1 + c_2 e^{-x}$$

4.8.3 The Euler Equation (Leonhard Euler 1707-1783)

- One well-known linear equation is the **Euler equation**:

$$a_n x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_0 y = q(x), \quad (0 < x < \infty)$$

- The Euler equation is important in solving the Laplace equation (electromagnetism, electricity, heat, fluids), but it also has methodological importance as a simple example of regular singular points.

Regular singular points (Wikipedia link)

- The substitution

$$x = e^t, \qquad t = \ln x \qquad (0 < x < \infty)$$

transforms the Euler equation into a linear equation with constant coefficients. We illustrate this with an equation of order 3:

$$\begin{aligned}
y' &= \tfrac{dy}{dx} = \tfrac{dy}{dt}\cdot\tfrac{dt}{dx} = \tfrac{dy}{dt}\cdot\tfrac{1}{x} \\
&= \tfrac{1}{x}\cdot\tfrac{dy}{dt} \\
y'' &= \tfrac{d}{dx}\left[\tfrac{1}{x}\cdot\tfrac{dy}{dt}\right] = -\tfrac{1}{x^2}\cdot\tfrac{dy}{dt} + \tfrac{1}{x^2}\cdot\tfrac{d^2y}{dt^2} \\
&= \tfrac{1}{x^2}\left[\tfrac{d^2y}{dt^2} - \tfrac{dy}{dt}\right] \\
y''' &= \tfrac{1}{x^3}\left[\tfrac{d^3y}{dt^3} - 3\tfrac{dy^2}{dt^2} + 2\tfrac{dy}{dt}\right]
\end{aligned}$$

After substituting these results into the Euler equation, all the powers x^k clearly cancel, and we obtain a linear equation in the function $y(t)$ with constant coefficients.

- The following operator formula can be proved by mathematical induction:

$$\frac{d^n y}{dx^n} = \frac{1}{x^n}\cdot\frac{d}{dt}\left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} - 2\right)\cdots\left(\frac{d}{dt} - n + 1\right)[y]$$

- On the interval $-\infty < x < 0$, the substitution is $x = -e^t$.

Example 25: Find a general solution of the following Euler equation:

$$x^2y'' - 2xy' + 2y = x^3 \qquad (0 < x < \infty)$$

Solution: We pass to the variable t using the substitution $x = e^t$:

$$\begin{aligned} x^2y'' - 2xy' + 2y &= x^2 \cdot \tfrac{1}{x^2}\left[\tfrac{d^2y}{dt^2} - \tfrac{dy}{dt}\right] - 2x \cdot \tfrac{1}{x} \cdot \tfrac{dy}{dt} + 2y(t) \\ &= \tfrac{dy^2}{dt^2} - 3\tfrac{dy}{dt} + 2y(t) \end{aligned}$$

We have obtained a new equation in the variables y and t:

$$y''(t) - 3y'(t) + 2y(t) = e^{3t}$$

which can, of course, be solved by the method of undetermined coefficients.

$$y(t) = \frac{1}{2}e^{3t} + c_1e^t + c_2e^{2t}$$

Returning to the original variables $y = y(x)$, we obtain

$$y(x) = \frac{1}{2}x^3 + c_1x + c_2x^2$$

It follows that the fundamental solutions of a homogeneous Euler equation are always of the form $y = x^r$. From now on, therefore, we will not use the substitution $x = e^t$ again, but will instead seek the fundamental solutions directly by substituting $y = x^r$ into the expression

$$L[y] = a_nx^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_0y = 0$$

which gives

$$L[x^r] = x^rP(r)$$

where $P(r)$ is a real polynomial of degree n.

For example, for a second-order Euler equation,

$$\begin{aligned} y &= x^r \\ y' &= rx^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned}$$

and therefore

$$\begin{aligned} L[y] &= a_2x^2y'' + a_1xy' + a_0y \\ &= a_2x^2r(r-1)x^{r-2} + a_1xrx^{r-1} + a_0x^r \\ &= a_2r(r-1)x^r + a_1rx^r + a_0x^r \\ &= x^r[a_2r(r-1) + a_1r + a_0] \\ &= 0 \end{aligned}$$

We have found that when $n = 2$,

$$P(r) = a_2r(r-1) + a_1r + a_0 = 0$$

- The last equation is called the **indicial equation** or the **characteristic equation** of the homogeneous Euler equation.
- For a homogeneous equation of order 3, the indicial equation is

$$P(r) = a_3r(r-1)(r-2) + a_2r(r-1) + a_1r + a_0 = 0$$

- If the indicial polynomial $P(r)$ has n distinct real roots

$$r_1, \quad r_2, \quad \ldots, \quad r_n$$

then a basis for the solution space of the homogeneous equation is

$$x^{r_1}, \quad x^{r_2}, \quad \ldots, \quad x^{r_n}$$

- If r is a real root of algebraic multiplicity k, then it corresponds to k linearly independent solutions:

$$x^r, \quad x^r \ln x, \quad x^r(\ln x)^2, \quad \ldots, \quad x^r(\ln x)^{k-1}$$

- If $r = \alpha \pm \beta i$ are two distinct complex roots of $P(r)$, then they correspond to 2 linearly independent solutions:

$$x^\alpha \cos(\beta \ln x), \qquad x^\alpha \sin(\beta \ln x)$$

- If $r = \alpha \pm \beta i$ are two complex roots of $P(r)$ of algebraic multiplicity k, then they correspond to $2k$ linearly independent solutions:

$$\begin{array}{ll}
x^\alpha \cos(\beta \ln x), & x^\alpha \sin(\beta \ln x) \\
x^\alpha \ln x \cos(\beta \ln x), & x^\alpha \ln x \sin(\beta \ln x) \\
x^\alpha (\ln x)^2 \cos(\beta \ln x), & x^\alpha (\ln x)^2 \sin(\beta \ln x) \\
\vdots & \vdots \\
x^\alpha (\ln x)^{k-1} \cos(\beta \ln x), & x^\alpha (\ln x)^{k-1} \sin(\beta \ln x)
\end{array}$$

Exercises: Find a general solution of each of the following equations:

(a) $x^2y'' - xy' + y = x^5$

(b) $2x^2y'' + 5xy' + y = 3x + 2$

(c) $x^2y'' - 6y = 1 + \ln x$

(d) $x^3y''' - x^2y'' + 2xy' - 2y = 1$

(e) $3(x+6)^2y'' + 25(x+6)y' - 16y = 0\,, \quad -6 < x < \infty$

Final Answers:

(a) $y(x) = \frac{x^4}{16} + c_1 x + c_2 \ln x$

(b) $y(x) = \frac{x^2}{2} + 2 + \frac{c_1}{x} + \frac{c_2}{\sqrt{x}}$

(c) $y(x) = -\frac{\ln x}{6} - \frac{1}{9} + c_1 x^{\frac{1-\sqrt{13}}{2}} + c_2 x^{\frac{1+\sqrt{13}}{2}}$

(d) $y(x) = -\frac{1}{2}c_1 x + c_2 x \ln x + c_3 x^2$

(e) $y(x) = c_1(x+6)^{-8} + c_2(x+6)^{\frac{2}{3}}$

Exercises for chapter 4

Guidance: A link at the end of the list leads to a Jupyter notebook containing final and/or complete solutions to a small number of the exercises. Before clicking the link, however, it is strongly recommended that you solve them on your own! Additional solutions will be added over time. We would be very happy to receive additional solutions from students for exercises that have not yet been solved!

1. Find a general solution:

(1) $y'' - 25y = 0$

(2) $y'' - 2y' + 2y = 0$

(3) $y'' - 5y' + 6y = 0$

(4) $y''' - y = 0$

(5) $y^{(4)} - y = 0$

(6) $y^{(4)} + 4y'' + 3y = 0$

(7) $y''' - 2y'' - 3y' = 0$

(8) $y''' - 3y'' + 3y' - y = 0$

(9) $y^{(4)} - 5y'' + 4y = 0$

2. Solve the following initial-value problems:

(a) $\begin{cases} y'' - 5y' + 4y = 0 \\ y(0) = 5 \\ y'(0) = 8 \end{cases}$ **(b)** $\begin{cases} y'' + 4y = 0 \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$

(c) $\begin{cases} y''' + y'' - 5y' + 3y = 0 \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = -2 \end{cases}$

3. Find a general solution:

(1) $y'' - 2y' - 3y = e^{4x}$
(2) $y'' - 5y' + 4y = 4x^2e^{2x}$
(3) $y'' - y = -x^2$
(4) $y'' + 3y' - 4y = xe^{-x}$
(5) $y'' - y' + y = x^3 + 6$
(6) $y'' - 8y' + 7y = 14$
(7) $y'' + y' - 2y = 3xe^x$
(8) $y'' - y = 2e^x$
(9) $y^{(4)} - 2y''' + y'' = e^x$
(10) $y'' - 4y' + 3y = 2 - \cos(2x) + 35\sin(2x)$
(11) $y'' + y' - 2y = 11\cos\frac{x}{2} - 7\sin\frac{x}{2}$
(12) $y'' + 5y' + 6y = -50\sin 4x$
(13) $y'' - 3y' + 2y = 2e^x\cos\frac{x}{2}$
(14) $y'' - 3y' + 2y = 2\sin x$
(15) $y'' + 4y = 12\cos 2x$
(16) $y'' + 9y = -18\cos 3x$
(17) $y'' - 4y' + 4y = x^2 + e^2x + \sin 2x$
(18) $y'' - 2y' - 8y = e^x - 8\cos 2x$
(19) $y'' - y = 2x - 1 - 3e^x$
(20) $y''' + y'' = 3xe^x + x^2 + 1$
(21) $y'' - 8y' + 20y = 5xe^{4x}\sin 2x$
(22) $y'' - 9y = e^{-3x}(x^2 + \sin 3x)$
(23) $y'' + 3y - 4y = e^{-4x} + xe^{-x}$

4. Consider a nonhomogeneous linear differential equation with constant coefficients:

$$L[y] = q(x)$$

The roots of the characteristic polynomial of the equation are given by

$$\begin{array}{lllll} r_1 = -2, & r_2 = -2, & r_3 = -2, & r_4 = 0, & r_5 = 3, \\ r_6 = 3, & r_7 = 5i, & r_8 = -5i, & r_9 = 6i, & r_{10} = -6i \end{array}$$

Write the form of the particular solution for each of the following cases:

(a) $q(x) = x^2 e^{3x}$

(b) $q(x) = 7xe^{-2x}$

(c) $q(x) = 11$

(d) $q(x) = 2\cos 4x - \sin 4x$

(e) $q(x) = \sin 5x$

(f) $q(x) = 4e^{3x} - x + \cos 6x + e^{-x}$

5. Construct a homogeneous linear differential equation with constant coefficients whose general solution is

$$y(x) = c_1 x + c_2 e^{-x} + c_3 e^{2x}$$

6. Solve the following initial-value problems:

(a) $\begin{cases} y'' - 2y' = e^{2x} + x^2 - 1 \\ y(0) = \frac{1}{8} \\ y'(0) = 1 \end{cases}$

(b) $\begin{cases} y'' + y = e^{2x} + x^2 - 1 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

(c) $\begin{cases} y''' - y' = 3(2 - x^2) \\ y(0) = 1 \\ y'(0) = 1 \\ y''(0) = 1 \end{cases}$

(d) $\begin{cases} y'' - 2y' + y = x \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$

(e) $\begin{cases} y'' - 6y' + 9y = 50\sin x \\ y(0) = 0 \\ y'(0) = 1 \\ y''(0) = -2 \end{cases}$

(f) $\begin{cases} y'' - 2y' + y = xe^{x} + 4 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

(g) $\begin{cases} y'' + 2y' + y = xe^{-x} \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

7. Suppose that $y_1(x)$ and $y_2(x)$ are particular solutions of the equation

$$y'' + 4y' + 8y = q(x)$$

where $q(x)$ is continuous on the entire real line $\mathbb{R}$.

(a) Prove that $\lim_{x\to\infty} y_1(x) = \lim_{x\to\infty} y_2(x)$.

(b) Suppose that $y_1(0) = y_2(0)$. Prove that $y_1(\frac{\pi}{2}) = y_2(\frac{\pi}{2})$.

8. Suppose that the three functions

$$y_1(x) = e^x, \quad y_2(x) = 2x, \quad y_3(x) = 3$$

are solutions of the equation

$$y''' + a_2y'' + a_1y' + a_0y = 0$$

What is the general solution of the equation?

9. Consider the equation

$$y^{(4)} + a_3y''' + a_2y'' + a_1y' + a_0y = 3e^{\alpha x}$$

and suppose that the roots of its characteristic polynomial are

$$r_1 = 1, \quad r_2 = 1, \quad r_3 = 1, \quad r_4 = -3$$

Find the form of the particular solution.

10. Consider the equation $y''' + 3y'' + 3y' + y = 0$. Find the solution that satisfies $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$.

11. Consider the equation $y'' + 4y = 0$. Find the solution that satisfies $y(0) = 0$, $y'(0) = 1$.

12. Find the general solution of each of the following equations:

(1) $x^2y'' + xy' + 4y = 10x$

(2) $x^2y'' - 3xy' + 5y = 3x^2$

(3) $x^2y'' - 6y = 5x^3 + 8x^2$

(4) $x^2y'' - 2y = \sin(\ln x)$

(5) $(1+x)^2y'' - 3(1+x)y' + 4y = (1+x)^3$

(6) $y'' - 2y' + y = \dfrac{e^x}{x}$

(7) $y'' + y = \dfrac{1}{\sin x}$

(8) $y'' + 2y' + 5y = e^{-x}(\cos^2 x + \tan x)$

(9) $y''' + y' = \sin x + x\cos x$

(10) $y^{(5)} + 4y' = x + 1 + \cos 2x$

13. Solve the following initial-value problems:

(a) $\begin{cases} x^2y'' - xy' + y = 2x \\ y(1) = 0 \\ y'(1) = 1 \end{cases}$ **(b)** $\begin{cases} x^2y'' - 3xy' + 13y = 0 \\ y(1) = 1 \\ y'(1) = -1 \end{cases}$

14. Find the general solution of each of the following equations:

(1) $y'' + y = \cot x$

(2) $y'' + 2y' + y = \dfrac{e^{-x}}{x}$

(3) $y'' + 5y' + 6y = \dfrac{1}{e^{2x} + 1}$

(4) $y'' - 4y' + 5y = \dfrac{e^{2x}}{\cos x}$

(5) $y''' + y = \dfrac{\sin x}{\cos^2 x}$

(6) $y''' + 4y = \dfrac{1}{\cos 2x}$

(7) $x^2y'' - xy' - 3y = 0$

(8) $x^2y'' - 2xy' + 2y = x^3 \sin x$

(9) $x^2y'' - xy' + y = \dfrac{\ln x}{x} + \dfrac{x}{\ln x}$

(10) $x^2y'' - 4xy' + 6y = 0$

(11) $(x-2)^2y'' - 3(x-2)y' + 4y = x$

(12) $x^3y''' + xy' - y = 0$

(13) $x^3y'' - 2xy = 6\ln x$

(14) $x^2y'' - 2y = \sin\ln x$

15. It is known that the following equation has a particular solution that is a real polynomial of degree 1. Find the general solution:

$$x^2y'' - x(x+2)y' + (x+2)y = 0$$

16. It is given that the function $y(x) = x^4 + x^4 \ln x$ is a particular solution of the differential equation

$$x^2y'' + a_1xy' + a_0y = 0$$

Find the coefficients a_0, a_1.

17. Two solutions $y_1(x)$ and $y_2(x)$ of the equation are given:

$$y'' - y' + ye^{2x} = 0$$

Their Wronskian is known to satisfy $W(0) = 1$. Find $W(x)$.

18. It is known that the three functions

$$\begin{cases} y_1(x) &= (x+4)\sin x \\ y_2(x) &= (x+5)\sin x \\ y_3(x) &= x\sin x + 3\cos x \end{cases}$$

are solutions of the equation

$$y'' + p_1(x)y' + p_0(x)y = q(x)$$

where $p_0(x)$, $p_1(x)$, and $q(x)$ are continuous on the entire real line $\mathbb{R}$. Find a solution $y(x)$ of the equation that satisfies

$$\begin{cases} y(0) = 1 \\ y'(0) = 1 \end{cases}$$

and explain why this solution is unique on the entire real line.

19. Consider the homogeneous linear differential equation

$$y'' + p_1(x)y' + p_0(x)y = 0$$

where $p_0(x)$ and $p_1(x)$ are continuous functions on the ray $[0, \infty)$.
Let $y_1(x)$ and $y_2(x)$ be two solutions of the equation such that $y_1(x) > 0$ throughout the domain and $y_1(0) = y_2(0)$. Prove or disprove each of the following assertions:
(a) If $y_1(3) = y_2(3)$, then $y_1(x) < y_2(x)$ throughout the ray $(0, \infty)$.
(b) If $y_1(3) < y_2(3)$, then $y_1(x) < y_2(x)$ throughout the ray $(0, \infty)$.
(c) If $y_1(3) > y_2(3)$, then $y_1(x) > y_2(x)$ throughout the ray $(0, \infty)$.
Hint: Study the function $h(x) = \frac{y_2(x)}{y_1(x)}$. Use the Wronskian and Rolle's theorem.

20. Consider the third-order homogeneous linear differential equation

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

where $p_0(x)$, $p_1(x)$, and $p_2(x)$ are continuous functions on the interval (α, β), and three solutions of the equation are given:

$$y_1(x) = xe^x, \quad y_2(x) = (x+1)e^x, \quad y_3(x) = x - 3$$

on the interval (α, β).

(a) Which of the following points does not belong to the interval (α, β)?

$$x = 1, \quad x = 2, \quad x = 3, \quad x = 4, \quad x = 5$$

(b) It is known that the function $p_2(x)$ vanishes at the point $x = x_0$, $\alpha < x_0 < \beta$. Find the point x_0.

Hint: Try to devote at least 10 minutes to thinking about the problem in terms of the Wronskian before reading the guidance below ...

A. Determine at which points the Wronskian of the three solutions vanishes. The Wronskian of the three functions should be $W(x) = (5-x)e^{2x}$.

B. After explicitly computing the Wronskian $W(x)$, use the formula $\frac{W'(x)}{W(x)} = -p_{n-1}(x)$ to compute x_0. $p_{n-1}(x)$ is the coefficient of $y^{(n-1)}$.

21. Consider the equation

$$x(x+1)y'' - 2y' - 2y = 0$$

(a) Show that the function $y_1(x) = \frac{1}{x-1}$ is a solution of the equation.

(b) Find a solution $y_2(x)$ of the equation that satisfies $y_2'(1) = \frac{3}{2}$.

22. Consider the equation

$$y^{(4)} + ay''' + by'' + cy' + ky = 0$$

where a, b, c, and k are real constants. It is given that $y_1(x) = 5x + 3\sin x$ is a solution of the equation. Prove or disprove the following assertions:

(a) $c = 0$, $k = 1$

(b) $c = 1$, $k = 1$

(c) $c = 0$, $k = 0$

(d) $k = a + b + c$

(e) $c = a + b$

23. Explain why the Wronskian of the equation

$$y''' - (e^{3x} + \cos 5x)y' + x^7 y = 0$$

is constant.

24. Suppose that $y(x)$ is the solution of the initial-value problem

$$\begin{cases} y'' - 3y' + 2y = \dfrac{1}{1+e^{-x}} \\ y(0) = \ln(4) \\ y'(0) = \ln(8) \end{cases}$$

Compute $y(1)$.

25. A linear equation together with a known particular solution $y_1(x)$ is given. Find the general solution.

$(x^2+1)y'' - 2xy' + 2y = 0\,, \quad y_1(x) = x$

$x^2(x+1)y'' - 2y = 0\,, \quad y_1(x) = 1 + \frac{1}{x}$

$xy'' + 2y' - xy = 0\,, \quad y_1(x) = \frac{e^x}{x}$

$y'' - 2(1+\tan^2 x)y = 0\,, \quad y_1(x) = \tan x$

$x^2(2x-1)y''' + (4x-3)xy'' - 2xy' + 2y = 0\,, \quad y_1(x) = x\,, \quad y_2(x) = \frac{1}{x}$

$xy'' - y' + 4x^3y = 0\,, \quad y_1(x) = \sin(x^2)$

$y'' + \frac{2}{x}y' + y = 0\,, \quad y_1(x) = \frac{\sin x}{x}$

26. It is known that $y_1(x) = e^x$ is a solution of the equation $(x-1)y'' - xy' + y = 0$. Use this solution and Abel's formula to find a second solution that is linearly independent of $y_1(x)$.

27. It is known that $\sin x$ is a solution of the differential equation

$$y^{(4)} - 2y''' + 3y'' - 2y' + 2y = 0$$

Find its general solution.

28. Two particular solutions of the equation $6x^3y''' - 24x^2y'' + 48xy' - 48y = 0$ are known:

$$y_1(x) = x^2, \qquad y_2(x) = x, \qquad 0 < x < \infty$$

Find the general solution.

29. For the equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0, \qquad 0 < x < \infty$$

the solution $y_1(x) = x$ is known. Find the general solution.

30. Two particular solutions of the equation $xy''' - y'' - xy' + y = 0$ are known:

$$y_1(x) = x, \qquad y_2(x) = e^x$$

Find the general solution.

Hint: In addition to the substitution method ($y_3(x) = v(x)y_1(x)$) for reducing the order, the Wronskian and Abel's formula can be used to find a third linearly independent solution.

31. Consider the equation: $x^2y'' - 2y = 21\sqrt{x}$. It is known that the corresponding homogeneous equation has the solution $y(x) = x^2$. Find the general solution of the equation.

32. Consider the equation: $x^2y'' - x(x+2)y' + (x+2)y = 2x^3$. It is known that the corresponding homogeneous equation has the solution $y_1(x) = x$. Find the general solution of the equation.

33. Consider the differential equation

$$y'' + p_1(x)y' + p_0(x)y = q(x), \qquad -\infty < x < \infty$$

where $p_0(x)$ and $p_1(x)$ are continuous on the entire real line. It is known that

$$y_1(x) = (x+3)\cos x, \qquad y_2(x) = x\cos x, \qquad y_3(x) = 3\sin x$$

are three solutions of the equation.

(a) Find the general solution of the equation.

(b) Compute $p_1(x)$.

34. It is known that the function $y(x)$ is the solution of the initial-value problem

$$\begin{cases} x^2y'' + 2xy' = x\cos x, \quad 0 < x < \infty \\ y(\frac{\pi}{2}) = 0 \\ y'(\frac{\pi}{2}) = \frac{2}{\pi} \end{cases}$$

Compute the value of $y(\pi)$.

35. Let $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ be four solutions of the differential equation

$$y^{(4)} + \frac{1}{x}y''' + \frac{1}{x^2}y'' + \frac{1}{x^3}y' + \frac{1}{x^4}y = 0$$

It is known that $W[y_1, y_2, y_3, y_4](1) = 8$. Compute $W[y_1, y_2, y_3, y_4](2)$.

Hint: Use the formula $W(x) = W(x_0)e^{-\int_{x_0}^{x} p_{n-1}(t)dt}$. The choice of the points x_0 and x is not difficult ...

36. Determine whether there exists a solution $y(x)$ of the differential equation

$$y^{(6)} + y^{(5)} - y^{(4)} - y''' = 3e^{-x}$$

that satisfies $y(0) = 0$, $y'(0) = 0$, and $\lim_{x\to\infty} y(x) = 0$.

37. Solve the initial-value problem $\begin{cases} x^2y'' - 2y = x^2 \\ y(1) = 1 \\ y'(1) = 1 \end{cases}$

What is the domain of the solution?

38. Solve the initial-value problem $\begin{cases} x^2y'' + xy' + 9y = 12\sin(3\ln x) \\ y(1) = 0 \\ y'(1) = 0 \end{cases}$

What is the domain of the solution you found?

39. Write the form of the particular solution according to the method of undetermined coefficients (there is no need to determine the constants) for the equation

$$y''' - 4y' = 3x + 10\cos(x)$$

40. A homogeneous linear differential equation with constant coefficients $L[y] = 0$ has the characteristic polynomial

$$P(r) = r^2(r^2 + 4)^2(r^2 + 3r - 4)$$

Find the form of a particular solution of the nonhomogeneous equation

$$L[y] = \sin 2x + xe^x + 2e^{-4x} + 5$$

41. Let $y(x)$ be a solution of the initial-value problem

$$\begin{cases} y'' - xy' + 3y = 0 \\ y(0) = 0 \\ y'(0) = 6 \end{cases}$$

Find $y'''(6)$.

42. It is given that the function $y(x) = 3xe^x\sin(2x)$ is a solution of the equation

$$y^{(4)} + a_3y''' + a_2y'' + a_1y' + a_0y = 0$$

where a_0, a_1, a_2, and a_3 are constant real coefficients. Compute a_1, a_2.

43. It is known that the functions $y_1(x) = \dfrac{\sqrt{\pi} + 3\cos x}{\sqrt{x}}$, $y_2(x) = \dfrac{\sqrt{\pi}}{\sqrt{x}}$ are two solutions of the equation

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = q(x), \quad 0 < x < \infty$$

where $q(x)$ is continuous on the ray $(0,\infty)$. Find the particular solution of the initial-value problem

$$\begin{cases} x^2y'' + xy' + (x^2 - \frac{1}{4})y = q(x), & 0 < x < \infty \\ y(\pi) = 0 \\ y'(\pi) = 1 \end{cases}$$

44. Write the general form of a particular solution of the equation

$$x^3y''' + xy' - y = (x + \sqrt{x})\ln x$$

45. Using the method of variation of parameters, find a formula for the general solution of the equation

$$y'' + y = q(x)$$

where $q(x)$ is any continuous function on a given interval.

46. Consider the homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x)$ and $q(x)$ are continuous functions on the entire real line. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the equation. For each assertion, prove it if it is true; otherwise, disprove it with a suitable example.

A. The graphs of $y_1(x)$ and $y_2(x)$ never intersect.

B. The graphs of $y_1(x)$ and $y_2(x)$ may intersect, but they cannot be tangent to one another.

C. If $y_1(x) > 0$ for every real x, then the graphs intersect at most once.

D. The graphs of $y_1(x)$ and $y_2(x)$ may intersect infinitely many times.

47. It is given that the function $y(x) = x^2\sin(3x)$ is a solution of the linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$$

where all the coefficients a_i are real constants.

A. Find the minimum possible value of n.

B. Find a basis for the solution space of the equation for the value of n found in the preceding part.

C. Find all the coefficients of the equation.

Link to solutions

Chapter 5

Solving Equations by Power Series

5.1 Review of Power Series

A concise review of power series. This topic should be covered in the Calculus 1 and Calculus 2 courses, and therefore only the main results are presented, without proofs or excessive detail. We have tried to include only the topics needed for solving differential equations.

Definition 5.1: A series of functions of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots \qquad (*)$$

is called a **power series**.
The point x_0 is called the **center of the series**.
The constants $\{a_n\}_{n=1}^{\infty}$ are called the **coefficients of the series**.

Fig. 5.1: Colin Maclaurin
1698-1746

Fig. 5.2: Brook Taylor
1685-1731

➤ In many examples, the center of the series is $x_0 = 0$. In this case, the power series has the simpler form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \qquad (**)$$

- **Example 1:** Every polynomial is in fact a power series in which all but finitely many coefficients vanish:

$$-7x^2 + 100x - 19 = -19 + 100x - 7x^2 + 0x^3 + 0x^4 + 0x^5 + \cdots$$

Thus, a power series is an infinite-dimensional generalization of the concept of a polynomial, and indeed, in certain contexts, a power series is sometimes called an "infinite polynomial".

- **Example 2:** The best-known power series is the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

The center of the series is $x_0 = 0$, and all its coefficients equal 1 (that is, for every n, $a_n = 1$). Recall that it converges on the open interval $(-1, 1)$.

- Reminder: Important examples of Maclaurin series (Taylor series about $x_0 = 0$)

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
\arctan x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} &&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} &&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
(1+x)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n &&= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots, (\alpha \in \mathbb{R}) \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \cdots \\
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n &&= 1 - x + x^2 - x^3 + \cdots
\end{aligned}$$

- Note that the geometric-series formula is a special case of the preceding Maclaurin formula ($\alpha = -1$, with x replaced by $-x$).

5.1.1 The Interval of Convergence of a Power Series

- Clearly, every power series $\sum_{n=0}^{\infty} a_n x^n$ converges at the point $x = 0$.
- The faster the sequence of coefficients a_n grows, the smaller the interval of convergence of the

series becomes.

- For example, the power series $\sum\limits_{n=0}^{\infty} n^n x^n$ converges only at the point $x=0$! ($n^n x^n = (nx)^n$ diverges if $x \neq 0$).
- In contrast, the series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ converges for every real x.

> **Theorem 5.1:** If a power series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ converges at the point $x = x_1$, then it converges uniformly and absolutely on every interval $[x_0 - r, x_0 + r]$ for every r satisfying $0 < r < |x_1 - x_0|$.

- The preceding theorem implies that the interval of convergence of a power series must be symmetric about its center x_0 (which explains the term ”center“).
- **Corollary:** The interval of convergence of a power series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ must have one of the following forms:
  (a) A single point $\{x_0\}$
  (b) An open interval of the form $(x_0 - R,\ x_0 + R)$
  (c) A closed interval of the form $[x_0 - R,\ x_0 + R]$
  (d) A half-open interval of the form $[x_0 - R,\ x_0 + R)$
  (e) A half-open interval of the form $(x_0 - R,\ x_0 + R]$
  (f) The entire real line $(-\infty,\ \infty)$
- The preceding theorem tells us nothing about the endpoints of a finite interval of convergence. For example, the series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n}$ converges at $x = -1$ but diverges at $x = 1$. Its interval of convergence is $[-1, 1)$ (the center of the series is $x = 0$).
- The number R is called the **radius of convergence of the series**, and it has a corresponding value in each of the forms A-F: In form A, $R = 0$, and in form F, $R = \infty$.

> **Theorem 5.2: (The Cauchy-Hadamard formula)**
> The radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ is
> $$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$
> (assuming that the limit exists and the coefficients do not vanish)

Theorem 5.3: (The ratio formula for computing the radius of convergence)
The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

(assuming that the limit exists and the coefficients do not vanish)

- In the Cauchy-Hadamard formula, if the limit does not exist, the limit superior of the sequence ($\limsup$) may be used.

$$R = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$$

- **Exercise 5.1:** If the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is R, what can you say about the radius of convergence of the series $\sum_{n=0}^{\infty} n a_n x^n$?

5.1.2 Uniform Convergence of a Power Series

As we saw in preceding lessons, the property of uniform convergence allows us to perform useful operations on series and obtain important results. It is therefore important to know the conditions under which a power series converges uniformly.

Theorem 5.4: Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series with radius of convergence R.

(a) If $R = 0$, then the series converges only at the point $x = x_0$.

(b) If $R = \infty$, then the series converges uniformly on every finite closed interval $[a,b]$.

(c) If $0 < R < \infty$, then the series converges uniformly on every closed interval $[a,b]$ that satisfies

$$x_0 - R < a < b < x_0 + R$$

- Note that the theorem does not rule out the possibility of uniform convergence on larger domains.
- For example, the series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ converges pointwise on the closed interval $[-1,1]$. The preceding theorem guarantees uniform convergence on every subinterval $[a,b]$ satisfying

$$-1 < a < b < 1$$

  but it is not difficult to verify that the series converges uniformly (and absolutely) on the entire interval $[-1,1]$.
- In contrast, the geometric series $\sum_{n=0}^{\infty} x^n$ converges pointwise on the open interval $(-1,1)$, but it does not converge uniformly on the entire interval $(-1,1)$. The assertion of the theorem is

therefore optimal for this series.

Theorem 5.5: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then $f(x)$ is continuous at every point x, $-R < x < R$. If the series converges at the point $x = R$, then $f(x)$ is left-continuous at that point. If the series converges at the point $x = -R$, then $f(x)$ is right-continuous at that point.

5.1.3 Differentiation and Integration of a Power Series

Theorem 5.6: (Integration of a power series)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R.

Then, for every point $|x| < R$, the following hold:

(a)

$$\int_0^x f(t)\,dt = \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

(b) The two series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ have the same radius of convergence R.

(c) If the series $\sum_{n=0}^{\infty} a_n x^n$ converges at the endpoint $x = R$, then the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also converges at that point.

(d) The analogous statement holds for the endpoint $x = -R$.

Example 3: If we integrate the geometric series,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

we obtain a practical formula for computing the natural logarithm:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

The interval of convergence of the geometric series is $(-1, 1)$, whereas the interval of convergence of the logarithmic series is $(-1, 1]$.

Theorem 5.7: (Differentiation of a power series)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R.

Then, for every point $|x| < R$, the following hold:

(a)

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

(b) The two series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ have the same radius of convergence R.

(c) If the series $\sum_{n=0}^{\infty} a_n x^n$ converges at the endpoint $x = R$, then the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also converges at that point.

(d) The analogous statement holds for the endpoint $x = -R$.

- The last theorem allows us to differentiate a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ as many times as we wish.

$$\begin{aligned}
f(x) &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + a_4(x-x_0)^4 + \cdots \\
f'(x) &= a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + 4a_4(x-x0)^3 + \cdots + na_n(x-x_0)^{n-1} + \cdots \\
f''(x) &= 2a_2 + 3\cdot 2a_3(x-x_0) + 4\cdot 3a_4(x-x0)^2 + \cdots + n(n-1)a_n(x-x_0)^{n-2} + \cdots \\
f'''(x) &= 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(x-x0) + \cdots + n(n-1)(n-2)a_n(x-x_0)^{n-3} + \cdots
\end{aligned}$$

Of course, all these identities are valid only on the interval $(-R, R)$ (where R is the radius of convergence).

- At the point x_0 (the center of the series), we obtain the following equalities:

$$\begin{aligned}
f(x_0) &= a_0 \\
f'(x_0) &= a_1 \\
f''(x_0) &= 2a_2 \\
f'''(x_0) &= 3\cdot 2a_3 \\
\vdots &= \vdots \\
f^{(k)}(x_0) &= k!a_k
\end{aligned}$$

- The conclusion of the last calculation is that the coefficients of the series can be expressed in terms of the values of the derivatives $f^{(k)}(x)$ at the point x_0.

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

Theorem 5.8: (Higher derivatives of a power series)
Let $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a power series with radius of convergence R, $R > 0$. Then its sum $f(x)$ has derivatives of every order at every point $|x| < R$. The derivative of order k of $f(x)$ can be obtained by differentiating the series term by term k times. In addition, the following relation holds between the derivative of order k and the coefficient a_k of the series:

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

5.1.4 Expanding a Function in a Power Series

Theorem 5.9: (Uniqueness of a power series)
If two power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ g(x) &= \sum_{n=0}^{\infty} b_n (x - x_0)^n \end{aligned}$$

satisfy $f(x) = g(x)$ for every x in an open interval $(x_0 - r, x_0 + r)$, where $r > 0$, then, for every natural number n, $a_n = b_n$.

- The preceding theorem means that if a function $f(x)$ can be represented by a power series about a point x_0 at all, then there is only one way to do so!
- The question that interested mathematicians such as Taylor and Maclaurin was this: Given a function defined on an interval $(x_0 - r, x_0 + r)$, does there exist a corresponding power series whose sum equals $f(x)$ for every x in this interval?

Definition 5.2: Let $f(x)$ be a function defined in a neighborhood of a point x_0. That is, suppose there exists $0 < r \leq \infty$ such that the function $f(x)$ is defined on the interval $(x_0 - r, x_0 + r)$. We say that $f(x)$ can be expanded in a power series on the interval $(x_0 - r, x_0 + r)$ if there exists a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ such that

$$\forall x \in (x_0 - r, x_0 + r): \quad f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

- The question is: Under what conditions can a function be expanded in a power series?
- We begin with necessary conditions that a function must satisfy in order to have a power-series expansion.

Theorem 5.10: (A necessary condition for the existence of a power series)

If a function $f(x)$ can be expanded in a power series about a point x_0, then

(a) The domain of $f(x)$ must be symmetric about x_0 (apart from the endpoints of the interval).

(b) The function $f(x)$ has continuous derivatives of every order at every interior point of its domain.

- The conditions above are necessary, but not sufficient! The best-known counterexample is the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f(x)$ is continuously differentiable on the entire real line, and satisfies $f^{(n)}(0) = 0$ for every order of differentiation n. Therefore, if it has a power-series expansion, then by the uniqueness theorem it must be a power series all of whose coefficients are zero, which is clearly absurd.

- To answer the existence question, we return to Taylor's formula, familiar from Calculus 1:

Theorem 5.11: (Taylor's formula)

Let $f(x)$ be a function differentiable $n+1$ times in a neighborhood of the point $x = x_0$, and let x be any point in this neighborhood. Then there exists a point c between x_0 and-x such that-

$$f(x) = f(x_0) + \tfrac{f'(x_0)}{1!}(x-x_0) + \tfrac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \tfrac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

is called the **Lagrange remainder formula**.

- Having recalled and reconsidered Taylor's formula from Calculus 1, we can formulate necessary and sufficient conditions for the existence of a power series.

Theorem 5.12: (Existence of a power series)

Let $f(x)$ be a function that is infinitely differentiable in a neighborhood $(x_0 - r, x_0 + r)$ of the point x_0. For every natural number n, let $R_n(x)$ denote the Lagrange remainder corresponding to the Taylor formula for $f(x)$ about x_0. Then $f(x)$ can be expanded in a power series in a neighborhood of x_0 if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for every x in the neighborhood $(x_0 - r, x_0 + r)$.

5.1.5 Arithmetic of Power Series

Shifting a summation index

- The summation index n is shifted according to the following simple rule: When replacing the index n by a new index $n+k$, subtract k from the limits of summation.
- For example, the series $\sum_{n=17}^{100} a_n x^n$ is equivalent to the series $\sum_{n=12}^{95} a_{n+5} x^{n+5}$. In this case:
  - The summation index is n.
  - The limits of summation are 17, 100.
  - The shift is $k=5$.
  - The series $\sum_{n=17}^{100} a_n x^n$ is equivalent to the series $\sum_{n=22}^{105} a_{n-5} x^{n-5}$. The shift here is $k=-5$.
- Usually, the upper limit of summation is ∞, and shifting has no effect on it. Thus, for example, the series $\sum_{n=17}^{\infty} a_n x^n$ is equivalent to the series $\sum_{n=12}^{\infty} a_{n+5} x^{n+5}$.
- In most cases, we want a standard power series in which the index n corresponds to the power $(x-x_0)^n$. In other cases, we want to ”align the powers“ of two different series.
- For example, if $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, then differentiation eliminates the first term, and therefore an index shift $k=-1$ must be made:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-x_0)^n$$

- The second derivative eliminates the first two terms of the series, and therefore a shift $k=2$ must be made:

$$f''(x) = \sum_{n=2}^{\infty} (n-1) n a_n (x-x_0)^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-x_0)^n$$

Multiplying a power series by an expression x^k

- By appropriate theorems from the theory of infinite series, the following operation is valid within the interval of convergence of the series:

$$x^k \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+k}$$

  One consequence of such multiplication is that the power of the nth term increases, and it is therefore advisable to ”align“ the series by shifting the index.

- **Example 4:**

$$x^3\sum_{n=0}^{\infty}(n+8)a_nx^n=\sum_{n=0}^{\infty}(n+8)a_nx^{n+3}=\sum_{n=3}^{\infty}(n+5)a_nx^n$$

An index shift $k=-3$ was performed.

Addition and subtraction of power series

- By appropriate theorems from the theory of infinite series, the following operations are valid within the **common** interval of convergence of the series involved:

$$\begin{aligned}\sum_{n=0}^{\infty}a_n(x-x_0)^n+\sum_{n=0}^{\infty}b_n(x-x_0)^n &= \sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^{n+k}\\ \sum_{n=0}^{\infty}a_n(x-x_0)^n-\sum_{n=0}^{\infty}b_n(x-x_0)^n &= \sum_{n=0}^{\infty}(a_n-b_n)(x-x_0)^{n+k}\end{aligned}$$

- A typical example combining all the operations above, which we will encounter in various exercises in solving differential equations, is

$$\begin{aligned}&x^2\sum_{n=3}^{\infty}(n-2)x^{n+1}+\sum_{n=2}^{\infty}n^2x^{n+1}\\ &=\sum_{n=3}^{\infty}(n-2)x^{n+3}+\sum_{n=2}^{\infty}n^2x^{n+1}\\ &=\sum_{n=3}^{\infty}(n-2)x^{n+3}+\sum_{n=0}^{\infty}(n+2)^2x^{n+3}\\ &=4x^3+9x^4+16x^5+\sum_{n=3}^{\infty}[(n+2)^2+(n-2)]x^{n+3}\\ &=4x^3+9x^4+16x^5+\sum_{n=6}^{\infty}[(n-1)^2+(n-5)]x^n\\ &=4x^3+9x^4+16x^5+\sum_{n=6}^{\infty}(n^2-n-4)x^n\end{aligned}$$

5.2 Solving Differential Equations by Power Series

In general, given an initial-value problem

$$\begin{cases}y'=f(x,y)\\ y(x_0)=y_0\end{cases}$$

we seek a solution of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. The problem, of course, is to determine the coefficients a_n for every natural number n.

- We begin with a simple example.

$$\begin{cases} y' = 2xy \\ y(0) = 1 \end{cases}$$

The form of our series is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

By a theorem from the preceding section, the particular solution is

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

Clearly, $a_0 = y(0) = 1$. Let us try to compute a_1:

$$a_1 = y'(0) = 2 \cdot 0 \cdot y(0) = 0$$

In the same way, we can also compute a_2. For this purpose, differentiate the equation with respect to x to obtain

$$y'' = 2y + 2xy'$$

and therefore

$$y''(0) = 2y(0) + 2 \cdot 0 \cdot y'(0) = 2$$

and therefore

$$a_2 = \frac{y''(0)}{2!} = 1$$

Clearly, we could continue in this manner and compute all the derivatives $y^{(n)}(0)$, but a general formula does not appear to be obtainable by this method.

- Let us substitute our series into the equation:

$$\begin{aligned} y' &= \left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ 2xy &= 2x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2a_n x^{n+1} = \sum_{n=1}^{\infty} 2a_{n-1} x^n \end{aligned}$$

The equality $y' = 2xy$ implies equality of the series, and hence equality of their coefficients:

$$(n+1)a_{n+1} = 2a_{n-1}$$

This is a recurrence relation (recursion) for the sequence of coefficients:

$$a_{n+2} = \frac{2a_n}{n+2}$$

We have already found that

$$a_0 = 1, \qquad a_1 = 0$$

and therefore

$$\begin{aligned}
a_0 &= \tfrac{1}{1} \\
a_1 &= 0 \\
a_2 &= \tfrac{2a_0}{2} = \tfrac{2}{2} \\
a_3 &= \tfrac{2a_1}{3} = 0 \\
a_4 &= \tfrac{2a_2}{4} = \tfrac{2\cdot 2}{2\cdot 4} \\
a_5 &= \tfrac{2a_3}{5} = 0 \\
a_6 &= \tfrac{2a_4}{6} = \tfrac{2\cdot 2\cdot 2}{2\cdot 4\cdot 6} \\
a_7 &= \tfrac{2a_5}{7} = 0 \\
a_8 &= \tfrac{2a_6}{8} = \tfrac{2\cdot 2\cdot 2\cdot 2}{2\cdot 4\cdot 6\cdot 8}
\end{aligned}$$

After writing about ten terms, the formula for the general coefficient of the series begins to emerge. For $n = 2k$, the coefficient formula is

$$a_n = \frac{2^k}{2\cdot 4\cdot 6\cdots 2k} = \frac{2^k}{2^k k!} = \frac{1}{k!} = \frac{1}{\frac{n}{2}!}$$

For $n = 2k+1$, the coefficient formula is

$$a_{2k+1} = 0$$

Therefore, we may write

$$a_n = \begin{cases} \dfrac{1}{k!}, & n = 2k \\ 0, & n = 2k+1 \end{cases}$$

Our power series has the form

$$y(x) = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$$

- In this particular case, the equation $y' = 2xy$ is an easy separable equation whose general solution is

$$y(x) = Ce^{x^2}$$

The particular solution satisfying the initial condition $y(0) = 1$ is $y(x) = e^{x^2}$. The Maclaurin series of e^x is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Replacing x by x^2 in the last series gives the result obtained above, thereby verifying it in this case. In general, we must rely on theorems of the following type to ensure the existence of a solution in the form of a power series. The theorem is stated for a differential equation of order 2, but analogous versions exist for every order.

Theorem 5.13: Consider the initial-value problem

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ y(x_0) = \alpha \\ y'(x_0) = \beta \end{cases}$$

If the functions $p(x)$ and $q(x)$ can be expanded in power series about the point x_0 on an open interval $(x_0 - R, x_0 + R)$, then there exists a unique solution of the problem that can also be expanded in a power series on the same interval.

Example 5: The Airy equation

Solve the initial-value problem

$$\begin{cases} y'' - xy = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Solution: In the present case, $p(x) = 0$, $q(x) = x$, and it is clear that these two functions are already in the form of (finite) power series. We seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

We compute the components of the equation:

$$\begin{aligned} y'' &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n \\ xy &= \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n \end{aligned}$$

Therefore,

$$y'' - xy = 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - a_{n-1}]x^n = 0$$

We have obtained the recurrence relation

$$\begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \\ a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots \end{cases}$$

It can be written in the more convenient form

$$\begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \\ a_{n+3} = \dfrac{a_n}{(n+3)(n+2)}, \quad n = 0, 1, 2, \ldots \end{cases}$$

Clearly, all coefficients of the forms a_{3k+1} and a_{3k+2} vanish. It remains to compute the coefficients of the form a_{3k}.

$$\begin{aligned} a_0 &= \frac{1}{1} \\ a_3 &= \frac{a_0}{3 \cdot 2} = \frac{1}{3 \cdot 2} \\ a_6 &= \frac{a_3}{6 \cdot 5} = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} \\ a_9 &= \frac{a_6}{9 \cdot 8} = \frac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} \end{aligned}$$

The particular solution of the problem is therefore

$$\begin{aligned} y(x) &= 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdots (3k-1) \cdot 3k} \end{aligned}$$

Example 6: General solution of the Airy equation

In the preceding example, we solved an initial-value problem, but sometimes we are asked to find a general solution of the equation

$$y'' - xy = 0$$

By the calculation performed in the preceding example,

$$\begin{cases} a_0 = \langle\text{free coefficient}\rangle \\ a_1 = \langle\text{free coefficient}\rangle \\ a_2 = 0 \\ a_{n+3} = \dfrac{a_n}{(n+3)(n+2)}, \quad n = 0, 1, 2, \ldots \end{cases}$$

We divide the sequence of coefficients into the subsequences a_{3k}, a_{3k+1}, and a_{3k+2}. The general form of the sequence of coefficients a_{3k} was already found in the preceding example:

$$\begin{aligned} a_0 &= \langle\text{free coefficient}\rangle \\ a_3 &= \frac{a_0}{2\cdot 3} \\ a_6 &= \frac{a_3}{6\cdot 5} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6} \\ a_9 &= \frac{a_6}{9\cdot 8} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} \\ &\vdots \\ a_{3k} &= \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3k-1)\cdot 3k}, \quad k = 1, 2, \ldots \end{aligned}$$

In the same way, we find the general formula for the sequence of coefficients a_{3k+1}:

$$\begin{aligned} a_1 &= \langle\text{free coefficient}\rangle \\ a_4 &= \frac{a_1}{3\cdot 4} \\ a_7 &= \frac{a_4}{7\cdot 6} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7} \\ a_{10} &= \frac{a_7}{10\cdot 9} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} \\ &\vdots \\ a_{3k+1} &= \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdots 3k\cdot(3k+1)}, \quad k = 1, 2, \ldots \end{aligned}$$

Since $a_2 = 0$, for every k, $a_{3k+2} = 0$. The general solution of the Airy equation is therefore

$$\begin{aligned} y(x) &= a_0 + \sum_{k=1}^{\infty} a_0 \frac{x^{3k}}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3k-1)\cdot 3k} + a_1 x + \sum_{k=1}^{\infty} a_1 \frac{x^{3k+1}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdots 3k\cdot(3k+1)} \\ &= a_0\left(1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3k-1)\cdot 3k}\right) + a_1\left(x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdots 3k\cdot(3k+1)}\right) \end{aligned}$$

Set

$$\begin{aligned} y_1(x) &= 1+\sum_{k=1}^{\infty}\frac{x^{3k}}{2\cdot3\cdot5\cdot6\cdot8\cdot9\cdots(3k-1)\cdot3k} \\ y_2(x) &= x+\sum_{k=1}^{\infty}\frac{x^{3k+1}}{3\cdot4\cdot6\cdot7\cdot9\cdot10\cdots3k\cdot(3k+1)} \end{aligned}$$

Therefore, the general solution we obtained is

$$y(x) = a_0y_1(x) + a_1y_2(x)$$

It is not difficult to see that

$$\begin{aligned} y_1(0) &= 1, \qquad y_2(0) = 0 \\ y_1'(0) &= 0, \qquad y_2'(0) = 1 \end{aligned}$$

and hence the Wronskian of y_1 and y_2 is nonzero:

$$W[y_1, y_2](0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Therefore, the set of solutions $\{y_1(x),\ y_2(x)\}$ is a fundamental set of solutions, and consequently the solution we obtained is indeed the general solution of the Airy equation.

Exercise 5.2: Solve the initial-value problem

$$\begin{cases} y'' - xy = 0 \\ y(0) = 3 \\ y'(0) = -2 \end{cases}$$

Solution: Of course, we seek a solution in the form of a power series.

$$y(x) = \sum_{n=0}^{\infty} a_nx^n$$

Clearly,

$$\begin{aligned} a_0 &= y(0) = 3 \\ a_1 &= y'(0) = -2 \end{aligned}$$

Therefore, by the preceding result, the particular solution is

$$\begin{aligned} y(x) &= a_0\left(1+\sum_{k=1}^{\infty}\frac{x^{3k}}{2\cdot3\cdot5\cdot6\cdot8\cdot9\cdots(3k-1)\cdot3k}\right) + a_1\left(x+\sum_{k=1}^{\infty}\frac{x^{3k+1}}{3\cdot4\cdot6\cdot7\cdot9\cdot10\cdots3k\cdot(3k+1)}\right) \\ &= 3-2x+\sum_{k=1}^{\infty}\frac{3x^{3k}}{2\cdot3\cdot5\cdot6\cdot8\cdot9\cdots(3k-1)\cdot3k} - \sum_{k=1}^{\infty}\frac{2x^{3k+1}}{3\cdot4\cdot6\cdot7\cdot9\cdot10\cdots3k\cdot(3k+1)} \end{aligned}$$

Remark: If one really wishes, the coefficients can be expressed by more "professional" formulas such as

$$\begin{aligned} a_{3k} &= \frac{a_0}{(3k)!}\prod_{i=0}^{k-1}(3i+1) \\ a_{3k+1} &= \frac{a_1}{(3k+1)!}\prod_{i=0}^{k-1}(3i+2) \end{aligned}$$

Exercise 5.3: The Airy equation about the point $x_0 = 1$:
Find a polynomial approximation of degree 7 for the initial-value problem

$$\begin{cases} y'' - xy = 0 \\ y(1) = 0 \\ y'(1) = 1 \end{cases}$$

Solution: This time, we must find the first eight terms of a power series about the point $x_0 = 1$.

$$y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$$

We compute the second derivative:

$$y''(x) = \sum_{n=2}^{\infty}(n-1)na_n(x-1)^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n$$

Computing the product xy requires a "small maneuver":

$$\begin{aligned} xy &= [1+(x-1)]y \\ &= \sum_{n=0}^{\infty} a_n(x-1)^n + (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n \\ &= \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1} \\ &= \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=1}^{\infty} a_{n-1}(x-1)^n \\ &= a_0 + \sum_{n=1}^{\infty}(a_n + a_{n-1})(x-1)^n \end{aligned}$$

Substituting the last two results into the Airy equation, we obtain

$$\begin{aligned} y'' - xy &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n - \left(a_0 + \sum_{n=1}^{\infty}(a_n + a_{n-1})(x-1)^n\right) \\ &= (2a_2 - a_0) + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} - a_n - a_{n-1}\right](x-1)^n \\ &= 0 \end{aligned}$$

We have obtained the recurrence relation

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = 0 \\ a_{n+2} = \dfrac{a_n + a_{n-1}}{(n+1)(n+2)}, \quad n = 1, 2, \ldots \end{cases}$$

The first two equalities follow from the initial conditions.

$$\begin{aligned} a_0 &= y(1) = 0 \\ a_1 &= y'(1) = 1 \end{aligned}$$

The third equality follows from $2a_2 - a_0 = 0$.

We can write the recurrence relation in the more convenient form

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = 0 \\ a_{n+3} = \dfrac{a_n + a_{n+1}}{(n+2)(n+3)}, \quad n = 0, 1, 2, \ldots \end{cases}$$

Unlike the preceding exercise, the coefficients cannot be separated into the subsequences a_{3k}, a_{3k+1}, and a_{3k+2}, because of the dependence on two preceding terms that are not of the same type.

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 0 \\ a_3 &= \frac{a_0 + a_1}{2 \cdot 3} = \frac{1}{2 \cdot 3} \\ a_4 &= \frac{a_1 + a_2}{3 \cdot 4} = \frac{1}{3 \cdot 4} \\ a_5 &= \frac{a_2 + a_3}{4 \cdot 5} = \frac{0 + \frac{1}{2 \cdot 3}}{4 \cdot 5} = \frac{1}{5!} \\ a_6 &= \frac{a_3 + a_4}{5 \cdot 6} = \frac{\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4}}{5 \cdot 6} = \frac{1}{5!} \\ a_7 &= \frac{a_4 + a_5}{6 \cdot 7} = \frac{\frac{1}{3 \cdot 4} + \frac{1}{5!}}{6 \cdot 7} = \frac{11}{7!} \end{aligned}$$

The polynomial approximation of degree 7 to the solution is therefore

$$y(x) = x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{120} + \frac{x^6}{120} + \frac{11x^7}{5040}$$

Exercise 5.4: The Hermite equation

An equation of the form

$$y'' - 2xy' + \lambda y = 0$$

where $\lambda = 0, 2, 4, \ldots$, is called the (Hermite) equation.

Solve the initial-value problem

$$\begin{cases} y'' - 2xy' + 6y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Solution: We seek a solution in the form of a power series:

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y''(x) &= \sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \\
-2xy' &= \sum_{n=1}^{\infty} - 2n a_n x^n
\end{aligned}$$

Therefore,

$$\begin{aligned}
y'' - 2xy' + 6y &= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=1}^{\infty} - 2n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n \\
&= 2a_2 + 6a_0 + \sum_{n=1}^{\infty} [(n+1)(n+2) a_{n+2} - 2n a_n + 6 a_n] x^n \\
&= 0
\end{aligned}$$

We obtain the recurrence relation

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = 0 \\ a_{n+2} = \dfrac{(2n-6) a_n}{(n+1)(n+2)}, \quad n = 1, 2, 3, \ldots \end{cases}$$

The first two equalities follow from the initial conditions.

$$\begin{aligned}
a_0 &= y(0) = 0 \\
a_1 &= y'(0) = 1
\end{aligned}$$

The third equality follows from $2a_2 + 6a_0 = 0$.

Let us examine the first coefficients of the series:

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 0 \\ a_3 &= \frac{(2\cdot 1-6)a_1}{2\cdot 3} = -\frac{2}{3} \\ a_4 &= \frac{(2\cdot 2-6)a_2}{3\cdot 4} = 0 \\ a_5 &= \frac{(2\cdot 3-6)a_3}{4\cdot 5} = 0 \end{aligned}$$

Clearly, all the coefficients vanish beginning with $n = 4$, and hence we have in fact obtained a finite power series (what we usually call a polynomial).

$$y(x) = x - \frac{2x^3}{3}$$

A simple verification shows that this is indeed the solution of our initial-value problem.

Exercise 5.5: A nonhomogeneous Hermite equation

Solve the following initial-value problem: Solve the initial-value problem

$$\begin{cases} y'' - 2xy' + 6y = 12 - 16x \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$$

Solution: The power-series method is also applicable to nonhomogeneous equations.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

We use the results obtained in the preceding exercise:

$$\begin{aligned} y'' - 2xy' + 6y &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} - 2na_n x^n + \sum_{n=0}^{\infty} 6a_n x^n \\ &= 2a_2 + 6a_0 + \sum_{n=1}^{\infty}[(n+1)(n+2)a_{n+2} - 2na_n + 6a_n]x^n \\ &= 12 - 16x \end{aligned}$$

By the initial conditions,

$$\begin{aligned} a_0 &= y(0) = 2 \\ a_1 &= y'(0) = -1 \end{aligned}$$

Equating coefficients also gives

$$\begin{aligned} 2a_2 + 6a_0 &= 12 \\ 6a_3 + 4a_1 &= -16 \end{aligned}$$

Therefore, our recurrence relation is

$$\begin{cases} a_0 = 2 \\ a_1 = -1 \\ a_2 = 0 \\ a_3 = -2 \\ a_{n+2} = \dfrac{(2n-6)a_n}{(n+1)(n+2)}, \quad n = 1, 2, 3, \ldots \end{cases}$$

As in the preceding exercise, it is easy to verify that all the remaining coefficients of the series vanish, and hence we have again obtained a solution that is in fact a polynomial:

$$y(x) = 2 - x - 2x^3$$

It is easy to verify that this is the solution of our initial-value problem. We conclude the present chapter with the following theorem.

Theorem 5.14: Let x_0 be a **regular point** of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

that is, $P(x_0) \neq 0$. If $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$ are analytic functions at the point x_0, then the general solution of the equation can be represented as

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary constants, and $y_1(x)$ and $y_2(x)$ are two linearly independent power-series solutions of the equation. The radius of convergence of all the solutions is the smaller of the radii of convergence of the power series of $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$.

Remark: A function $f(x)$ over the complex field $\mathbb{C}$ is called analytic at the point x_0 if it has a power-series expansion in a neighborhood of x_0 with a positive radius of convergence.

Exercises for chapter 5

Guidance: A link at the end of the list leads to a Jupyter notebook containing final and/or complete solutions to a small number of the exercises. Before clicking the link, however, it is strongly recommended that you solve them on your own! Additional solutions will be added over time. We would be very happy to receive additional solutions from students for exercises that have not yet been solved!

1. Find a solution in the form of a power series for each of the following problems:

(a) $(1+x^2)y'' + x^3 y = 0$

(b) $\begin{cases} y'' - 2x^2 y' + 4xy = x^2 + 2x + 2 \\ y(0) = 3 \\ y'(0) = 12 \end{cases}$

(c) $\begin{cases} (1-x^2)y'' - 2xy' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

(d) $(x^2+4)y'' + xy = x + 2$

(e) $\begin{cases} y'' + xy' + (2x-1)y = 0 \\ y(-1) = 2 \\ y'(-1) = -2 \end{cases}$

(f) $\begin{cases} y'' - 2xy' + 8y = 1 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

(g) $\begin{cases} (1-x^2)y'' + 12y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

(h) $\begin{cases} y'' - (x-1)y' - y = 0 \\ y(1) = 1 \\ y'(1) = 0 \end{cases}$

2. Consider the differential equation $y'' - xy' + 5y = x + 1$

(a) Find a general solution in the form of a power series.
Write all the terms of the series through the sixth power (x^6).

(b) Find a complete particular solution satisfying $y(0) = 0$, $y'(0) = 3$.

(c) What is the sum of the coefficients $a_3 + a_5$ when $y(0) = 0$, $y'(0) = 3$?

3. Let $y(x)$ be the solution of the initial-value problem

$$\begin{cases} y'' - 2xy' + 10y = 0 \\ y(0) = 0 \\ y'(0) = 3 \end{cases}$$

Compute $y(\frac{1}{2})$.

4. Let $y(x)$ be the solution of the initial-value problem

$$\begin{cases} y'' - xy' - y = 0 \\ y(1) = 0 \\ y'(1) = 1 \end{cases}$$

Compute the power-series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ through the seventh power $(x-1)^7$.

5. Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be the solution of the initial-value problem

$$\begin{cases} y'' + x^4 y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Determine whether each of the following assertions is true or false:

(a) $a_1 = 0$
(b) $a_n = \frac{(-1)^n}{n! 6^n}$
(c) The radius of convergence of the series is $R = 1$.
(d) $a_1 = a_2 = a_3 = a_4 = a_5 = 0$
(e) $a_6 = \frac{1}{30}$
(f) $a_7 = \frac{1}{210}$
(g) $y(x) - 1$ is an odd function.

6. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series that solves the initial-value problem

$$\begin{cases} y'' - 2(x^2+1)y' - 6xy = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Find the recurrence relation for the series, and find the first four nonzero terms of the series.

7. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series that solves the initial-value problem

$$\begin{cases} (10 - x^3)y'' + 5xy = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

Compute the coefficients a_3, a_4, a_5.

8. Let $\sum_{n=0}^{\infty} a_n (x-1)^n$ be a power series that solves the initial-value problem

$$\begin{cases} (x^2 - 2x)y'' - 2y = 0 \\ y(1) = 0 \\ y'(1) = -1 \end{cases}$$

Find the recurrence relation for the coefficient a_n.

9. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series that solves the initial-value problem $\begin{cases} y'' - 2xy' + my = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$.

Find the values of m for which the solution of the equation is a polynomial of finite degree, and determine the degree of the corresponding polynomial for each such value of m.

10. Construct a solution in the form of a power series about the point $x = -2$ for the equation $y'' + xy = 0$. Write the solutions for the following initial conditions through the fourth power:

(a) $y(-2) = 0$, $y'(-2) = 1$

(b) $y(-2) = 1$, $y'(-2) = 0$

Link to solutions

Chapter 6

The Laplace Transform

Fig. 6.1: Pierre-Simon, marquis de Laplace 1749-1827

The Laplace transform is an advanced and ingenious mathematical tool that helps solve a variety of mathematical problems, including differential equations to which it applies. In this part of the course, we study the basic properties of the Laplace transform and use it to solve various types of ordinary differential equations. The theorems and examples in this summary are based on the following two sources:

1. Elementary Differential Equations and Boundary Value Problems, Boyce and Diprima
2. Elementary Differential Equations with Linear Algebra, Ross L. Finney and Donald R. Ostberg

6.1 What Is the Laplace Transform?

- The Laplace transform is a special case of the broader concept of ***integral transforms***, studied in advanced courses on functional analysis. Another familiar transform is, for example, the ***Fourier transform***, which is studied in a special course on Fourier series.
- An integral transform is essentially a method for converting a real-valued function $f(t)$ (satisfying the required conditions) into another real-valued function $F(s)$. The general form of an integral transform is
$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t)\, dt$$
The function $K(s,t)$ is called the kernel of the transform, and the limits of integration α, β must be defined accordingly.
- In many cases, one or both limits are infinite ($\alpha = -\infty$ and/or $\beta = \infty$).

- The new function $F(x)$ is called the ***transform*** of $f(x)$.
- The idea underlying the transform method is:
  1. Apply the transform to the given problem in the unknown $f(t)$ (a differential equation, an integral equation, or another type of problem).
  2. Obtain a new equation for the function $F(s)$ that is easier to solve.
  3. Solve the new problem for the unknown $F(s)$.
  4. Return to the original function $f(t)$ by applying the ***inverse transform*** to $F(s)$.

Example 1: For example, let us see what a typical solution of an initial-value problem using the Laplace transform looks like (the required formulas will be developed later).

$$\begin{cases} y'' - y = 1 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Apply the Laplace transform to both sides of the equation:

$$\mathscr{L}[y''] - \mathscr{L}[y] = \mathscr{L}[1]$$

Later, we will prove the following three formulas:

$$\begin{aligned} \mathscr{L}[y''] &= s^2\mathscr{L}[y] - sy(0) - y'(0) \\ \mathscr{L}[1] &= \frac{1}{s} \\ \mathscr{L}\left[e^{at}\right] &= \frac{1}{s-a} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathscr{L}[y''] - \mathscr{L}[y] &= s^2\mathscr{L}[y] - sy(0) - sy'0) - \mathscr{L}[y] \\ &= s^2\mathscr{L}[y] - 0 - 1 - \mathscr{L}[y] \\ &= (s^2-1)\mathscr{L}[y] - 1 \\ &= \frac{1}{s} \end{aligned}$$

We have obtained

$$\mathscr{L}[y(t)] = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s}$$

and therefore

$$y = \mathscr{L}^{-1}\left[\frac{1}{s-1}\right] - \mathscr{L}^{-1}\left[\frac{1}{s}\right] = e^t - 1$$

- This process will become clear later, when we solve differential equations using the Laplace transform. For now, patience is required.
- We begin by describing the properties required of a function in order for it to have a Laplace transform. The first property is called "piecewise continuity" and guarantees integrability.

Definition 6.1: A function $f : [a, b] \to \mathbb{R}$ is called ***piecewise continuous*** on the interval $[a, b]$ if

(a) $f(t)$ is continuous on the interval $[a, b]$ except possibly at finitely many points:

$$t_1, \quad t_2, \quad \dots, \quad t_n$$

(b) At every point of discontinuity, the left- and right-hand limits exist and are etinfi.

The points of discontinuity t_1, t_2, $\dots$, t_n are also called jump discontinuities.

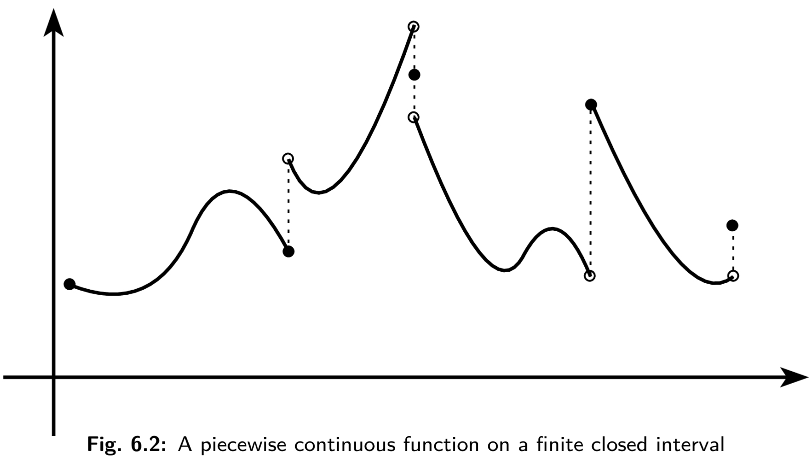

Fig. 6.2: A piecewise continuous function on a finite closed interval

Definition 6.2: A function $f : [a, \infty] \to \mathbb{R}$ is called ***piecewise continuous*** on the ray $[a, \infty]$ if $f(t)$ is piecewise continuous on every closed interval $[a, b]$, where $a < b < \infty$.

Definition 6.3: (The Laplace transform)

Let $f : [0, \infty] \to \mathbb{R}$ be a piecewise continuous function on the ray $[0, \infty]$. The Laplace transform, denoted by $F(s)$ or $\mathscr{L}[f](s)$, is defined by the improper integral

$$\mathscr{L}[f](s) = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

- The kernel of the Laplace transform is

$$K(s, t) = e^{-st}$$

- The definition does not guarantee the existence of the function $F(s)$. Since this is an improper integral, we must verify that the function $f(t)$ satisfies sufficient conditions to guarantee its convergence.
- Understanding the Laplace transform therefore requires a review of improper integrals. Calculus 1 textbooks cover this topic at a level sufficient for our purposes. We recall several concepts and theorems that we need.

Definition 6.4: If the limit $I = \lim_{b\to\infty} \int_a^b f(t)\,dt$ exists, then I is called ***the improper integral*** of $f(t)$ on the interval $[a, \infty)$, and we denote it by

$$I = \int_a^\infty f(t)dt$$

We then say that the improper integral of $f(t)$ on the interval $[a, \infty)$ ***exists*** or ***converges***. Otherwise, we say that the integral ***does not exist*** or ***diverges***. Equivalently, we may write

$$\int_a^\infty f(t)dt = \lim_{b\to\infty} \int_a^b f(t)dt$$

- All functions considered in this chapter are piecewise continuous and therefore integrable on every closed interval on which they are defined.
- Integrability on a half-line $[a, \infty)$ requires an additional property, which we describe below.

Definition 6.5: Let $f(t)$ be a function integrable on the interval $[a, b]$ for every $b > a$, where a is a fixed point. We say that $f(t)$ is ***absolutely integrable*** on the interval $[a, \infty)$ if the improper integral $\int_a^\infty |f(t)|dt$ converges. We also say that the integral $\int_a^\infty f(t)dt$ ***converges absolutely***.

Theorem: ***(Calculus 1, Chapter 11, Theorem 2)***
If $f(t)$ is absolutely integrable on the interval $[a, \infty)$, then it is integrable on this interval.

Theorem: (***The comparison test, Calculus 1, Chapter 11, Theorem 4***)
Let $f(t)$ and-$g(t)$ be two nonnegative functions on the interval $[a, \infty)$, integrable on the interval $[a, b]$ for every $b > a$. Suppose that there exists a real number b_0 such that, for every $t \geq b_0$, $f(t) \leq g(t)$. Then

(a) If the integral $\int_a^\infty g(t)dt$ converges, then the integral $\int_a^\infty f(t)dt$ also converges.

(b) If the integral $\int_a^\infty f(t)dt$ diverges, then the integral $\int_a^\infty g(t)dt$ also diverges.

Theorem 6.1: Let $f(t)$ be a piecewise continuous function on the ray $[a, \infty)$. If there exists a point c such that $|f(t)| \leq g(t)$ for every $t \geq c$, and if $\int_c^\infty g(t)dt$ exists, then the following improper integral also exists: $\int_a^\infty f(t)dt$.

If, on the other hand, $f(t) \geq g(t)$ for every $t \geq c$, and if $\int_c^\infty g(t)dt$ diverges, then $\int_a^\infty f(t)dt$ also diverges.

Proof: The proof follows immediately from the comparison test for improper integrals (Calculus 1, Chapter 11, Theorem 4) and the theorem on absolute integrability (Calculus 1, Chapter 11, Theorem 2). ∎

Definition 6.6: We say that a function $f : [0, \infty) \to \mathbb{R}$ is ***of exponential order*** if there exist evitisop constants K, c, and a such that, for every $t \geq c$, $|f(t)| \leq Ke^{at}$.

- In other words, a function $f(t)$ is of exponential order if it is bounded in absolute value by an exponential function $g(t) = Ke^{at}$ on the ray $[c, \infty)$.
- **Exercise 6.1:** Prove that the function $f(t) = t^n e^{at} \sin bt$ is of exponential order for every natural number n, real number a, and real number b.
  Solution: Prove that $f(t)$ is exponentially bounded by e^{2at}.
$$\left|\frac{t^n e^{at} \sin bt}{e^{2at}}\right| \leq \frac{t^n}{e^{at}}$$
  It is easy to prove by L'Hopital's rule that
$$\lim_{t\to\infty} \frac{t^n}{e^{at}} = 0$$
- It turns out that most functions encountered in practical problems are of this type.
- **Exercise 6.2:** Prove that the function $f(t) = e^{t^2}$ is not of exponential order.
  Guidance: Show that for every K, a,
$$\lim_{t\to\infty} \frac{Ke^{at}}{e^{t^2}} = 0$$

- The following theorem states that if a function is of exponential order, then it has a Laplace transform. Thus, most functions arising in practical applications have a Laplace transform.

Theorem 6.2: Let $f(t)$ be a piecewise continuous function on the ray $[0, \infty)$. If $f(t)$ is of exponential order, then the Laplace transform of $f(t)$ exists for every $s > a$, where a is the constant in Definition 6: $|f(t)| \leq Ke^{at}$.

Proof: We must prove that for every $s > a$, the improper integral $\int_0^\infty e^{-st} f(t) dt$ converges. We use Theorem 1. For every $t \geq c$,

$$\left|e^{-st} f(t)\right| \leq e^{-st} \cdot Ke^{at} = Ke^{(a-s)t}$$

The assumption $s > a$ implies that $a - s < 0$, and hence the integral $\int_c^\infty e^{(a-s)t} f(t) dt$ converges (as is easily verified from the definition). Therefore, by Theorem 1, the improper integral $\int_c^\infty e^{-st} f(t) dt$ also converges.
The equality

$$\int_0^\infty e^{-st} f(t) dt = \int_0^c e^{-st} f(t) dt + \int_c^\infty e^{-st} f(t) dt$$

yields the conclusion of the theorem. ∎

- Theorem 2 implies that if $f(t)$ is exponentially bounded, then not only does the Laplace transform $\mathscr{L}[f]$ exist, but it is also defined on a half-line (a, ∞).
- In fact, more can be said. For every $s > a$,

$$|\mathscr{L}[f](s)| \leq \frac{K}{s-a}$$

This follows immediately from the inequality encountered in the proof of the theorem:

$$\left|e^{-st} f(t)\right| \leq Ke^{(a-s)t}$$

and therefore

$$\begin{aligned} |\mathscr{L}[f](s)| &= \left|\int_0^\infty e^{-st} f(t)\, dt\right| \\ &\leq \int_0^\infty \left|e^{-st} f(t)\right| dt \\ &\leq \int_0^\infty \left|Ke^{(a-s)t}\right| dt \\ &= \frac{K}{s-a} \end{aligned}$$

- **Corollary:** If $f(t)$ is a piecewise continuous function on the ray $[0, \infty)$ and is of exponential order, then

$$\lim_{s\to\infty} \mathscr{L}\,[f]\,(s) = 0$$

- This conclusion implies that functions such as $\frac{s}{s+1}$ cannot be Laplace transforms of piecewise continuous functions of exponential order. This fact will be significant when we reach the inverse transform.
- We now compute the Laplace transforms of several elementary functions.

6.2 Elementary Examples

- $f(t) = 1\,,\ t \geq 0$

$$\mathscr{L}\,[f] = \int_0^\infty e^{-st} dt = \frac{1}{s}, \quad s > 0$$

- $f(t) = e^{at}\,,\ t \geq 0$

$$\mathscr{L}\,[f] = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \quad s > a$$

- $f(t) = \cos at\,,\ t \geq 0$

$$\mathscr{L}\,[f]\,(s) = \int_0^\infty e^{-st} \cos at\, dt = \lim_{b\to\infty} \int_0^b e^{-st} \cos at\, dt$$

Set

$$I(b) = \int_0^b e^{-st} \cos at\, dt$$

We compute $I(b)$ by applying integration by parts twice:

$$\begin{aligned}
I(b) &= \int_0^b e^{-st} \cos at\, dt \\
&= \frac{1}{a} e^{-st} \sin at \Big|_0^b + \frac{s}{a} \int_0^b e^{-st} \sin at\, dt \\
&= \frac{1}{a} e^{-sb} \sin ab + \frac{s}{a} \left[-\frac{1}{a} e^{-st} \cos at \Big|_0^b - \frac{s}{a} \int_0^b e^{-st} \cos at\, dt \right] \\
&= \frac{1}{a} e^{-sb} \sin ab - \frac{s}{a^2} e^{-sb} \cos ab + \frac{s}{a^2} - \frac{s^2}{a^2} \int_0^b e^{-st} \cos at\, dt \\
&= \frac{1}{a} e^{-sb} \sin ab - \frac{s}{a^2} e^{-sb} \cos ab + \frac{s}{a^2} - \frac{s^2}{a^2} I(b)
\end{aligned}$$

We have obtained

$$\frac{a^2+s^2}{a^2}\cdot I(b)=\frac{1}{a}e^{-sb}\sin ab-\frac{s}{a^2}e^{-sb}\cos ab+\frac{s}{a^2}$$

and therefore

$$I(b)=\frac{ae^{-sb}\sin ab}{a^2+s^2}-\frac{se^{-sb}\cos ab}{a^2+s^2}+\frac{s}{a^2+s^2}$$

The first two terms on the right-hand side tend to zero as b tends to infinity for every $s>0$, and therefore, for every $s>0$,

$$\mathscr{L}\left[f\right](s)=\lim_{b\to\infty}I(b)=\frac{s}{a^2+s^2}$$

We can summarize the result by the following formula:

$$\mathscr{L}\left[\cos at\right]=\frac{s}{a^2+s^2}\qquad(s>0)$$

- Similarly, we prove the formula

$$\mathscr{L}\left[\sin at\right]=\frac{a}{a^2+s^2}\qquad(s>0)$$

- The following is a short list of formulas, which we will expand later:

$$\mathscr{L}\left[1\right]=\frac{1}{s}\qquad s>0\tag{6.1}$$

$$\mathscr{L}\left[e^{at}\right]=\frac{1}{s-a}\qquad s>a\tag{6.2}$$

$$\mathscr{L}\left[\sin at\right]=\frac{a}{s^2+a^2}\qquad s>0\tag{6.3}$$

$$\mathscr{L}\left[\cos at\right]=\frac{s}{s^2+a^2}\qquad s>0\tag{6.4}$$

$$\mathscr{L}\left[t^n\right]=\frac{n!}{s^{n+1}}\qquad s>0\tag{6.5}$$

6.3 The Inverse Laplace Transform $\mathscr{L}^{-1}$

The following theorem states that the Laplace transform is a one-to-one linear operator (in a certain sense), and therefore has an inverse transform.

Theorem 6.3: (Lerch)

If $f(t)$ and $g(t)$ are piecewise continuous functions of exponential order on the ray $[0, \infty)$, and if

$$\mathscr{L}[f] = \mathscr{L}[g]$$

then $f(t) = g(t)$ at every point of continuity $0 \leq t < \infty$.

- Lerch's theorem states that the linear operator $\mathscr{L}$ is one-to-one in the sense that two original functions may differ only at points of discontinuity (and the difference is therefore negligible).
- Thus, Lerch's theorem implies that the Laplace transform is invertible; the inverse operator is called the "inverse Laplace transform" and is denoted by $\mathscr{L}^{-1}$.
- For example, we may write the following formulas:

$$\mathscr{L}^{-1}\left[\frac{1}{s}\right] = 1 \tag{6.6}$$

$$\mathscr{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at} \tag{6.7}$$

$$\mathscr{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at \tag{6.8}$$

$$\mathscr{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at \tag{6.9}$$

$$\mathscr{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n \tag{6.10}$$

6.4 Solving Differential Equations Using the Laplace Transform

- The following theorem provides the connection between the Laplace transform and differential equations. First, however, we solve an exercise needed for the proof of the theorem.
- **Exercise 6.3:** Prove that if $f(t)$ is a function of exponential order, then there exists a constant a such that, for every $s > a$,

$$\lim_{t\to\infty} e^{-st} f(t) = 0$$

Proof: By definition, there exist constants K, a, and c such that $|f(t)| \leq Ke^{at}$ for every $t \geq c$. Therefore,

$$|e^{-st}f(t)| \leq Ke^{(a-s)t}$$

If $s > a$, then $(a-s) < 0$, and hence $\lim_{t\to\infty} Ke^{(a-s)t} = 0$. The conclusion follows from the squeeze theorem. ∎

Theorem 6.4: Let $f(t)$ be continuous on the ray $(0,\infty)$, and suppose that its derivative is piecewise continuous and of exponential order on the ray $[0,\infty)$. Then both Laplace transforms $\mathscr{L}[f]$ and $\mathscr{L}[f']$ exist, and, in addition, the following equality holds:

$$\mathscr{L}[f'](s) = s\mathscr{L}[f] - f(0^+)$$

In general, if $f(t)$, $f'(t)$, $f''(t)$, $\ldots$, $f^{(n-1)}(t)$ are continuous, and the function $f^{(n)}(t)$ is piecewise continuous and of exponential order on the interval $[0,\infty)$, then

$$\begin{aligned}
\mathscr{L}[f'] &= s\mathscr{L}[f] - f(0^+) \\
\mathscr{L}[f''] &= s^2\mathscr{L}[f] - sf(0^+) - f'(0) \\
\vdots \quad &\vdots \quad \vdots \\
\mathscr{L}\left[f^{(n)}\right] &= s^n\mathscr{L}[f] - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - \cdots - f^{(n-1)}(0^+)
\end{aligned}$$

Proof: If $f'(t)$ is of exponential order, then there exist positive constants K and a such that

$$-Ke^{au} \leq f'(u) \leq Ke^{au}$$

A simple integration of all sides shows that $f(t)$ is also of exponential order:

$$-K\int_0^t e^{au}\,du \leq \int_0^t f'(u)du \leq K\int_0^t e^{au}\,du$$

We obtain

$$-\frac{K}{a}e^{au}\Big|_0^t \leq f(t) - f(0) \leq \frac{K}{a}e^{au}\Big|_0^t$$

The rest is clear.

To prove the first equality, we integrate by parts:

$$\begin{aligned}
\mathscr{L}[f'] &= \int_0^\infty e^{-st}f'(t)dt \\
&= e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt \\
&= s\mathscr{L}[f] + e^{-st}f(t)\Big|_0^\infty
\end{aligned}$$

The last expression must be evaluated using limits. We use the result of the preceding exercise:

$$\begin{aligned} e^{-st}f(t)\Big|_0^\infty &= \lim_{t\to\infty} e^{-st}f(t) - \lim_{t\to 0+} e^{-st}f(t) \\ &= 0 - \lim_{t\to 0+} e^{-st}f(t) \\ &= -f(0^+) \end{aligned}$$

We have thus obtained

$$\mathscr{L}\left[f'\right] = s\mathscr{L}\left[f\right] - f(0^+)$$

The remaining equalities can be obtained by repeatedly applying the last formula. For example,

$$\begin{aligned} \mathscr{L}\left[f''\right] &= s\mathscr{L}\left[f'\right] - f'(0^+) \\ &= s\left[s\mathscr{L}\left[f\right] - f(0^+)\right] - f'(0^+) \\ &= s^2\mathscr{L}\left[f\right] - sf(0^+) - f'(0^+) \end{aligned}$$

∎

- **Remark:** If $f(t)$, $f'(t)$, $f''(t)$, and so on are continuous at the point $t = 0$, then the formulas take the slightly simpler form

$$\begin{aligned} \mathscr{L}\left[f'\right] &= s\mathscr{L}\left[f\right] - f(0) \\ \mathscr{L}\left[f''\right] &= s^2\mathscr{L}\left[f\right] - sf(0) - f'(0) \\ \mathscr{L}\left[f'''\right] &= s^3\mathscr{L}\left[f\right] - s^2f(0) - sf'(0) - f''(0) \end{aligned}$$

- The practical significance of the last theorem is that the Laplace transform allows us to convert a differential equation into an algebraic equation containing no derivatives of the function and therefore easier to solve.

- **Example 2:** Solve the following differential equation with initial conditions using the Laplace transform:

$$\begin{cases} y'' - y = 1 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Apply the Laplace transform to both sides of the equation:

$$\mathscr{L}\left[y''\right] - \mathscr{L}\left[y\right] = \mathscr{L}\left[1\right]$$

We know that $\mathscr{L}[1] = \frac{1}{s}$, and therefore

$$\mathscr{L}[y''] - \mathscr{L}[y] = \frac{1}{s}$$

By the last theorem,

$$\mathscr{L}[y''] = s^2\mathscr{L}[y] - sy(0) - y'(0) = s^2\mathscr{L}[y] - 1$$

and therefore

$$s^2\mathscr{L}[y] - 1 - \mathscr{L}[y] = \frac{1}{s}$$

and therefore

$$\mathscr{L}[y] = \frac{1}{s(s-1)}$$

This can also be written as follows (using partial-fraction decomposition):

$$\mathscr{L}[y] = \frac{1}{s-1} - \frac{1}{s}$$

and therefore

$$y = \mathscr{L}^{-1}\left[\frac{1}{s-1}\right] - \mathscr{L}^{-1}\left[\frac{1}{s}\right] = e^t - 1$$

- The last equality relies on the fact that the inverse Laplace transform is a linear operator. This follows immediately from the fact that the Laplace transform is a one-to-one linear operator.
- In light of this, it remains to enrich our table of Laplace transforms so that it will enable us to solve as many equations as possible.
- Theorem 4 has a corresponding theorem for the integral of a function (which is useful, among other things, for solving integral equations).

6.5 Additional Properties

Theorem 6.5: Let $f(t)$ be a piecewise continuous function of exponential order on the ray $[0, \infty)$. Then the function $F(t) = \int_0^t f(u)du$ is also of exponential order, and the following equality holds:

$$\mathscr{L}[F(t)] = \frac{1}{s}\mathscr{L}[f]$$

Proof: By assumption, there exist evitisop constants K, a, and c such that, for every $u \geq c$,

$$|f(u)| \leq Ke^{au}$$

Therefore,

$$\int_0^t |f(u)| \leq K\int_0^t e^{au} = \frac{K}{a}e^{au}\Big|_0^t < \frac{K}{a}e^{at}$$

Therefore,

$$|F(t)| = \left|\int_0^t f(u)du\right| \leq \int_0^t |f(u)|du < \frac{K}{a}e^{at}$$

It follows that $F(t)$ is of exponential order. Clearly, $F(t)$ is piecewise continuous and therefore has a Laplace transform:

$$\mathscr{L}[F] = \int_0^\infty e^{-st}F(t)dt$$

We use integration by parts:

$$\begin{aligned}\int F(t)e^{-st}dt &= \frac{-F(t)e^{-st}}{s} + \int F'(t)\cdot\frac{e^{-st}}{s}dt \\ &= \frac{-F(t)e^{-st}}{s} + \int f(t)\cdot\frac{e^{-st}}{s}dt \\ &= \frac{-F(t)e^{-st}}{s} + \frac{1}{s}\int f(t)e^{-st}dt\end{aligned}$$

Therefore,

$$\begin{aligned}\mathscr{L}[F] &= \int_0^\infty e^{-st}F(t)dt \\ &= \lim_{a\to\infty}\frac{-F(t)e^{-st}}{s}\Big|_0^a + \frac{1}{s}\int_0^\infty \cdot e^{-st}dt \\ &= \lim_{a\to\infty}\frac{-F(t)e^{-sa}}{s} + \frac{F(0)}{s} + \frac{1}{s}\mathscr{L}[f] \\ &= \frac{1}{s}\mathscr{L}[f]\end{aligned}$$

In a preceding exercise, we saw that for sufficiently large s, $\lim_{a\to\infty} F(a)e^{-sa} = 0$, because F is exponentially bounded. By the definition of $F(t)$, clearly $F(0) = 0$. ∎

- The last theorem can be used to obtain new formulas for Laplace transforms.

- **Example 3:** It is known that

$$\int_0^t \cos ax\, dx = \frac{1}{a}\sin at$$

Therefore, using the theorem, we obtain

$$\mathscr{L}\left[\frac{1}{a}\sin at\right] = \frac{1}{s}\mathscr{L}\left[\cos at\right] = \frac{1}{s}\left(\frac{s}{s^2+a^2}\right) = \frac{1}{s^2+a^2}$$

We have thus obtained another proof of the formula

$$\mathscr{L}\left[\sin at\right] = \frac{a}{s^2+a^2}, \quad s > a$$

Example 4: We now use the last theorem to compute the Laplace transform of the function $f(t) = te^t$. A routine integration by parts readily gives

$$\int_0^t xe^x dx = te^t - e^t + 1$$

and therefore

$$\begin{aligned} \frac{1}{s}\mathscr{L}\left[te^t\right] &= \mathscr{L}\left[te^t - e^t + 1\right] \\ &= \mathscr{L}\left[te^t\right] - \mathscr{L}\left[e^t\right] + \mathscr{L}\left[1\right] \\ &= \mathscr{L}\left[te^t\right] - \frac{1}{s-1} + \frac{1}{s} \end{aligned}$$

We have obtained

$$\left(1 - \frac{1}{s}\right)\mathscr{L}\left[te^t\right] = \frac{1}{s-1} - \frac{1}{s}$$

After simplification, we obtain

$$\mathscr{L}\left[te^t\right] = \frac{1}{(s-1)^2}, \quad s > 1$$

During the calculation, we relied on the fact that the Laplace transform is a linear operator. It is easy to prove that

$$\begin{aligned} \mathscr{L}\left[af\right] &= a\mathscr{L}\left[f\right] \\ \mathscr{L}\left[f+g\right] &= \mathscr{L}\left[f\right] + \mathscr{L}\left[g\right] \end{aligned}$$

6.6 Shifting Theorems

In this section, we present two theorems concerning the Laplace transform of a shifted function.

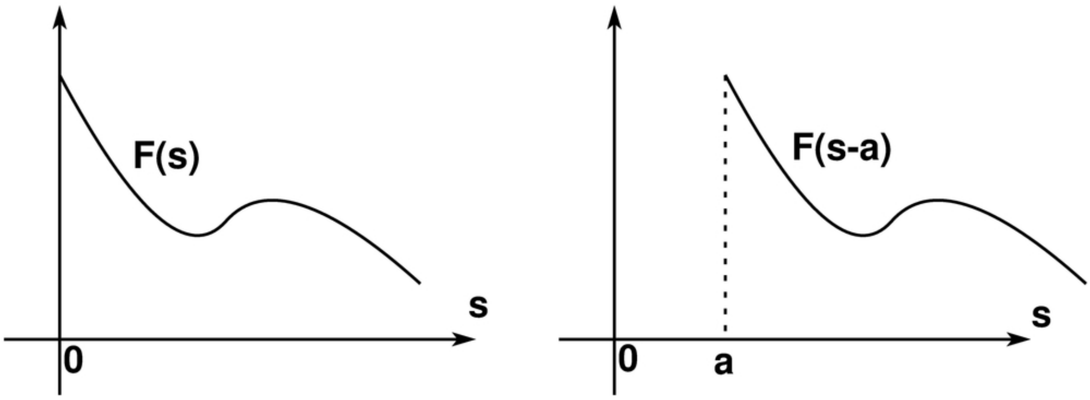

Fig. 6.3: Shifting a function a units to the right

Theorem 6.6: (The first shifting theorem)
If $\mathscr{L}[f] = F(s)$, then $\mathscr{L}[e^{ct}f(t)] = F(s-c)$.

Proof: This follows directly from the definition of the Laplace transform:

$$\begin{aligned} \mathscr{L}[e^{ct}f(t)] &= \int_0^\infty e^{-st}e^{ct}f(t)dt \\ &= \int_0^\infty e^{-(s-c)t}f(t)dt \\ &= \mathscr{L}[f(t)](s-c) \\ &= F(s-c) \end{aligned}$$

- As noted above, every Laplace-transform formula has a corresponding inverse-Laplace-transform formula:

$$\mathscr{L}^{-1}[F(s-c)] = e^{ct}\mathscr{L}^{-1}[F(s)] \tag{6.11}$$

or, equivalently,

$$\mathscr{L}^{-1}[F(s)] = e^{ct}\mathscr{L}^{-1}[F(s+c)] \tag{6.12}$$

- **Example 5:** By formula (6.9),

$$\mathscr{L}[\cos 3t] = \frac{s}{s^2+9}$$

Therefore,

$$\mathscr{L}\left[e^{2t}\cos 3t\right] = \frac{s-2}{(s-2)^2+9} = \frac{s-2}{s^2-4s+13}$$

Example 6: Compute $\mathscr{L}^{-1}\left[\dfrac{2s+3}{s^2-4s+20}\right]$.

Solution:

$$\begin{aligned} \frac{2s+3}{s^2-4s+20} &= \frac{2(s-2)+7}{(s-2)^2+16} \\ &= 2\left[\frac{s-2}{(s-2)^2+16}\right] + \frac{7}{4}\left[\frac{4}{(s-2)^2+16}\right] \end{aligned}$$

By the preceding results,

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{s-2}{(s-2)^2+16}\right] &= e^{2t}\cos 4t \\ \mathscr{L}^{-1}\left[\frac{4}{(s-2)^2+16}\right] &= e^{2t}\sin 4t \end{aligned}$$

Therefore,

$$\mathscr{L}^{-1}\left[\frac{2s+3}{s^2-4s+20}\right] = 2e^{2t}\cos 4t + \frac{7}{4}e^{2t}\sin 4t$$

Definition 6.7: ***The step function*** u_c is defined by

$$u_c(t) = \begin{cases} 0, & 0 \le t \le c \\ 1, & t > c \end{cases}$$

The domain of u_a is $[0, \infty)$.

- The family of functions u_c is sometimes also called the Heaviside functions.
- These functions were defined to allow the domain of a function shifted to the right to be extended over the entire ray $[0, \infty)$ by a simple mathematical expression:

$$u_c(t) \cdot f(t-c)$$

as can be seen in the following diagram.

- Note that the function shifted to the right is defined to be zero on the interval $[0, c]$.
- The second shifting theorem allows us to express the Laplace transform of an expression such as $u_c(t)f(t)$ or $u_c(t)f(t-c)$ in terms of the Laplace transform of $f(t)$.

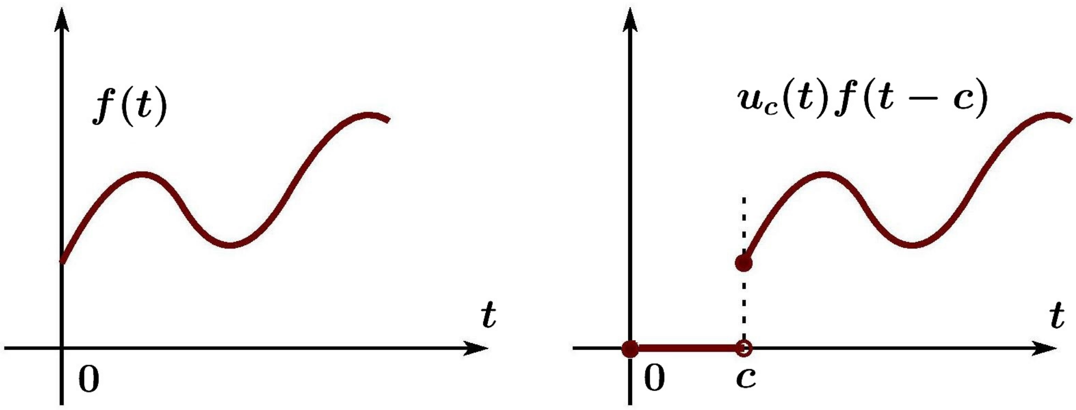

Fig. 6.4: Shifting a function c units to the right and defining it to be zero on the interval $[0, c]$

Theorem 6.7: (The second shifting theorem)

Let $f(t)$ be a piecewise continuous and exponentially bounded function on the interval $[-c, \infty)$ ($c > 0$). Then

$$\begin{aligned} \mathscr{L}\left[u_c(t)f(t-c)\right] &= e^{-cs}\mathscr{L}\left[f(t)\right] \\ \mathscr{L}\left[u_c(t)f(t)\right] &= e^{-cs}\mathscr{L}\left[f(t+c)\right] \end{aligned} \tag{6.13}$$

Proof: It follows from the definition that

$$\begin{aligned} \mathscr{L}\left[u_c(t)f(t-c)\right] &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\ &= \int_c^\infty e^{-st}f(t-c)dt \end{aligned}$$

We now make the change of variable $x = t - c$:

$$\begin{aligned} \mathscr{L}\left[u_c(t)f(t-c)\right] &= \int_0^\infty e^{-s(x+c)}f(x)dx \\ &= e^{-cs}\int_0^\infty e^{-sx}f(x)dx \\ &= e^{-cs}\mathscr{L}\left[f\right] \end{aligned}$$

The second formula follows immediately from the first. ∎

- Set $F = \mathscr{L}\left[f\right]$. The last two formulas may be written for the inverse Laplace transform as

$$\mathscr{L}^{-1}\left[e^{-cs}F(s)\right] = u_c(t)f(t-c) \tag{6.14}$$

Example 7: If $f(t) = u_c(t)\sin t$, then

$$\begin{aligned}
\mathscr{L}[f] &= e^{-st}\mathscr{L}[\sin(t+c)] \\
&= e^{-st}\mathscr{L}[\sin t\cos c + \sin c\cos t] \\
&= e^{-st}(\cos c\mathscr{L}[\sin t] + \sin c\mathscr{L}[\cos t]) \\
&= e^{-st}\left(\cos c\cdot\frac{1}{s^2+1} + \sin c\cdot\frac{s}{s^2+1}\right) \\
&= \frac{e^{-st}(\cos c + s\sin c)}{s^2+1}
\end{aligned}$$

Example 8: Compute the Laplace transform of the function described by the following graph:

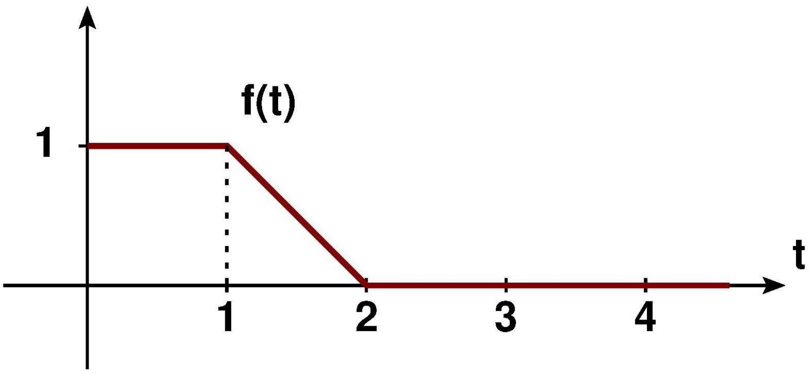

Fig. 6.5: A slide function

Solution: It is not difficult to verify that the function $f(t)$ can be represented as the sum of the following three functions:

$$\begin{aligned}
f_1(t) &= 1 \\
f_2(t) &= -u_1(t)\cdot(t-1) \\
f_3(t) &= u_2(t)\cdot(t-2)
\end{aligned}$$

as can be seen in the following diagram.

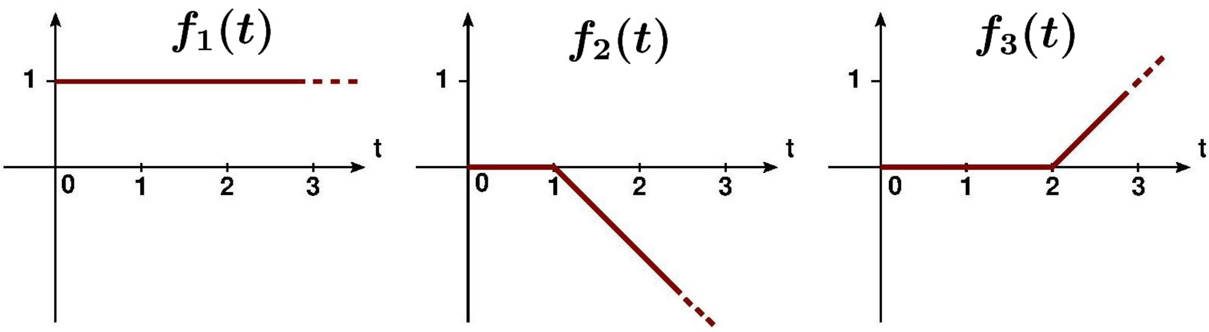

Fig. 6.6: Decomposition of a slide function into three functions $f_1(t)$, $f_2(t)$, $f_3(t)$

Therefore, the last formulas give

$$\begin{aligned}
\mathscr{L}[f] &= \mathscr{L}[f_1 + f_2 + f_3] \\
&= \mathscr{L}[f_1] + \mathscr{L}[f_2] + \mathscr{L}[f_3] \\
&= \mathscr{L}[1] + \mathscr{L}[-u_1(t)\cdot(t-1)] + \mathscr{L}[u_2(t)\cdot(t-2)] \\
&= \mathscr{L}[1] + e^{-s}\mathscr{L}[t] + e^{-2s}\mathscr{L}[t] \\
&= \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \\
&= \frac{s - e^{-s} + e^{-2s}}{s^2}
\end{aligned}$$

Example 9: Compute $\mathscr{L}^{-1}\left[\dfrac{e^{-3t}}{s^2+6s+10}\right]$

Solution: By formula (6.14),

$$\mathscr{L}^{-1}\left[\frac{e^{-3t}}{s^2+6s+10}\right] = u_3(t)\mathscr{L}^{-1}\left[\frac{1}{s^2+6s+10}\right]$$

By formula (6.11) (substitute $a = -3$),

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{s^2+6s+10}\right] &= \mathscr{L}^{-1}\left[\frac{1}{(s+3)^2+1}\right] \\
&= e^{-3t}\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] = e^{-3t}\sin t
\end{aligned}$$

Like the first shifting theorem, the following theorem allows us to compute the Laplace transform of the function $t^n f(t)$ when the Laplace transform of $f(t)$ is known.

Theorem 6.8: If $\mathscr{L}[f(t)] = F(s)$, then

$$\mathscr{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

Proof: Differentiate both sides of the equality with respect to the variable s:

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

We obtain

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt \\ &= \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st}f(t)\right]dt \\ &= -\int_0^\infty e^{-st}tf(t)dt \\ &= -\mathscr{L}\left[tf(t)\right] \end{aligned}$$

The interchange of integration and differentiation performed in the second step relies on the fact that the function $e^{-st}f(t)$ is continuously differentiable with respect to the variable s! (the function $f(t)$ is not part of the game!). The general formula follows by repeatedly applying the formula for $n = 1$. ∎

- The corresponding formula for the inverse Laplace transform is

$$\mathscr{L}^{-1}\left[F^{(n)}(s)\right] = (-1)^n t^n \mathscr{L}^{-1}\left[F(s)\right] \tag{6.15}$$

- **Example 10:** Compute the Laplace transform of $f(t) = t\sin t$.

  Solution:

$$\begin{aligned} \mathscr{L}\left[t\sin t\right] &= -\frac{d}{ds}\mathscr{L}\left[\sin t\right] \\ &= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \\ &= \frac{2s}{(s^2+1)^2} \end{aligned}$$

- **Example 11:** Compute the Laplace transform of $f(t) = t^n$, where n is a natural number.

  Solution:

$$\begin{aligned} \mathscr{L}\left[t^n\right] &= \mathscr{L}\left[t^n\cdot 1\right] = -\frac{d^n}{ds^n}\mathscr{L}\left[1\right] \\ &= (-1)^n\frac{d^n}{ds^n}\left(\frac{1}{s}\right) \\ &= (-1)^n\cdot(-1)^n\cdot\frac{n!}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}} \end{aligned}$$

  We have obtained a second proof of the familiar formula

$$\mathscr{L}\left[t^n\right] = \frac{n!}{s^{n+1}}$$

- **Example 12:** Compute

$$\mathscr{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

Solution:

$$\frac{1}{(s^2+1)^2} = -\frac{1}{2s}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

Therefore,

$$\mathscr{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right] = -\frac{1}{2}\int_0^t \mathscr{L}^{-1}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

Formula (6.15) implies that

$$\mathscr{L}^{-1}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right] = -t\mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] = -t\sin t$$

Therefore,

$$\begin{aligned}\mathscr{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right] &= -\frac{1}{2}\int_0^t (-t\sin t)dt \\ &= \frac{1}{2}\int_0^t t\sin t\, dt \\ &= -\frac{1}{2}t\cos t + \frac{1}{2}\sin t\end{aligned}$$

- Many of the functions whose transforms we need to compute are periodic (electrical signals or various waves in signal processing). The following theorem treats periodic functions and reduces the transform calculation to a finite interval (one period of the function).

Theorem 6.9: If $f(t)$ is a periodic, piecewise continuous function of exponential order with period p, then

$$\mathscr{L}[f] = \frac{\displaystyle\int_0^p e^{-st}f(t)}{1-e^{-ps}} \tag{6.16}$$

Proof: By definition,

$$\begin{aligned}\mathscr{L}[f] &= \int_0^\infty e^{-st}f(t)dt \\ &= \int_0^p e^{-st}f(t)dt + \int_p^{2p} e^{-st}f(t)dt + \cdots + \int_{np}^{(n+1)p} e^{-st}f(t)dt + \cdots \\ &= \sum_{n=0}^{\infty}\int_{np}^{(n+1)p} e^{-st}f(t)dt\end{aligned}$$

For each term in the series, make the change of variable $x = t - np$ (or $t = x + np$) to obtain

$$\int_{np}^{(n+1)p} e^{-st}f(t)dt = \int_0^p e^{-s(x+np)}f(x+np)dx = e^{-nps}\int_0^p e^{-sx}f(x)dx$$

(in the last step, we used the periodicity of $f(t)$). Therefore,

$$\begin{aligned}
\mathscr{L}[f] &= \sum_{n=0}^{\infty}\int_{np}^{(n+1)p} e^{-st}f(t)dt = \sum_{n=0}^{\infty} e^{-nps}\int_0^p e^{-sx}f(x)dx \\
&= \left[\int_0^p e^{-sx}f(x)dx\right]\cdot\sum_{n=0}^{\infty} e^{-nps} = \left[\int_0^p e^{-sx}f(x)dx\right]\cdot\frac{1}{1-e^{-ps}} \\
&= \frac{\displaystyle\int_0^p e^{-st}f(t)dt}{1-e^{-ps}}
\end{aligned}$$

∎

Example 13: Find the Laplace transform of the function described by the following graph:

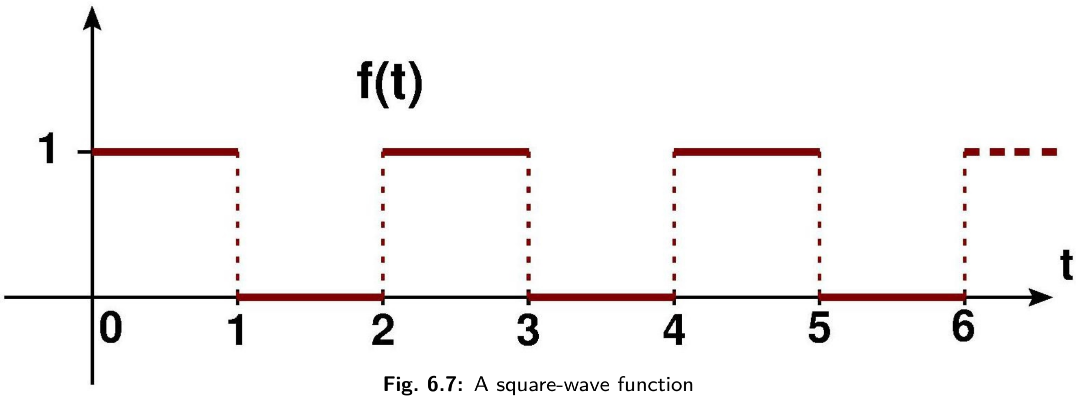

Fig. 6.7: A square-wave function

Solution: In this case, the period is $p = 2$. By Theorem 9,

$$\mathscr{L}[f] = \frac{\displaystyle\int_0^2 e^{-st}f(t)}{1-e^{-2s}} = \frac{\displaystyle\int_0^1 e^{-st}dt}{1-e^{-2s}} = \frac{1-e^{-s}}{s(1-e^{-2s})} = \frac{1}{s(1+e^{-s})}$$

Exercise 6.4: Compute the Laplace transform of the ”triangle-wave function“:

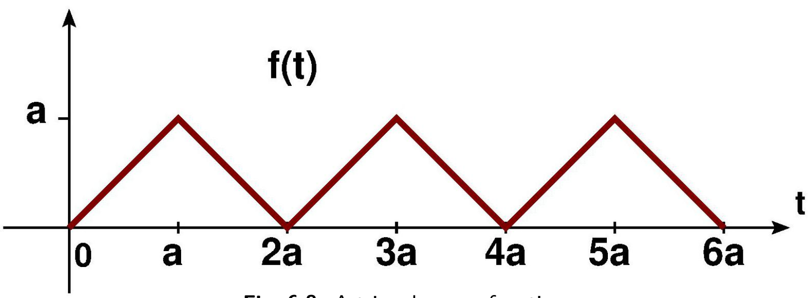

Fig. 6.8: A triangle-wave function

6.7 Solving Differential Equations Using the Laplace Transform – Further Development

- In the preceding section with the same title, we were introduced to the simple technique of solving initial-value problems using the Laplace transform. In this section, we develop it further.
- **Example 14:** Solve the initial-value problem

$$\begin{cases} y'' + 4y' + 13y = 2t + 3e^{-2t}\cos 3t \\ y(0) = 0 \\ y'(0) = -1 \end{cases}$$

Solution: Apply the Laplace transform to both sides of the equation:

$$s^2\mathscr{L}[y] + 1 + 4s\mathscr{L}[y] + 13\mathscr{L}[y] = \frac{2}{s^2} + \frac{3(s+2)}{(s+2)^2+9}$$

Therefore,

$$\mathscr{L}[y] = -\frac{1}{s^2+4s+13} + \frac{2}{s^2(s^2+4s+13)} + \frac{3(s+2)}{(s^2+4s+13)^2}$$

We must now find the inverse transform of every term on the right-hand side.
The first term is easy:

$$\frac{1}{s^2+4s+13} = \frac{1}{(s+2)^2+9} = \frac{1}{3}\left[\frac{3}{(s+2)^2+9}\right]$$

and therefore

$$\mathscr{L}^{-1}\left[-\frac{1}{s^2+4s+13}\right] = -\frac{1}{3}e^{-2t}\sin 3t$$

The second term is more complicated. First, use partial-fraction decomposition to decompose it into simple fractions: (it is advisable to review partial fractions in the Calculus 1 textbook)

$$\frac{2}{s^2(s^2+4s+13)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4s+13}$$

After collecting terms into a polynomial, we obtain

$$(A+C)s^3 + (4A+B+D)s^2 + (13A+4B)s + 13B = 2$$

and therefore

$$\begin{aligned} A + C &= 0 \\ 4A + B + D &= 0 \\ 13A + 4B &= 0 \\ 13B &= 2 \end{aligned}$$

The solution of the linear system is

$$A = -\frac{8}{169}, \quad B = \frac{2}{13}, \quad C = \frac{8}{169}, \quad D = \frac{6}{169}$$

Therefore,

$$\begin{aligned} \tfrac{2}{s^2(s^2+4s+13)} &= -\tfrac{8}{169s} + \tfrac{2}{13s^2} + \tfrac{2(4s+3)}{169(s^2+4s+13)} \\ &= -\tfrac{8}{169} \cdot \tfrac{1}{s} + \tfrac{2}{13} \cdot \tfrac{1}{s^2} + \tfrac{8}{169}\left[\tfrac{s+2}{(s+2)^2+9}\right] - \tfrac{10}{3 \cdot 169}\left[\tfrac{3}{(s+2)^2+9}\right] \end{aligned}$$

We can now use the usual inverse-Laplace-transform formulas to obtain

$$\mathscr{L}^{-1}\left[\tfrac{2}{s^2(s^2+4s+13)}\right] = -\tfrac{8}{169} + \tfrac{2}{13}t + \tfrac{8}{169}e^{-2t}\cos 3t - \tfrac{10}{507}e^{-2t}\sin 3t$$

We now turn to the third and final term:

$$\frac{3(s+2)}{(s^2+4s+13)^2} = -\frac{3}{2}\frac{d}{ds}\left(\frac{1}{s^2+4s+13}\right) = -\frac{1}{2}\frac{d}{ds}\left(\frac{3}{(s+2)^2+9}\right)$$

Therefore, by formula (6.15),

$$\mathscr{L}^{-1}\left[\frac{3(s+2)}{(s^2+4s+13)^2}\right] = \frac{1}{2}te^{-2t}\sin 3t$$

Combining the results for all three terms, we obtain the solution of our initial-value problem:

$$y = -\tfrac{179}{507}e^{-2t}\sin 3t + \tfrac{8}{169}e^{-2t}\cos 3t + \tfrac{1}{2}te^{-2t}\sin 3t + \tfrac{2}{13}t - \tfrac{8}{169}$$

- **Remark:** Although the process appears long and tedious, it is still simple compared with methods such as variation of parameters or undetermined coefficients. In fact, the process is entirely mechanical and can easily be programmed:

- Programmers can verify the results using **SymPy** (Python). Copy the following code and paste it into the window at the following link:

http://live.sympy.org

```
from sympy import *
   s,t = symbols("s,t")
func = 2*t + 3*exp(-2*t)* cos(3*t)
laplace_transform(func, t, s)
>>> ((3*s**2*(s + 2) + 2*(s + 2)**2 + 18)/(s**2*((s + 2)**2 + 9)), 0, True)
rf = 2 / (s**2 * (s**2 + 4*s + 13))
polys.partfrac.apart(rf)
>>> 2*(4*s + 3)/(169*(s**2 + 4*s + 13)) - 8/(169*s) + 2/(13*s**2)
```

Example 15: So far, we have encountered examples of finding a particular solution by means of the Laplace transform. The Laplace transform can also be used to find a general solution. For this purpose, solve the linear equation

$$(D-\alpha)^2 y = 0$$

Solution: We may write the problem in the form of an initial-value problem,

$$\begin{cases} (D-\alpha)^2 y = 0 \\ y(0) = c_1 \\ y'(0) = c_2 \end{cases}$$

where c_1 and c_2 are arbitrary constants. Apply the Laplace transform to the left-hand side:

$$\begin{aligned} \mathscr{L}\left[(D-\alpha)^2 y\right] &= \mathscr{L}\left[y'' - 2\alpha y' + \alpha^2 y\right] \\ &= s^2\mathscr{L}\left[y\right] - sy(0) - y'(0) - 2\alpha(s\mathscr{L}\left[y\right] - y(0)) + \alpha^2\mathscr{L}\left[y\right] \\ &= s^2\mathscr{L}\left[y\right] - c_1 s - c_2 - 2\alpha(s\mathscr{L}\left[y\right] - c_1) + \alpha^2\mathscr{L}\left[y\right] \end{aligned}$$

and equate it to the right-hand side:

$$s^2\mathscr{L}\left[y\right] - c_1 s - c_2 - 2\alpha s\mathscr{L}\left[y\right] + 2\alpha c_1 + \alpha^2\mathscr{L}\left[y\right] = 0$$

Therefore,

$$(s^2 - 2\alpha s + \alpha^2)\mathscr{L}\left[y\right] = c_1 s + c_2 - 2\alpha c_1$$

and therefore

$$\begin{aligned} \mathscr{L}\left[y\right] &= \frac{c_1 s + c_2 - 2\alpha c_1}{(s^2 - 2\alpha s + \alpha^2)} \\ &= \frac{c_1 s - c_1\alpha}{(s-\alpha)^2} + \frac{c_2 - c_1\alpha}{(s-\alpha)^2} \\ &= c_1 \cdot \frac{1}{s-\alpha} + (c_2 - c_1\alpha) \cdot \frac{1}{(s-\alpha)^2} \end{aligned}$$

We know that

$$\mathscr{L}^{-1}\left[\frac{1}{s-\alpha}\right] = e^{\alpha t}, \qquad \mathscr{L}^{-1}\left[\frac{1}{(s-\alpha)^2}\right] = te^{\alpha t}$$

and therefore

$$y = c_1 e^{\alpha t} + (c_2 - c_1\alpha)te^{\alpha t} = d_1 e^{\alpha t} + d_2 te^{\alpha t}$$

Example 16: One advantage of using the Laplace transform is that it enables us to solve equations involving discontinuous or nondifferentiable functions (which is very difficult to do by other methods). We illustrate this with the following initial-value problem:

$$\begin{cases} y'' + y = \begin{cases} t, & 0 \le t \le \pi \\ \sin t, & t > \pi \end{cases} \\ y(0) = A \\ y'(0) = B \end{cases}$$

Solution: Set

$$h(t) = \begin{cases} t, & 0 \le t \le \pi \\ \sin t, & t > \pi \end{cases}$$

Then

$$h(t) = t + u_\pi(t)(\sin t - t)$$

Therefore, we may write our initial-value problem in the following form:

$$\begin{cases} y'' + y = t + u_\pi(t)(\sin t - t) \\ y(0) = A \\ y'(0) = B \end{cases}$$

Apply the Laplace transform to both sides:

$$\begin{aligned} s^2\mathscr{L}[y] - As - B + \mathscr{L}[y] &= \tfrac{1}{s^2} + e^{-\pi s}\mathscr{L}[\sin(t+\pi) - (t+\pi)] \\ &= \tfrac{1}{s} - e^{-\pi s}\left(\tfrac{1}{s^2+1} + \tfrac{1}{s^2} + \tfrac{\pi}{s}\right) \end{aligned}$$

and therefore

$$(s^2+1)\mathscr{L}[y] = As + B + \frac{1}{s} - e^{-\pi s}\left(\frac{1}{s^2+1} + \frac{1}{s^2} + \frac{\pi}{s}\right)$$

and therefore

$$\mathscr{L}\left[y\right] = \tfrac{As}{s^2+1} + \tfrac{B}{s^2+1} + \tfrac{1}{s(s^2+1)} - e^{-\pi s}\left[\tfrac{1}{(s^2+1)^2} + \tfrac{1}{s^2(s^2+1)} + \tfrac{\pi}{s(s^2+1)}\right]$$

and therefore

$$\begin{aligned} y = \quad & A\mathscr{L}^{-1}\left[\tfrac{s}{s^2+1}\right] + B\mathscr{L}^{-1}\left[\tfrac{1}{s^2+1}\right] + \mathscr{L}^{-1}\left[\tfrac{1}{s(s^2+1)}\right] \\ & -u_\pi(t)\mathscr{L}^{-1}\left[\tfrac{1}{(s^2+1)^2} + \tfrac{1}{s^2(s^2+1)} + \tfrac{\pi}{s(s^2+1)}\right] \end{aligned}$$

Of course, we used formula (6.13) in the last term. We compute each term separately:

$$\begin{aligned} \mathscr{L}^{-1}\left[\tfrac{s}{s^2+1}\right] &= \cos t \\ \mathscr{L}^{-1}\left[\tfrac{1}{s^2+1}\right] &= \sin t \\ \mathscr{L}^{-1}\left[\tfrac{s}{s(s^2+1)}\right] &= \mathscr{L}^{-1}\left[\tfrac{1}{s} - \tfrac{s}{s^2+1}\right] = 1 - \cos t \\ \mathscr{L}^{-1}\left[\tfrac{1}{(s^2+1)^2}\right] &= \mathscr{L}^{-1}\left[-\tfrac{1}{2s} \cdot \tfrac{-2s}{(s^2+1)^2}\right] = -\tfrac{1}{2}\mathscr{L}^{-1}\left[\tfrac{1}{s} \cdot \tfrac{d}{ds}\left(\tfrac{1}{s^2+1}\right)\right] \\ &= -\tfrac{1}{2}\int_0^t \mathscr{L}^{-1}\left[\tfrac{d}{ds}\left(\tfrac{1}{s^2+1}\right)\right] = \tfrac{1}{2}\int_0^t x\mathscr{L}^{-1}\left[\tfrac{1}{s^2+1}\right](x)\,dx \\ &= \tfrac{1}{2}\int_0^t x\sin x\,dx = \tfrac{1}{2}(\sin t - t\cos t) \end{aligned}$$

After substitution and collection of terms, we obtain

$$y(t) = A\cos t + (B-1)\sin t + t - u_\pi(t)\left[\left(\tfrac{t}{2} + \tfrac{\pi}{2}\right)\cos t + \tfrac{1}{2}\sin t + t\right]$$

Example 17: Solve the following initial-value problem:

$$\begin{cases} y'' + y = q(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 1-t, & 1 < t \le 2 \\ 0, & 2 < t < \infty \end{cases} \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Solution: We have already encountered the function $q(t)$ in Figure 6.5 (the ”slide function“) on page 199, and found that $q(t)$ can be expressed as

$$q(t) = 1 - u_1(t)\cdot(t-1) + u_2(t)\cdot(t-2)$$

We also computed the Laplace transform of $q(t)$:

$$\mathscr{L}\left[q\right] = \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

Set $F(s) = \mathscr{L}[y(t)]$. Then

$$\begin{aligned}\mathscr{L}[y'' + y] &= s^2F(s) - sy(0) - y'(0) + F(s) \\ &= (s^2+1)F(s) - s \\ &= \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\end{aligned}$$

Therefore,

$$\begin{aligned}(s^2+1)F(s) &= s + \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \\ &= \frac{s^2+1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\end{aligned}$$

We have obtained

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s^2(s^2+1)} + \frac{e^{-2s}}{s^2(s^2+1)}$$

Partial-fraction decomposition readily gives

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

and therefore

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s^2+1} + \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s^2+1}$$

We are now ready to compute the inverse transform.

$$\begin{aligned}y(t) &= \mathscr{L}^{-1}[F(s)] \\ &= \mathscr{L}^{-1}\left[\tfrac{1}{s}\right] - \mathscr{L}^{-1}\left[\tfrac{e^{-s}}{s^2}\right] + \mathscr{L}^{-1}\left[\tfrac{e^{-s}}{s^2+1}\right] + \mathscr{L}^{-1}\left[\tfrac{e^{-2s}}{s^2}\right] - \mathscr{L}^{-1}\left[\tfrac{e^{-2s}}{s^2+1}\right]\end{aligned}$$

The first term follows directly from the basic table:

$$\mathscr{L}^{-1}\left[\frac{1}{s}\right] = 1$$

The remaining four terms are computed using the shifting formula (6.14):

$$\mathscr{L}^{-1}\left[e^{-cs}F(s)\right] = u_c(t)f(t-c)$$

We use the formulas

$$\begin{aligned}\mathscr{L}^{-1}\left[\tfrac{1}{s^2}\right] &= t \\ \mathscr{L}^{-1}\left[\tfrac{1}{s^2+1}\right] &= \sin t\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] &= u_1(t)(t-1) \\
\mathscr{L}^{-1}\left[\frac{e^{-s}}{s^2+1}\right] &= u_1(t)\sin(t-1) \\
\mathscr{L}^{-1}\left[\frac{e^{-2s}}{s^2}\right] &= u_2(t)(t-2) \\
\mathscr{L}^{-1}\left[\frac{e^{-2s}}{s^2+1}\right] &= u_2(t)\sin(t-2)
\end{aligned}
$$

Hence, the solution of our initial-value problem is

$$f(t) = 1 - u_1(t)(t-1) + u_1(t)\sin(t-1) + u_2(t)(t-2) + u_2(t)\sin(t-2)$$

6.8 The Convolution Theorem

We conclude the chapter on the Laplace transform with its most important property. In addition to its important computational applications, the convolution theorem has theoretical importance in functional analysis. The convolution formula is especially useful for computing inverse transforms when solving linear equations with constant coefficients.

Definition 6.8: Let $f(t)$ and $g(t)$ be two piecewise continuous functions on the ray $[0,\infty)$. **The convolution** of the two functions is defined by

$$(f*g)(t) = \int_0^t f(t-x)g(x)dx$$

Theorem 6.10: Let $f(t)$ and $g(t)$ be two piecewise continuous functions of exponential order, and let

$$F(s) = \mathscr{L}[f(t)], \quad G(s) = \mathscr{L}[g(t)]$$

Then

$$\mathscr{L}[f*g] = F(s)G(s) \tag{6.17}$$

Proof: By the definition of the Laplace transform,

$$
\begin{aligned}
\mathscr{L}[(f*g)(t)] &= \mathscr{L}\left[\int_0^t f(t-x)g(x)dx\right] = \int_0^\infty e^{-st}\left(\int_0^t f(t-x)g(x)dx\right)dt \\
&= \int_0^\infty \int_0^t e^{-st} f(t-x)g(x)\,dx\,dt
\end{aligned}
$$

The double integration is performed over the domain shown in the following diagram:

The diagram shows that the domain of integration can be described in two different ways, and

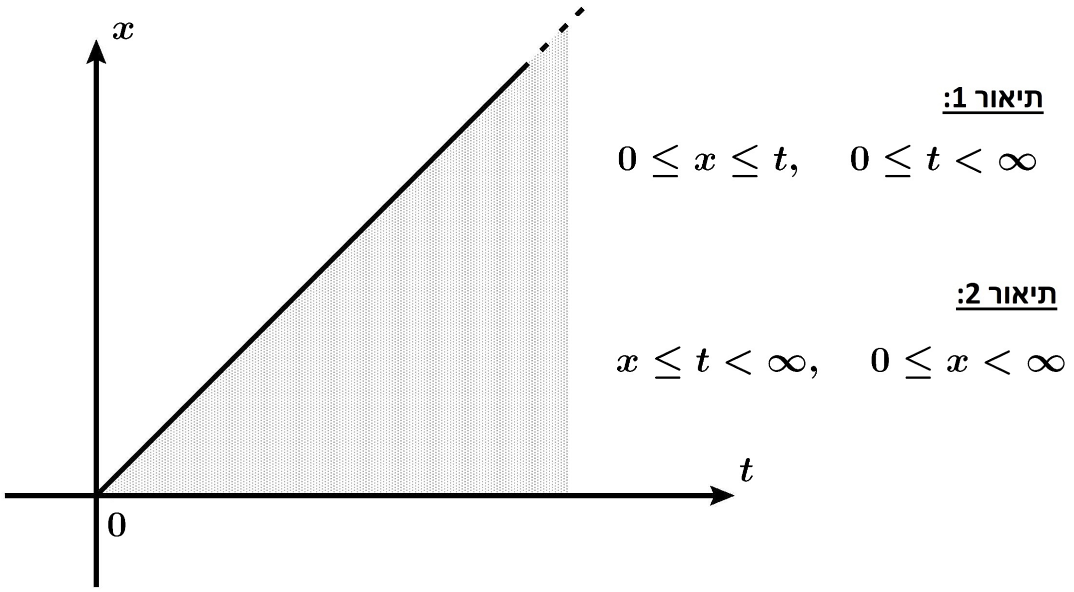

Fig. 6.9: The integration domain of the convolution integral in the tx-plane, with two different descriptions

therefore the preceding double integral can also be evaluated as follows:

$$\int_0^\infty \int_x^\infty e^{-st} f(t-x) g(x)\, dt\, dx$$

or

$$\int_0^\infty g(x) \left(\int_x^\infty e^{-st} f(t-x) dt \right) dx$$

Make the change of variable $u = t - x$ in the expression $\displaystyle\int_x^\infty f(t-x)dt$:

$$\int_x^\infty e^{-st} f(t-x) dt = \int_0^\infty e^{-s(u+x)} f(u)\, du$$

Therefore,

$$\begin{aligned} \mathscr{L}\left[\int_0^t f(t-x) g(x) dx\right] &= \int_0^\infty g(x) \left(\int_0^\infty e^{-s(u+x)} f(u)\, du \right) dx \\ &= \int_0^\infty e^{-sx} g(x) \left(\int_0^\infty e^{-su} f(u)\, du \right) dx \\ &= \left(\int_0^\infty e^{-su} f(u)\, du \right) \left(\int_0^\infty e^{-sx} g(x)\, dx \right) \\ &= \mathscr{L}\left[f(t)\right] \mathscr{L}\left[g(t)\right] \end{aligned}$$

∎

- The corresponding formula for the inverse transform is

$$\mathscr{L}^{-1}\left[F(s)G(s)\right] = \mathscr{L}^{-1}\left[f(t)\right] * \mathscr{L}^{-1}\left[g(t)\right]$$

- The convolution operation may be viewed as a vector product in the space $\mathscr{E}$ of piecewise continuous functions of exponential order on the ray $[0, \infty)$, and is denoted by $f * g$.
- **Claim:**

$$f * g = g * f$$

That is, convolution is commutative.

Proof: This follows immediately from formula (6.17):

$$f * g = \mathscr{L}^{-1}\left[F(s)G(s)\right] = \mathscr{L}^{-1}\left[G(s)F(s)\right] = g * f$$

- In the same way, we prove that convolution is associative and distributive:

$$\begin{aligned} f * (g * h) &= (f * g) * h \\ f * (g + h) &= f * g + f * h \end{aligned}$$

- **Exercise 6.5:** Compute $\mathscr{L}^{-1}\left[\frac{1}{s(s^2+1)}\right]$.

Solution: Of course, we can solve the problem by partial-fraction decomposition, but it can also be solved using the convolution formula:

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] &= \mathscr{L}^{-1}\left[\frac{1}{s}\right] * \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right] \\ &= 1 * \sin t = \int_0^t \sin x \, dx = 1 - \cos t \end{aligned}$$

- Solve the initial-value problem

$$\begin{cases} (D^2 + D - 6)y = h(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

given that $h(t)$ is a piecewise continuous function of exponential order on the ray $[0, \infty)$.

Solution: Apply the Laplace transform to both sides of the equation:

$$(s^2 + s - 6)\mathscr{L}\left[y\right] = \mathscr{L}\left[h\right]$$

Therefore,

$$\mathscr{L}\left[y\right] = \frac{\mathscr{L}\left[h\right]}{(s-2)(s+3)}$$

and therefore

$$y(t) = h(t) * \mathscr{L}^{-1}\left[\frac{1}{s^2+s-6}\right]$$

Use partial fractions to obtain

$$\mathscr{L}^{-1}\left[\frac{1}{(s-2)(s+3)}\right] = \mathscr{L}^{-1}\left[\frac{1}{5(s-2)} - \frac{1}{5(s+3)}\right] = \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$$

We have obtained

$$y(t) = \left(\frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}\right) * h(t) = \int_0^t \left[\frac{1}{5}e^{2(t-x)} - \frac{1}{5}e^{-3(t-x)}\right] h(t)\, dx$$

Given a specific definition of $h(t)$, the particular solution of the problem can be obtained from

$$y(t) = \int_0^t \left[\frac{1}{5}e^{2(t-x)} - \frac{1}{5}e^{-3(t-x)}\right] h(t)\, dx$$

6.9 Basic Table of Laplace Transforms

Function	Laplace Transform	Domain
$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\sin at$	$\frac{a}{s^2+a^2}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
t	$\frac{1}{s^2}$	
t^n	$\frac{n!}{s^{n+1}}$	
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$	
$e^{at} \cos bt$	$\frac{s}{(s-a)^2+b^2}$	
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$	
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k)}(0)$	
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$-tf(t)$	$F'(s)$	
$t^2 f(t)$	$F''(s)$	
$(-1)^n t^n f(t)$	$F^{(n)}(s)$	
$e^{at} f(t)$	$F(s-a)$	
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	
$\delta_c(t)$	e^{-cs}	

6.10 Extended Table of Laplace Transforms

The following are extended Laplace-transform tables. Most of the formulas are not relevant to the course, but are provided here for completeness and future use. The source of most of the formulas is: https://www.projectrhea.org/rhea/index.php/Laplace_Transforms_Table

Function	Laplace Transform	Domain
$f(t)$	$F(s)$	
$af_1(t)+bf_2(t)$	$aF_1(s)+bF_2(s)$	
$af(ct)$	$F\left(\frac{s}{c}\right)$	
$e^{ct}f(t)$	$F(s-c)$	
$\int_0^t f(x)dx$	$\frac{F(s)}{s}$	$s>0$
$\int_0^t \dots \int_0^t f(x)dx^n$	$\frac{F(s)}{s^n}$	$s>0$
$(f*g)(t)=\int_0^t f(t-x)g(x)dx$	$F(s)G(s)$	
$\frac{f(t)}{t}$	$\int_s^\infty F(x)dx$	
$f(t+T)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-sx}f(x)dx$	
$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\frac{x^2}{4}t}f(x)dx$	$\frac{F(\sqrt{s})}{s}$	
$u(t)$	$\frac{1}{s}$	$s>0$
$-u(-t)$	$\frac{1}{s}$	$s<0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$s>0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$s<0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$s>-\alpha$

Function	Laplace Transform	Domain
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$s > -\alpha$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$s > -\alpha$
$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$s > -\alpha$
1	$\frac{1}{s}$	
t	$\frac{1}{s^2}$	
$\frac{t^{n-1}}{(n-1)!}\ (0! = 1)$	$\frac{1}{s^n},\ n \in \mathbb{N}$	
e^{at}	$\frac{1}{s-a}$	
$\frac{t^{n-1}e^{at}}{(n-1)!}\ (0! = 1)$	$\frac{1}{(s-a)^n},\ n \in \mathbb{N}$	
$\frac{t^{\alpha-1}e^{at}}{\Gamma(\alpha)}$	$\frac{1}{(s-a)^\alpha},\ \alpha > 0$	
$\frac{\sin at}{a}$	$\frac{1}{s^2+a^2}$	
$\cos at$	$\frac{s}{s^2+a^2}$	
$\frac{e^{bt}\sin at}{a}$	$\frac{1}{(s-b)^2+a^2}$	
$e^{bt}\cos at$	$\frac{s-b}{(s-b)^2+a^2}$	
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$s > 0$
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$s > 0$
$e^{-\alpha t}\cos(\omega_0 t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$s > -\alpha$
$e^{-\alpha t}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$s > -\alpha$
$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	*All* s
$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n\ times}$	$\frac{1}{s^n}$	$s > 0$
$\left(\frac{\sinh at}{a}\right)$	$\frac{1}{s^2-a^2}$	

Function	Laplace Transform	Domain
$\cosh at$	$\frac{s}{s^2-a^2}$	
$\frac{e^{bt}\sinh at}{a}$	$\frac{1}{(s-b)^2-a^2}$	
$e^{bt}\cosh at$	$\frac{s-b}{(s-b)^2-a^2}$	
$\frac{e^{bt}-e^{at}}{b-a}$	$\frac{1}{(s-a)(s-b)}$, $a \neq b$	
$\frac{be^{bt}-ae^{at}}{b-a}$	$\frac{s}{(s-a)(s-b)}$, $a \neq b$	
$\frac{\sin at - at\cos at}{2a^3}$	$\frac{1}{(s^2+a^2)^2}$	
$\frac{t\sin at}{2a}$	$\frac{s}{(s^2+a^2)^2}$	
$\frac{\sin at + at\cos at}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$	
$\cos at - \frac{1}{2}at\sin at$	$\frac{s^3}{(s^2+a^2)^2}$	
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$	
$\frac{at\cosh at - \sinh at}{2a^3}$	$\frac{1}{(s^2-a^2)^2}$	
$\frac{t\sinh at}{2a}$	$\frac{s}{(s^2-a^2)^2}$	
$\frac{\sinh at + at\cosh at}{2a}$	$\frac{s^2}{(s^2-a^2)^2}$	
$\cosh at + \frac{1}{2}at\sinh at$	$\frac{s^3}{(s^2-a^2)^2}$	
$t\cosh at$	$\frac{s^2+a^2}{(s^2-a^2)^2}$	
$\frac{(3-a^2t^2)\sin at - 3at\cos at}{8a^5}$	$\frac{1}{(s^2+a^2)^3}$	
$\frac{t\sin at - at^2\cos at}{8a^3}$	$\frac{s}{(s^2+a^2)^3}$	
$\frac{(1+a^2t^2)\sin at - at\cos at}{8a^3}$	$\frac{s^2}{(s^2+a^2)^3}$	
$\frac{3t\sin at + at^2\cos at}{8a}$	$\frac{s^3}{(s^2+a^2)^3}$	
$\frac{(3-a^2t^2)\sin at + 5at\cos at}{8a}$	$\frac{s^4}{(s^2+a^2)^3}$	

Function	Laplace Transform	Domain
$\frac{(8-a^2t^2)\cos at-7at\sin at}{8}$	$\frac{s^5}{(s^2+a^2)^3}$	
$\frac{t^2\sin at}{2a}$	$\frac{3s^2-a^2}{(s^2+a^2)^3}$	
$\frac{1}{2}t^2\cos at$	$\frac{s^3-3a^2s}{(s^2+a^2)^3}$	
$\frac{1}{6}t^3\cos at$	$\frac{s^4-6a^2s^2+a^4}{(s^2+a^2)^4}$	
$\frac{t^3\sin at}{24a}$	$\frac{s^3-a^2s}{(s^2+a^2)^4}$	
$\frac{3+a^2t^2\sinh at-3at\cosh at}{8a^5}$	$\frac{1}{(s^2-a^2)^3}$	
$\frac{at^2\cosh at-t\sinh at}{8a^3}$	$\frac{s}{(s^2-a^2)^3}$	
$\frac{at\cosh at+(a^2t^2-1)\sinh at}{8a^3}$	$\frac{s^2}{(s^2-a^2)^3}$	
$\frac{3t\sinh at+at^2\cosh at}{8a}$	$\frac{s^3}{(s^2-a^2)^3}$	
$\frac{(3+a^2t^2)\sinh at+5at\cosh at}{8a}$	$\frac{s^4}{(s^2-a^2)^3}$	
$\frac{(8+a^2t^2)\cosh at+7at\sinh at}{8}$	$\frac{s^5}{(s^2-a^2)^3}$	
$\frac{t^2\sinh at}{2a}$	$\frac{3s^2+a^2}{(s^2-a^2)^3}$	
$\frac{1}{2}t^2\cosh at$	$\frac{s^3+3a^2s}{(s^2-a^2)^3}$	
$\frac{1}{6}t^3\cosh at$	$\frac{s^4+6a^2s^2+a^4}{(s^2-a^2)^4}$	
$\frac{t^3\sinh at}{24a}$	$\frac{s^3+a^2s}{(s^2-a^2)^4}$	
$\frac{e^{at/2}}{3a^2}\left(\sqrt{3}\sin\frac{\sqrt{3}at}{2}-\cos\frac{\sqrt{3}at}{2}+e^{-3at/2}\right)$	$\frac{1}{s^3+a^3}$	
$\frac{e^{at/2}}{3a^2}\left(\cos\frac{\sqrt{3}at}{2}+\sqrt{3}\sin\frac{\sqrt{3}at}{2}-e^{-3at/2}\right)$	$\frac{s}{s^3+a^3}$	
$\frac{1}{3}\left(e^{-at}+2e^{at/2}\cos\frac{\sqrt{3}at}{2}\right)$	$\frac{s^2}{s^3+a^3}$	
$\frac{e^{-at/2}}{3a^2}\left(e^{3at/2}-\cos\frac{\sqrt{3}at}{2}-\sqrt{3}\sin\frac{\sqrt{3}at}{2}\right)$	$\frac{1}{s^3-a^3}$	
$\frac{e^{-at/2}}{3a}\left(\sqrt{3}\sin\frac{\sqrt{3}at}{2}-\cos\frac{\sqrt{3}at}{2}+e^{3at/2}\right)$	$\frac{s}{s^3-a^3}$	

Function	Laplace Transform	Domain
$\frac{1}{3}\left(e^{at}+2e^{-at/2}\cos\frac{\sqrt{3}at}{2}\right)$	$\frac{s^2}{s^3-a^3}$	
$\frac{1}{4a^3}\left(\sin at\cosh at-\cos at\sinh at\right)$	$\frac{1}{s^4+4a^4}$	
$\frac{\sin at\sinh at}{2a^2}$	$\frac{s}{s^4+4a^4}$	
$\frac{1}{2a}\left(\sin at\cosh at+\cos at\sinh at\right)$	$\frac{s^2}{s^4+4a^4}$	
$\cos at\cosh at$	$\frac{s^3}{s^4+4a^4}$	
$\frac{1}{2a^3}\left(\sinh at-\sin at\right)$	$\frac{1}{s^4-a^4}$	
$\frac{1}{2a^2}\left(\cosh at-\cos at\right)$	$\frac{s}{s^4-a^4}$	
$\frac{1}{2a}\left(\sinh at+\sin at\right)$	$\frac{s^2}{s^4-a^4}$	
$\frac{1}{2}\left(\cosh at+\cos at\right)$	$\frac{s^3}{s^4-a^4}$	
$\frac{e^{-bt}-e^{-at}}{2(b-a)\sqrt{\pi t^3}}$	$\frac{1}{\sqrt{s+a}+\sqrt{s+b}}$	
$\frac{\operatorname{erf}\sqrt{at}}{\sqrt{a}}$	$\frac{1}{s\sqrt{s+a}}$	
$\frac{e^{at}\operatorname{erf}\sqrt{at}}{\sqrt{a}}$	$\frac{1}{\sqrt{s}(s-a)}$	
$e^{at}\left(\frac{1}{\sqrt{\pi t}}-be^{b^2t}\operatorname{erfc}(b\sqrt{t})\right)$	$\frac{1}{\sqrt{s-a}+b}$	
$J_0(at)$	$\frac{1}{\sqrt{s^2+a^2}}$	
$I_0(at)$	$\frac{1}{\sqrt{s^2-a^2}}$	
$a^nJ_n(at)$	$\frac{\left(\sqrt{s^2+a^2}-s\right)^n}{\sqrt{s^2+a^2}},\quad n>-1$	
$a^nI_n(at)$	$\frac{\left(s-\sqrt{s^2-a^2}\right)^n}{\sqrt{s^2-a^2}},\quad n>-1$	
$J_0(a\sqrt{t(t+2b)})$	$\frac{e^{b\left(s-\sqrt{s^2+a^2}\right)}}{\sqrt{s^2+a^2}}$	
$\begin{cases} J_0(a\sqrt{t^2-b^2}) & t>b \\ 0 & t<b \end{cases}$	$\frac{e^{-b\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}$	
$tJ_0(at)$	$\frac{1}{(s^2+a^2)^{3/2}}$	

Function	Laplace Transform	Domain
$J_0(at) - atJ_1(at)$	$\frac{s^2}{(s^2+a^2)^{3/2}}$	
$\frac{tI_1(at)}{a}$	$\frac{1}{(s^2-a^2)^{3/2}}$	
$I_0(at) + atI_1(at)$	$\frac{s}{(s^2+a^2)^{3/2}}$	
$f(t) = n,\ n \leq t\ < n+1,\ n = 0, 1, 2, \ldots$	$\frac{1}{s(e^s-1)} = \frac{e^{-s}}{s(1-e^{-s})}$	
$f(t) = \sum_{k=1}^{[t]} r^k$	$\frac{1}{s(e^s-r)} = \frac{e^{-s}}{s(1-re^{-s})}$	
$f(t) = r^n,\ n \leqq t < n+1,\ n = 0, 1, 2, \ldots$	$\frac{s^s-1}{s(e^s-r)} = \frac{1-e^{-s}}{s(1-re^{-s})}$	
$\frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$	$\frac{s^{-a/s}}{\sqrt{s}}$	
$\frac{\sin 2\sqrt{at}}{\sqrt{\pi a}}$	$\frac{e^{-a/s}}{s^{3/2}}$	
$\left(\frac{t}{a}\right)^{n/2} J_n(2\sqrt{at})$	$\frac{e^{-a/s}}{s^n+1} \quad n > -1$	
$\frac{e^{-a^2/4t}}{\sqrt{\pi t}}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	
$\frac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$	
$\operatorname{erf}(a/2\sqrt{t})$	$\frac{1-e^{-a\sqrt{s}}}{s}$	
$\operatorname{erfc}(a/2\sqrt{t})$	$\frac{e^{-a\sqrt{s}}}{s}$	
$e^{b(bt+a)} \operatorname{erfc}\left(b\sqrt{t} + \frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s}+b)}$	
$\frac{1}{\sqrt{\pi t a^{2n+1}}} \int_0^\infty u^n e^{-u^2/4a^2t} J_{2n}(2\sqrt{u})du$	$\frac{e^{-a\sqrt{s}}}{s^{n+1}} \quad n > -1$	
$\frac{e^{-bt}-e^{-at}}{t}$	$\ln\left(\frac{s+a}{s+b}\right)$	
$\operatorname{Ci}(at)$	$\frac{\ln[(s^2+a^2)/a^2]}{2s}$	
$Ei(at)$	$\frac{\ln[(s+a)/a]}{s}$	
$\ln t$	$-\frac{(\gamma+\ln s)}{s}\ (\gamma = \text{Euler const})$	

Function	Laplace Transform	Domain
$\frac{2(\cos at-\cos bt)}{t}$	$\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)$	
$\ln^2 t$	$\frac{\pi^2}{6s}+\frac{(\gamma+\ln s)^2}{s}$ $(\gamma=\text{Euler constant})$	
$-(\ln t+\gamma)$ $(\gamma=\text{Euler const})$	$\frac{\ln s}{s}$	
$(\ln t+\gamma)^2-\frac{1}{6}\pi^2$ $(\gamma=\text{Euler const})$	$\frac{\ln^2 s}{s}$	
$t^n\ln t$	$\frac{\Gamma'(n+1)-\Gamma(n+1)\ln s}{s^{n+1}}$ $n>-1$	
$\frac{t^{n-1}}{\Gamma(n)}$	$\frac{1}{s^n}, n>0$	
$\frac{\sin at}{t}$	$\arctan(a/s)$	
$\mathrm{Si}(at)$	$\frac{\arctan(a/s)}{s}$	
$\frac{e^{-2\sqrt{at}}}{\sqrt{\pi t}}$	$\frac{e^{a/s}}{\sqrt{s}}$ $\mathrm{erfc}(\sqrt{a/s})$	
$\frac{2a}{\sqrt{\pi}}e^{-a^2t^2}$	$e^{s^2/4a^2}$ $\mathrm{erfc}(s/2a)$	
$\mathrm{erf}(at)$	$\frac{e^{s^2/4a^2}\ \mathrm{erfc}(s/2a)}{s}$	
$\frac{1}{\sqrt{\pi(t+a)}}$	$\frac{e^{as}\,\mathrm{erfc}\sqrt{as}}{\sqrt{s}}$	
$\frac{1}{t+a}$	$e^{as}Ei(as)$	
$\frac{1}{t^2+a^2}$	$\frac{1}{a}\left[\cos as\left(\frac{\pi}{2}-\mathrm{Si}(as)\right)-\sin as\ \ \mathrm{Ci}(as)\right]$	
$\frac{t}{t^2+a^2}$	$\sin as\left(\frac{\pi}{2}-\mathrm{Si}(as)\right)+\cos as\ \ \mathrm{Ci}(as)$	
$\arctan(t/a)$	$\frac{\cos as\left(\frac{\pi}{2}-\mathrm{Si}(as)\right)-\sin as\ \ \mathrm{Ci}(as)}{s}$	
$\frac{1}{2}\ln\left(\frac{t^2+a^2}{a^2}\right)$	$\frac{\sin as\left(\frac{\pi}{2}-\mathrm{Si}(as)\right)+\cos as\ \ \mathrm{Ci}(as)}{s}$	
$\frac{1}{t}\ln\left(\frac{t^2+a^2}{a^2}\right)$	$\left[\frac{\pi}{2}-\mathrm{Si}(as)\right]^2+\mathrm{Ci}^2(as)$	
$\delta(t)$	1	

Function	Laplace Transform	Domain
$\delta(t-a)$	e^{-as}	
$\mu(t-a)$	$\frac{e^{-as}}{s}$	
$\int_0^\infty J_0(2\sqrt{ut})f(u)du$	$\frac{1}{s}F\left(\frac{1}{s}\right)$	
$t^{\frac{n}{2}}\int_0^\infty u^{-\frac{n}{2}}J_n(2\sqrt{ut})f(u)du$	$\frac{1}{s^{n+1}}F\left(\frac{1}{s}\right)$	
$\int_0^t J_0(2\sqrt{u(t-u)})f(u)du$	$\frac{F(s+\frac{1}{s})}{s^2+1}$	
$f(t^2)$	$\frac{1}{2\sqrt{\pi}}\int_0^\infty u^{-\frac{3}{2}}e^{-\frac{s^2}{4u}}F(u)du$	
$\int_0^\infty \frac{t^u f(u)}{\Gamma(u+1)}du$	$\frac{F(\ln s)}{s\ln s}$	
$\sum_{k=1}^{N}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$	$\frac{P(s)}{Q(s)}$	
$\delta(t-T)$	e^{-sT}	All

Exercises for chapter 6

Guidance: A link at the end of the list leads to a Jupyter notebook containing final and complete solutions to some of the exercises. Additional solutions will be added over time (I hope the students will help). We would be very happy to receive additional solutions from students for exercises that have not yet been solved!

1. Does the function $f(t) = e^{t^2}$ have a Laplace transform? If it does, find it and determine where it is defined. If not, explain why it does not exist.

2. Find the Laplace transform of $f(t) = \begin{cases} \sin t, & 0 \le t \le 2\pi \\ 0, & 2\pi < t \end{cases}$.

3. Prove that if $f : [0, \infty) \to \mathbb{C}$ is piecewise continuous and periodic with period p $(p > 0)$, then

$$\mathscr{L}[f](s) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$$

4. Let $f : [0, \infty) \to \mathbb{R}$ be a function of period 4 such that-

$$f(t) = \begin{cases} 1, & 0 \le t \le 2 \\ -1, & 2 < t \le 4 \end{cases}$$

Prove that
$$\mathscr{L}\left[f\right](s)=\frac{\tanh s}{s}$$

5. Compute the Laplace transform of each of the following functions:

(o) $e^{-t}\cos 2t$

(p) $e^{-4t}\cosh 2t$

(q) $(t^2+1)^2$

(r) $3\cosh t-4\sinh 5t$

(s) $t^n\sin t$

(t) $\dfrac{1}{(t-a)(t-b)}$

(u) $\dfrac{3}{2t-1}$

6. Express each function in terms of Heaviside-type functions, and then find the Laplace transforms of:

(a) $f(t)=\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<2\\ 1, & 2\le t<3\\ 0, & 3\le t<\infty\end{cases}$

(b) $f(t)=\begin{cases}1, & 0\le t<2\\ (t-2)^2, & 2\le t<\infty\end{cases}$

(c) $f(t)=\begin{cases}1, & 0\le t<1\\ t^2-2t+2, & 1\le t<\infty\end{cases}$

(d) $f(t)=\begin{cases}0, & 0\le t<\pi\\ t-\pi, & \pi\le t<2\pi\\ 1, & 2\pi\le t<\infty\end{cases}$

(e) $f(t)=\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<\infty\end{cases}$

(f) $f(t)=\begin{cases}1, & 0\le t<2\\ t-3, & 2\le t<3\\ -1, & 3\le t<\infty\end{cases}$

7. Compute the Laplace transform of $f(t)=\dfrac{\sin t}{t}$. (Hint: compute $\frac{d}{ds}\mathscr{L}\left[f\right](s)$.)

8. Find f if it is known that $\mathscr{L}\left[f\right](s)=\dfrac{d^n}{ds^n}\left[\dfrac{1}{s^2-a^2}\right]$, $\quad a>0$.

9. Prove each of the following formulas:

(a) $\mathscr{L}\left[\sin^2 t\right](s)=\dfrac{2}{s(s^2+4)}$

(b) $\mathscr{L}\left[A\cos(\omega t+\theta)\right](s)=\dfrac{A(s\cos\theta-\omega\sin\theta)}{s^2+\omega^2}$

(c) $\mathscr{L}\left[\cos at\cosh at\right](s)=\dfrac{s^3}{s^4+4a^4}$

(d) $\mathscr{L}\left[(t^2-5t+6)e^{2t}\right] = \dfrac{6s^2-29s+36}{(s-2)^3}$

10. Using the definition and properties of the Laplace transform, compute each of the following integrals:

(a) $\displaystyle\int_0^\infty te^{-2t}\cos t\,dt$

(b) $\displaystyle\int_0^\infty t^3e^{-t}\sin t\,dt$

(c) $\displaystyle\int_0^\infty x^4e^{-x}dx$

(d) $\displaystyle\int_0^\infty x^6e^{-3x}dx$

11. For every $t \geq 0$, let $f(t) = \displaystyle\int_0^t \frac{\sin u}{u}\,du$. Prove that

$$\mathscr{L}\left[f\right](s) = \frac{1}{s}\arctan\frac{1}{s}$$

12. For every $t \geq 0$, let $f(t) = \displaystyle\int_t^\infty \frac{\cos u}{u}\,du$. Prove that

$$\mathscr{L}\left[f\right](s) = \frac{\ln(1+s^2)}{2s}$$

13. For every $t \geq 0$, let $f(t) = \displaystyle\int_t^\infty \frac{e^{-u}}{u}\,du$. Prove that

$$\mathscr{L}\left[f\right](s) = \frac{\ln(1+s)}{s}$$

14. For every $t \geq 0$, let $f(t) = \displaystyle\int_0^t \frac{1-e^{-u}}{u}\,du$. Prove that

$$\mathscr{L}\left[f\right](s) = \frac{1}{s}\ln\left(1+\frac{1}{s}\right)$$

15. Compute the inverse Laplace transform of each of the following functions:

(a) $\dfrac{3s-14}{s^2-4s+18}$

(b) $\dfrac{1}{s^2(s^2+1)}$

(c) $\dfrac{1}{s^2-3s+2}$

(d) $\dfrac{s^2}{(s^2+4)^2}$

(e) $\dfrac{s}{(s^2+4)^2}$

(f) $\ln\left(1+\dfrac{1}{s^2}\right)$

(g) $\dfrac{1}{(s^2+4)^2}$

(h) $\dfrac{s^3}{s^4-16}$

(i) $\dfrac{1}{s^3+1}$ **(j)** $\dfrac{2s^2+s-10}{(s-4)(s^2+2s+2)}$

16. For each of the following differential equations, find a particular solution on the interval $[0,\infty)$ that satisfies the accompanying initial conditions.

(a) $y''+3y'-4y=0, \quad y(0)=3, \quad y'(0)=-2$
(b) $y''-y=0, \quad y(0)=1, \quad y'(0)=-1$
(c) $y''+4y'+4y=0, \quad y(0)=1, \quad y'(0)=0$
(d) $y''+4y=0, \quad y(0)=2, \quad y'(0)=1$
(e) $y''+4y'+7y=0, \quad y(0)=1, \quad y'(0)=1$
(f) $ty''(t)+2y'(t)+ty(t)=0, \quad y(0)=1, \quad y'(0)=0$

17. Compute the Laplace transform of each of the following functions:

(a) $f(t)=\begin{cases} 1, & 0\le t\le T \\ 0, & t>T \end{cases}$ **(b)** $f(t)=\begin{cases} 0, & 0\le t\le 1 \\ (t-1)^2, & t>1 \end{cases}$

(c) $f(t)=\begin{cases} t^2, & 0\le t<2 \\ 4t, & t\ge 2 \end{cases}$ **(d)** $f(t)=\begin{cases} t, & 0\le t\le 7 \\ 3-t, & 7<t<8 \\ 1, & 8\le t \end{cases}$

(e) $f(t)=\begin{cases} 5, & a\le t<b \\ 0, & t<a \text{ or } t\ge b \end{cases}$ **(f)** $f(t)=\begin{cases} 0, & 0\le t<1 \\ 1, & 1\le t<4 \\ 4, & t\ge 4 \end{cases}$

(g) $f(t)=\begin{cases} 0, & 0\le t<1 \\ t-1, & 1\le t<3 \\ 8-2t, & 3\le t<4 \\ 0, & 4\le t \end{cases}$ **(h)** $u_1(t)+3u_5(t)$

(i) $f(t)=\begin{cases} \cos 2t, & 0\le t<\pi \\ 0, & \pi\le t<\infty \end{cases}$

18. Compute the inverse Laplace transform of each of the following functions.

(a) $\dfrac{e^{-s}(1-e^{-s})}{s(s^2+1)}$ **(b)** $\dfrac{e^{-16s}}{s(s^2+2s+4)}$

19. For each of the following differential equations, find a particular solution on the interval $[0,\infty)$ that satisfies the accompanying initial conditions.

(a) $y''(t)+4y'(t)+7y(t)=u_1(t), \quad y(0)=1, \quad y'(0)=0$
(b) $y''(t)-2y'(t)+y(t)=(-1)^{[t]}, \quad y(0)=0, \quad y'(0)=0$
(c) $y'''(t)-y(t)=h(t), \quad y(0)=y'(0)=0, \quad y''(0)=1$

where

$$h(t) = \begin{cases} 1, & \pi \le t \le 2\pi \\ 0, & \text{esiwrehto} \end{cases}$$

20. Compute the Laplace transform of the function

$$f(t) = \int_0^t (u^2 - u + e^{-u})\, du$$

21. For each of the following differential equations, find a particular solution on the interval $[0, \infty)$ that satisfies the accompanying initial conditions.

(a) $y''(t) + y(t) = g(t), \qquad y(0) = 0, \quad y'(0) = 0$
where

$$g(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t \end{cases}$$

(b) $y''(t) + 2y'(t) - 3y(t) = f(t), \qquad y(0) = 1, \quad y'(0) = 0$
where

$$f(t) = \begin{cases} 0, & 0 \le t \le 2\pi \\ \sin t, & t > 2\pi \end{cases}$$

(c) $y'''(t) - y''(t) + 4y'(t) - 4y(t) = 68e^x \sin 2x\,,$
$y(0) = 1, \quad y'(0) = -19, \quad y''(0) = -37$

(d) $y''(t) + 9y(t) = f_c(t), \qquad y(0) = a, \quad y'(0) = b$
where, for $c > 0$,

$$f_c(t) = \begin{cases} 0, & 0 \le t \le c \\ t - c, & c < t \end{cases}$$

(e) $y'' - 2y' + 2y = \cos t\,,\ y(0) = 1\,,\ y'(0) = 0$

22. Solve the following initial-value problems using the Laplace transform:

(a) $\begin{cases} y'' + 2y' + y = 4e^t \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$

(b) $\begin{cases} y'' + y = u_{3\pi}(t) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

(c) $\begin{cases} y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

(d) $\begin{cases} y'' + y' + \frac{5}{4}y = t - u_{\frac{\pi}{2}}(t)(t - \frac{\pi}{2}) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

(e) $\begin{cases} y'' + y = \begin{cases} 1, & 0 \le t < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le t < \infty \end{cases} \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

(f) $\begin{cases} y'' - 6y' + 15y = 2\sin 3t \\ y(0) = -1 \\ y'(0) = -4 \end{cases}$

(g) $\begin{cases} y'' - y' - 2y = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t \end{cases} \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$

(h) $\begin{cases} y'' - y = \begin{cases} 0, & 0 \le t < \frac{\pi}{2} \\ \sin t, & \frac{\pi}{2} \le t \end{cases} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

(i) $\begin{cases} y'' + 3y' + 2y = \begin{cases} 1, & 0 \le t < 10 \\ 0, & 10 \le t \end{cases} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$

(j) $\begin{cases} y^{(4)} - y = u_1(t) - u_2(t) \\ y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 0 \\ y'''(0) = 0 \end{cases}$

23. Let $y_1(t)$ be the solution of the problem

$$y''(t) - 4y'(t) + 4y(t) = f_1(t), \qquad y(0) = 1, \quad y'(0) = 1, \quad (t \ge 0)$$

and let $y_2(t)$ be the solution of the problem

$$y''(t) - 4y'(t) + 4y(t) = f_2(t), \qquad y(0) = 1, \quad y'(0) = 1, \quad (t \geq 0)$$

where it is given that- $f_1(t) \leq f_2(t)$ for every $t \geq 0$, and that the functions f_1 and- f_2 are continuous and bounded on the interval $[0, \infty)$. Prove that $y_1(t) \leq y_2(t)$ for every $t \geq 0$.

24. Using the inverse-transform formula, compute

(a) $\mathscr{L}^{-1}\left[\dfrac{\cosh a\sqrt{s}}{s\cosh\sqrt{s}}\right]$, $0 < a < 1$

(b) $\mathscr{L}^{-1}\left[\dfrac{1}{(1+s^2)^2}\right]$

(c) $\mathscr{L}^{-1}\left[\dfrac{1}{s^3(1+s^2)}\right]$

(d) $\mathscr{L}^{-1}\left[\dfrac{s}{(1+s^2)^3}\right]$

(e) $\mathscr{L}^{-1}\left[\dfrac{1}{s\sqrt{1+s}}\right]$

(f) $\mathscr{L}^{-1}\left[\dfrac{\sqrt{s}}{s-1}\right]$

(g) $\mathscr{L}^{-1}\left[\dfrac{1}{s(e^s+1)}\right]$

(h) $\mathscr{L}^{-1}\left[\dfrac{1}{(s+2)^2(s^2+4)}\right]$

Link to solutions

Chapter 7

Systems of Linear Equations

- Systems of differential equations arise in physics, electronics, population interactions in ecology, biology, the social sciences, and other fields. A system of differential equations is a mathematical method for representing and constructing a mathematical model of systems in which a collection of quantities (functions) depend on and influence one another according to physical or other laws.
- For example, to construct a model of a complex electrical circuit, the currents, voltages, capacitances, inductances (inductance), and resistances across the many components of the circuit must be represented by suitable mathematical functions, and the relationships among them must be described by differential equations that may contain the derivatives of all the different functions in the model.

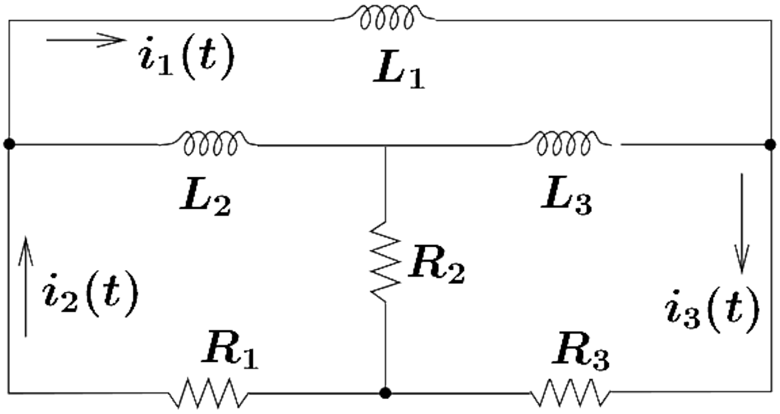

Fig. 7.1: Electrical circuit

- A system of equations of this type may, for example, look like this

$$\begin{aligned}
i_1'(t) &= -\frac{R_2}{L_1} i_2(t) - \frac{R_3}{L_1} i_3(t) \\
i_2'(t) &= -\left(\frac{R_2}{L_2} + \frac{R_2}{L_1}\right) i_2(t) - \left(\frac{R_1}{L_2} - \frac{R_3}{L_1}\right) i_3(t) \\
i_3'(t) &= -\left(\frac{R_1}{L_3} - \frac{R_2}{L_1}\right) i_2(t) - \left(\frac{R_1}{L_3} - \frac{R_3}{L_1} + \frac{R_3}{L_3}\right) i_3(t)
\end{aligned}$$

where the unknown functions $i_1(t)$, $i_2(t)$, $i_3(t)$ denote the intensities of the electric currents flowing through different wires in the circuit as functions of time t.

- We present several more examples of first-order systems, while indicating the type of each system **in its normal form**. In most cases, the system will consist of n first-order differential equations in n unknown functions: $x_1(t)$, $x_2(t)$, $\ldots$, $x_n(t)$, where t is usually the time variable.

- **Homogeneous System with Constant Coefficients:**

$$\begin{cases} x_1'(t) &= 5x_1(t) - 3x_2(t) \\ x_2'(t) &= x_1(t) + 4x_2(t) \end{cases}$$

- **Homogeneous System with Variable Coefficients:**

$$\begin{cases} x_1'(t) &= t^2x_1(t) - 3tx_2(t) \\ x_2'(t) &= (1-t)x_1(t) + t^7x_2(t) \end{cases}$$

- **Nonhomogeneous System with Constant Coefficients:**

$$\begin{cases} x_1'(t) &= 5x_1(t) - 3x_2(t) + \cos t \\ x_2'(t) &= x_1(t) + 4x_2(t) - e^t \end{cases}$$

- An initial-value problem requires n initial conditions (here $n = 2$)

$$\begin{cases} x_1'(t) &= 5x_1(t) - 3x_2(t) \\ x_2'(t) &= x_1(t) + 4x_2(t) \end{cases} \quad \Bigg| \quad \begin{aligned} x_1(0) &= 8 \\ x_2(0) &= -6 \end{aligned}$$

- **An Example of an Ecological System**
  - $x_1(t)$ will denote the number of rabbits at time t in a given forest
  - $x_2(t)$ will denote the number of foxes in the forest
  - It turns out that the two functions $x_1(t)$, $x_2(t)$ satisfy the system

$$\begin{cases} x_1'(t) &= a_1x_1(t) - b_1x_1(t)x_2(t) \\ x_2'(t) &= b_2x_1(t)x_2(t) - a_2x_2(t) \end{cases}$$

  This is, of course, a nonlinear system.
  - a_1 is the growth rate of the rabbit population
  - a_2 is the mortality rate of the fox population
  - b_1, b_2 are interaction constants

A Physical Example: Motion of a Particle in the 2D Plane

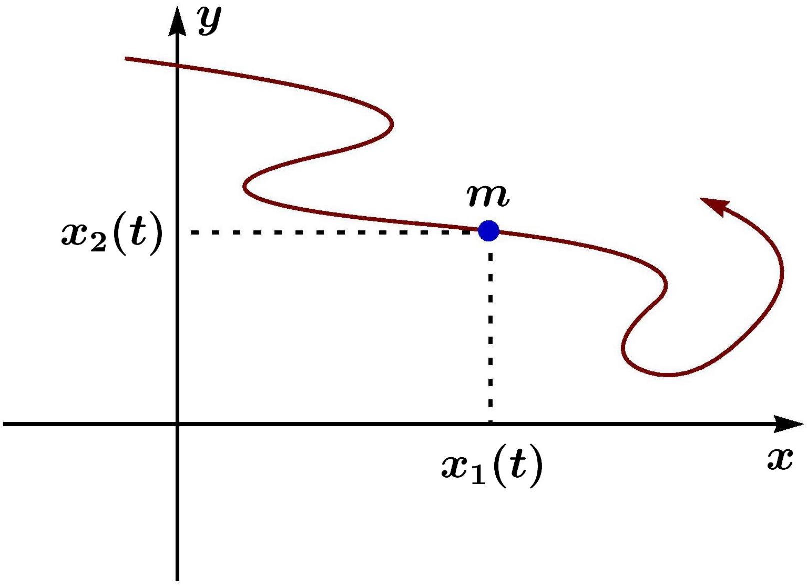

Fig. 7.2: Motion of a particle of mass m in the plane

- $x_1(t)$ will denote the coordinate of the particle m on the x-axis at time t
- $x_2(t)$ will denote the coordinate of the particle m on the y-axis at time t
- The position functions $x_1(t)$, $x_2(t)$ of the particle satisfy the system of second-order differential equations

$$\begin{cases} mx_1''(t) &= F_1(t, x_1(t), x_2(t), x_1'(t), x_2'(t)) \\ mx_2''(t) &= F_2(t, x_1(t), x_2(t), x_1'(t), x_2'(t)) \end{cases}$$

- F_1 is the horizontal force acting on the particle m, depending on the time, position, and instantaneous velocity of m at time t
- F_2 is the vertical force acting on the particle m, depending on the time, position, and instantaneous velocity of m at time t
- The system in fact describes the decomposition of Newton's second law into a vertical component and a horizontal component.

In the same way, we can model the motion of a particle with mass in three-dimensional space $\mathbb{R}^3$ by means of three functions $x_1(t)$, $x_2(t)$, $x_3(t)$.

7.1 The Existence and Uniqueness Theorem for Systems of First-Order Differential Equations

Theorem 7.1: Consider a system of n differential equations in n unknown functions $x_1(t)$, $x_2(t)$, $\dots$, $x_n(t)$, **in the following normal form**

$$\left\{\begin{array}{rcl} x_1'(t) & = & F_1(t, x_1, x_2, \dots, x_n) \\ x_2'(t) & = & F_2(t, x_1, x_2, \dots, x_n) \\ \vdots & \vdots & \vdots \\ x_n'(t) & = & F_n(t, x_1, x_2, \dots, x_n) \end{array}\right. \quad \left|\quad \begin{array}{rcl} x_1(t_0) & = & \alpha_1 \\ x_2(t_0) & = & \alpha_2 \\ \vdots & \vdots & \vdots \\ x_n(t_0) & = & \alpha_n \end{array}\right.$$

where

A. F_1, F_2, $\dots$, F_n are functions of $n+1$ variables whose first partial derivatives $\frac{\partial F_i}{\partial x_j}$ all exist and are continuous on an open $n+1$-dimensional domain $\mathscr{D}$ in the space $\mathbb{R}^{n+1}$ (differentiability with respect to the variable t is not required).

B. The point $(t_0, x_1(t_0), x_2(t_0), \dots, x_n(t_0))$ is an interior point of the domain $\mathscr{D}$.

Then there exists a neighborhood (α, β) of the point t_0 in which there exists a unique solution $(x_1(t), x_2(t), \dots, x_n(t))$ that satisfies both the system of equations and the initial conditions.

Remark: The existence and uniqueness theorem for ordinary differential equations of order 1 is the special case ($n = 1$) of the theorem above.

The normal form of a first-order **nonhomogeneous linear system** with initial conditions is

$$\left\{\begin{array}{rcl} x_1'(t) & = & p_{11}(t)x_1(t) + p_{12}(t)x_2(t) + \cdots + p_{1n}(t)x_n(t) + q_1(t) \\ x_2'(t) & = & p_{21}(t)x_1(t) + p_{22}(t)x_2(t) + \cdots + p_{2n}(t)x_n(t) + q_2(t) \\ \vdots & \vdots & \vdots \\ x_n'(t) & = & p_{n1}(t)x_1(t) + p_{n2}(t)x_2(t) + \cdots + p_{nn}(t)x_n(t) + q_n(t) \end{array}\right. \quad \left|\quad \begin{array}{rcl} x_1(t_0) & = & \alpha_1 \\ x_2(t_0) & = & \alpha_2 \\ \vdots & \vdots & \vdots \\ x_n(t_0) & = & \alpha_n \end{array}\right.$$

where all the functions $p_{ij}(t)$, $q_i(t)$ are defined and continuous on an open interval (α, β), and the point t_0 belongs to the interval. If all the functions $q_i(t)$ vanish, we obtain the normal form of a **homogeneous linear system**

$$\left\{\begin{array}{rcl|rcl} x_1'(t) & = & p_{11}(t)x_1(t)+p_{12}(t)x_2(t)+\cdots+p_{1n}(t)x_n(t) & x_1(t_0) & = & \alpha_1 \\ x_2'(t) & = & p_{21}(t)x_1(t)+p_{22}(t)x_2(t)+\cdots+p_{2n}(t)x_n(t) & x_2(t_0) & = & \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n'(t) & = & p_{n1}(t)x_1(t)+p_{n2}(t)x_2(t)+\cdots+p_{nn}(t)x_n(t) & x_n(t_0) & = & \alpha_n \end{array}\right.$$

7.1.1 The Existence and Uniqueness Theorem for First-Order Linear Systems

Theorem 7.2: If all the functions $p_{ij}(t)$, $q_i(t)$ are defined and continuous on an open interval (α, β) that contains the point t_0, then there exists a unique solution

$$x_1(t),\ x_2(t),\ \ldots,\ x_n(t)$$

that satisfies the system throughout the entire domain (α, β), and satisfies all the initial conditions.

Note that, in contrast to the general existence and uniqueness theorem, for linear systems, the existence and uniqueness of the solution are guaranteed on the **eritne** domain, rather than on a subdomain.

7.1.2 Systems of Higher-Order Equations

We note that, unlike ordinary differential equations, the treatment of first-order differential systems is sufficiently general, since every system of higher-order equations can easily be converted into a first-order system. We explain the idea by means of an example. For example, the following second-order system

$$\left\{\begin{array}{rcl} x_1''(t) & = & x_1(t)+2x_2(t)-3x_1'(t)+4x_2'(t) \\ x_2''(t) & = & 5x_1(t)-6x_2(t)+7x_1'(t)+8x_2'(t) \end{array}\right.$$

can be converted into a first-order system by adding two more functions $x_3(t)$, $x_4(t)$, and adding two equations, as follows

$$\left\{\begin{array}{rcl} x_1'(t) & = & x_3(t) \\ x_2'(t) & = & x_4(t) \\ x_3'(t) & = & x_1(t)+2x_2(t)-3x_3(t)+4x_4(t) \\ x_4'(t) & = & 5x_1(t)-6x_2(t)+7x_3(t)+8x_4(t) \end{array}\right.$$

Therefore, focusing on first-order differential systems is justified and entirely sufficient for an introductory course in differential equations.

Exercise 7.1: Using the preceding procedure, convert an ordinary linear differential equation of order

n

$$y^{(n)} + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_0(t)y = q(x)$$

into an $n \times n$ linear system of order 1.

7.1.3 Vector Notation

By using suitable theorems from linear algebra, we can investigate systems of linear differential equations by means of tools from linear algebra. We begin with the vector notation for a nonhomogeneous linear system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix}$$

We will also use the vector representation

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \vec{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix}, \quad P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix}$$

We can also encode the initial conditions

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

by the vector equality

$$\vec{x}(t_0) = \vec{\alpha}$$

Therefore, our system reduces to the following vector system

$$\begin{cases} \vec{x}'(t) = P(t)\vec{x}(t) + \vec{q}(t) \\ \vec{x}(t_0) = \vec{\alpha} \end{cases}$$

7.1.4 Solving a Linear System by the Elimination Method

In the elimination method, one tries to express one of the unknown functions in terms of the remaining functions, thereby reducing the system. A side effect of this method is that the order of differentiation may increase.

We present a simple solution method by means of the following example

$$\begin{cases} x_1'(t) &=& x_1(t) + x_2(t) \\ x_2'(t) &=& 4x_1(t) - 2x_2(t) \end{cases}$$

The first equation implies that $x_2(t) = x_1'(t) - x_1(t)$, and therefore

$$x_2'(t) = x_1''(t) - x_1'(t)$$

We replace $x_2(t)$ and $x_2'(t)$ in the second equation by the corresponding expressions

$$x_1''(t) - x_1'(t) = 4x_1(t) - 2[x_1'(t) - x_1(t)]$$

We have obtained an ordinary second-order linear differential equation

$$x_1''(t) + x_1'(t) - 6x_1(t) = 0$$

The roots of the characteristic polynomial $r^2 + r - 6r = 0$ are $r = -3, 2$ and therefore

$$\begin{array}{rcl} x_1(t) &=& c_1e^{-3t} + c_2e^{2t} \\ x_2(t) &=& x_1'(t) - x_1(t) \\ &=& -3c_1e^{-3t} + 2c_2e^{2t} - c_1e^{-3t} - c_2e^{2t} \\ &=& -4c_1e^{-3t} + c_2e^{2t} \end{array}$$

It is customary to write the general solution of the system as a linear combination of two solutions in vector form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-3t} \\ -4e^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

This notation suggests a possible connection between systems of differential equations and solution methods from linear algebra.

7.2 Homogeneous Linear System with Constant Coefficients

The general form of a homogeneous linear system with constant coefficients is

$$\begin{cases} x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ \vdots & \vdots \quad\quad \vdots \\ x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) \end{cases}$$

where a_{ij} are real constants. The vector representation of the system is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

We denote the **coefficient matrix** by A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

and then we can represent our system by

$$\vec{x}'(t) = A\vec{x}$$

A vector of n functions $\vec{u}(t)$

$$\vec{u}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is called a **solution of the system** if it satisfies

$$\vec{u}'(t) = A\vec{u}(t)$$

for every real t (the domain of existence of a system with constant coefficients is $\mathbb{R}$).

In the following theorem, we prove that a **linear combination** of any two solutions is also a solution

Theorem 7.3: If $\vec{u}(t)$, $\vec{v}(t)$ are two solutions of the system

$$\vec{x}'(t) = A\vec{x}$$

then the linear combination $c_1\vec{u} + c_2\vec{v}$ is also a solution of the system.

Proof: We note that proving this theorem in the original notation may be long and tedious. Vector notation allows us to prove assertions of this type very easily. We rely, of course, on suitable theorems from linear algebra

$$A(c_1\vec{u} + c_2\vec{v}) = c_1A\vec{u} + c_2A\vec{v} = c_1\vec{u}'(t) + c_2\vec{v}'(t) = \frac{d}{dt}[c_1\vec{u}(t) + c_2\vec{v}(t)]$$

We have therefore obtained that

$$\frac{d}{dt}[c_1\vec{u}(t) + c_2\vec{v}(t)] = A(c_1\vec{u} + c_2\vec{u})$$

and therefore the linear combination $c_1\vec{u} + c_2\vec{u}$ is also a solution of the system. ∎

For example, in the preceding example,

$$\begin{cases} x_1'(t) &= x_1(t) + x_2(t) \\ x_2'(t) &= 4x_1(t) - 2x_2(t) \end{cases}$$

we can write the system in vector form

$$\begin{bmatrix} x_1't) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Above, we found the general solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1e^{-3t} + c_2e^{2t} \\ -4c_1e^{-3t} + c_2e^{2t} \end{bmatrix}$$

It can be represented as a linear combination of two fundamental solutions

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-3t} \\ -4e^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

The question is whether this is the general solution of the system. We will prove later that the solution space of the system is a vector space of dimension 2 (the size of the system). Since the two vectors

$$\vec{u} = \begin{bmatrix} e^{-3t} \\ -4e^{-3t} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

are linearly independent, they form a basis for the solution space, and therefore the solution we found is indeed the general solution.

Without loss of generality, we formulate the theorems and examples that follow in this chapter in terms of a 2×2 or 3×3 system of equations, in order to convey the ideas clearly. They can easily be generalized to $n \times n$ systems of equations.

Theorem 7.4: Let $\vec{u}_1(t)$, $\vec{u}_2(t)$, $\vec{u}_3(t)$ be three linearly independent solutions of the 3×3 homogeneous system of equations

$$\vec{u}'(t) = A\vec{u}(t)$$

on the domain (α, β). Then every solution of the system is a linear combination of the three solutions above. That is, the general solution of the system is given by

$$\vec{u}(t) = c_1\vec{u}_1(t) + c_2\vec{u}_2(t) + c_3\vec{u}_3(t)$$

Suppose that

$$\vec{u}_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \vec{u}_2(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}, \quad \vec{u}_3(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \end{bmatrix}$$

are three solutions of a linear system of equations. Then their linear dependence or independence can be tested by means of the Wronskian

$$W[\vec{u}_1, \vec{u}_2, \vec{u}_3](t) = \begin{vmatrix} x_1(t) & y_1(t) & z_1(t) \\ x_2(t) & y_2(t) & z_2(t) \\ x_3(t) & y_3(t) & z_3(t) \end{vmatrix}$$

If the Wronskian $W[\vec{u}_1, \vec{u}_2, \vec{u}_3](t) \neq 0$, then the vectors $\vec{u}_1(t)$, $\vec{u}_2(t)$, $\vec{u}_3(t)$ are linearly independent.

Theorem 7.5: If $\vec{u}_1(t)$, $\vec{u}_2(t)$, $\vec{u}_3(t)$ are solutions of the system

$$\vec{u}'(t) = A\vec{u}(t)$$

on the domain (α, β), then either the Wronskian $W[\vec{u}_1, \vec{u}_2, \vec{u}_3]$ vanishes identically on the entire domain (α, β), or for every t in the domain (α, β), $W(t) \neq 0$.

7.2.1 Abel's Formula (Niels Henrik Abel 1802-1829)

If $\vec{u}_1(t)$, $\vec{u}_2(t)$, $\vec{u}_3(t)$ is a fundamental set of solutions of a system with variable coefficients, then

$$W[\vec{u}_1, \vec{u}_2, \vec{u}_3](t) = W(t_0)e^{\int_{t_0}^{t}[p_{11}(t)+p_{22}(t)+p_{33}(t)]dt}$$

For a system of equations with constant coefficients, the formula takes the form

$$W[\vec{u}_1, \vec{u}_2, \vec{u}_3](t) = W(0)e^{(a_{11}+a_{22}+a_{33})t}$$

The sum $a_{11} + a_{22} + a_{33}$ is the sum of the diagonal entries of the coefficient matrix A, and is called the trace (trace) of A and is sometimes denoted by $\mathbf{trace}(A)$. It follows from the formula, of course, that the Wronskian does not depend on the particular solutions and is the same for every fundamental set of solutions.

For example, in the following 2×2 system,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

the value of the Wronskian is

$$W = W(0)e^{\int_0^t 8dt} = W(0)e^{8t}$$

We found above that the following two vectors form a fundamental set of solutions

$$\vec{u} = \begin{bmatrix} e^{-3t} \\ -4e^{-3t} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

and therefore

$$W(0) = \begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} = 2$$

and therefore the Wronskian of the system is

$$W = 2e^{8t}$$

Proof for $n = 2$:

Suppose that $\vec{u}$, $\vec{v}$ are two solutions of a homogeneous linear system of equations

$$\vec{u}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \vec{v}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},$$

The Wronskian of $\boldsymbol{u}$, $\boldsymbol{v}$ is

$$W[\vec{u},\vec{v}](t)=\begin{vmatrix} x_1(t) & y_1(t)\\ x_2(t) & y_2(t)\end{vmatrix}=x_1(t)y_2(t)-x_2(t)y_1(t)$$

The derivative of the Wronskian with respect to t is

$$\begin{aligned}
\tfrac{d}{dt}W[\vec{u},\vec{v}] &= [x_1'(t)y_2(t)-x_1(t)y_2'(t)]-[x_2'(t)y_1(t)-x_2(t)y_1'(t)]\\
&= \begin{vmatrix} x_1'(t) & y_1'(t)\\ x_2(t) & y_2(t)\end{vmatrix}+\begin{vmatrix} x_1(t) & y_1(t)\\ x_2'(t) & y_2'(t)\end{vmatrix}\\
&= \begin{vmatrix} a_{11}x_1+a_{12}x_2 & a_{11}y_1+a_{12}y_2\\ x_2 & y_2\end{vmatrix}+\begin{vmatrix} x_1 & y_1\\ a_{21}x_1+a_{22}x_2 & a_{21}y_1+a_{22}y_2\end{vmatrix}\\
&= \begin{vmatrix} a_{11}x_1 & a_{11}y_1\\ x_2 & y_2\end{vmatrix}+\begin{vmatrix} x_1 & y_1\\ a_{22}x_2 & a_{22}y_2\end{vmatrix}
\end{aligned}$$

The last equality was obtained by row operations

$$\begin{array}{lcll} R_1 & \leftarrow & R_1-a_{12}R_2 & \text{(determinant 1)}\\ R_2 & \leftarrow & R_2-a_{21}R_1 & \text{(determinant 2)}\end{array}$$

We have therefore obtained

$$W'=a_{11}\begin{vmatrix} x_1 & y_1\\ x_2 & y_2\end{vmatrix}+a_{22}\begin{vmatrix} x_1 & y_1\\ x_2 & y_2\end{vmatrix}=(a_{11}+a_{22})W$$

We have obtained a separable differential equation

$$\frac{dW}{W}=(a_{11}+a_{22})dt$$

whose general solution is

$$W=W(0)e^{(a_{11}+a_{22})t}$$

We note that in the general case one obtains a separable differential equation

$$\boxed{\frac{dW}{W}=\text{trace}\,(A)\,dt}$$

whose general solution is Abel's formula written above

$$W = e^{\int_{t_0}^{t} \mathrm{trace}(A)dt}$$

Theorem 7.6: Every homogeneous linear system of order $n \times n$

$$\vec{u}'(t) = A\vec{u}(t)$$

on the domain (α, β) has a fundamental set of n solutions.

Proof: We prove the theorem for a 3×3 system of equations. Let t_0 be any point in the domain (α, β). By the existence and uniqueness theorem for linear systems, there exists a solution $\vec{u}_1(t)$ of the system that satisfies the initial conditions

$$\vec{u}_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, there exist solutions $\vec{u}_2(t)$, $\vec{u}_3(t)$ that satisfy

$$\vec{u}_2(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{u}_3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

By the preceding theorem, the set $\vec{u}_1(t)$, $\vec{u}_2(t)$, $\vec{u}_3(t)$ is linearly independent because its Wronskian does not vanish at the point t_0

$$W[\vec{u}_1, \vec{u}_2, \vec{u}_3](t_0) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

If $\vec{u}(t)$ is any solution of the system, set

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{u}(t_0)$$

Set

$$\vec{v}(t) = c_1\vec{u}_1(t) + c_2\vec{u}_2(t) + c_3\vec{u}_3(t)$$

Then

$$\begin{aligned}\vec{v}(t_0) &= c_1\vec{u}_1(t_0) + c_2\vec{u}_2(t_0) + c_3\vec{u}_3(t_0) \\ &= c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}\end{aligned}$$

By the existence and uniqueness theorem, $\vec{u}(t) = \vec{v}(t)$ for every t in the domain (α, β), that is, $\vec{u}$ is a linear combination of the solutions $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, and therefore the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for the solution space of the system. ∎

The preceding theorem proves the theoretical existence of a basis for the solution space of a homogeneous linear system of equations, but gives no indication of how such a basis of solutions can be found. In the next stage of our discussion, we describe a method for finding such a fundamental set of solutions in the case of a homogeneous linear system with constant coefficients.

7.3 Methods for Solving a Homogeneous System with Constant Coefficients

Consider a homogeneous linear system of equations

$$\vec{u}'(t) = A\vec{u}(t) \tag{7.1}$$

where A is a constant $n \times n$ coefficient matrix. For convenience, we focus on the case of a 3×3 system. As in the case of ordinary homogeneous linear equations, we seek solutions of the form

$$\vec{u} = \begin{bmatrix}c_1e^{\lambda t}\\c_2e^{\lambda t}\\c_3e^{\lambda t}\end{bmatrix} = e^{\lambda t}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = e^{\lambda t}\vec{c}$$

We denoted the vector of constants by $\vec{c}$. The derivative of $\vec{u}(t)$ is

$$\vec{u}'(t) = \lambda e^{\lambda t}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \lambda e^{\lambda t}\vec{c}$$

and therefore our system of equations becomes the vector equation

$$A \cdot e^{\lambda t}\vec{c} = \lambda e^{\lambda t}\vec{c}$$

and after canceling the expression $e^{\lambda t}$,

$$A\vec{c} = \lambda\vec{c} \tag{7.2}$$

we obtain a homogeneous linear system of equations

$$(A - \lambda I)\vec{c} = \vec{0} \tag{7.3}$$

where I is the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The system (7.3) is familiar from linear algebra, from the topic of eigenvectors and eigenvalues. By a fundamental theorem of linear algebra, the system (7.3) has a nontrivial solution $\vec{c} \neq \vec{0}$ if and only if

$$|A - \lambda I| = 0 \tag{7.4}$$

The determinant on the left-hand side is precisely the **characteristic polynomial** of the matrix A, and every root λ of the characteristic polynomial is called an **eigenvalue** of the matrix A. Every nontrivial solution $\vec{c}$ of the system (7.3) is called an **eigenvector corresponding to λ**. As noted, each eigenvector $\vec{c}$ gives a fundamental solution $e^{\lambda t}\vec{c}$ of the differential system (7.1). The vectors $\vec{c}$ are the eigenvectors corresponding to the values of λ. By the last theorem, we need exactly three linearly independent eigenvectors in order to obtain a basis for the solution space of the differential system (7.1). We explain this by means of a simple example of a 2×2 system, in which we must find a basis consisting of two eigenvectors.

Example 1: Find the general solution of the system of equations

$$\begin{cases} x_1'(t) &= 2x_1(t) - x_2(t) \\ x_2'(t) &= 3x_1(t) + 6x_2(t) \end{cases}$$

Solution: We pass to vector notation

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We compute the eigenvalues of the coefficient matrix

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) + 3 = \lambda^2 - 8\lambda + 15 = 0$$

The roots of the characteristic polynomial are $\lambda = 3, 5$. We compute the eigenvectors corresponding to these values

$$(A-3I)\vec{c} = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A-5I)\vec{c} = \begin{bmatrix} -3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It is not difficult to find the two eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

We have therefore obtained two solutions of the system

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}, \qquad \vec{u}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{5t}$$

The Wronskian of these two solutions at the point $t = 0$ is

$$W[\vec{u}_1, \vec{u}_2](0) = \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix}$$

It does not vanish, since a set of eigenvectors is always linearly independent (therefore, checking the Wronskian is not necessary, but is recommended as a double check). The general solution of the system is therefore

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ -3e^{5t} \end{bmatrix}$$

Example 2: Find a particular solution of the initial-value problem

$$\left\{ \begin{array}{lcl} x_1'(t) & = & x_1(t) + x_2(t) - x_3(t) \\ x_2'(t) & = & 2x_1(t) + 3x_2(t) - 4x_3(t) \\ x_3'(t) & = & 4x_1(t) + x_2(t) - 4x_3(t) \end{array} \right. \quad \left| \quad \begin{array}{lcl} x_1(0) & = & 5 \\ x_2(0) & = & 10 \\ x_3(0) & = & 15 \end{array} \right.$$

Solution: The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix}$$

The characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 2 & 3-\lambda & -4 \\ 4 & 1 & -4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 3-\lambda & -4 \\ 1 & -4-\lambda \end{vmatrix} - \begin{vmatrix} 2 & -4 \\ 4 & -4-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 3-\lambda \\ 4 & 1 \end{vmatrix}$$

$$= (1-\lambda)[(\lambda-3)(\lambda+4)+4] - (-8-2\lambda+16) - (2-12+4\lambda)$$

$$= (1-\lambda)(\lambda^2+\lambda-8) + 8 + 2\lambda - 16 - 2 + 12 - 4\lambda$$

$$= \lambda^2 + \lambda - 8 - \lambda^3 - \lambda^2 + 8\lambda + 6 - 2\lambda$$

$$= -\lambda^3 + 7\lambda - 6 = 0$$

We have obtained a third-degree polynomial, which we solve by guessing

$$\begin{aligned} \lambda^3 - 7\lambda + 6 &= \lambda^3 - \lambda - 6\lambda + 6 \\ &= \lambda(\lambda^2-1) - 6(\lambda-1) \\ &= \lambda(\lambda+1)(\lambda-1) - 6(\lambda-1) \\ &= (\lambda-1)(\lambda^2+\lambda-6) \\ &= (\lambda-1)(\lambda+3)(\lambda-2) \\ &= 0 \end{aligned}$$

We have obtained three distinct eigenvalues

$$\lambda = 1, \quad 2, \quad -3$$

Finding eigenvectors

$$(A-I)\vec{c} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -4 \\ 4 & 1 & -5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A-2I)\vec{c} = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -4 \\ 4 & 1 & -6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A+3I)\vec{c} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 6 & -4 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using row reduction, we find the eigenvectors

$$\lambda = 1 : \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \lambda = 2 : \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \lambda = -3 : \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}$$

Therefore, the general solution of the system is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} e^{-3t}$$

Finding a particular solution

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$$

We obtain a system of linear equations

$$\begin{cases} c_1 + c_2 + c_3 &= 5 \\ c_1 + 2c_2 + 7c_3 &= 10 \\ c_1 + c_2 + 11c_3 &= 15 \end{cases}$$

Solving by row reduction yields the solution $c_1 = 5$, $c_2 = -1$, $c_3 = 1$. Therefore, the solution of the initial-value problem is

$$\begin{cases} x_1(t) &= 5e^t - e^{2t} + e^{-3t} \\ x_2(t) &= 5e^t - 2e^{2t} + 7e^{-3t} \\ x_3(t) &= 5e^t - e^{2t} + 11e^{-3t} \end{cases}$$

7.3.1 Eigenvalues with Equal Algebraic and Geometric Multiplicities k > 1

In the preceding two examples, we obtained simple eigenvalues. That is, eigenvalues whose algebraic multiplicity is 1. In other cases, we may obtain eigenvalues λ with algebraic multiplicity k greater than one. The geometric multiplicity of an eigenvalue λ is the dimension of its eigenspace. This number ranges from 1 to k.

At this stage, we consider the case in which the geometric multiplicity equals the algebraic multiplicity k. That is, the eigenvalue λ has k linearly independent eigenvectors $\vec{v}_1$, $\vec{v}_2$, $\ldots$, $\vec{v}_k$. In this case, the eigenvalue λ corresponds to k fundamental solutions of the differential system (7.1):

$$\vec{v}_1 e^{\lambda t}, \quad \vec{v}_2 e^{\lambda t}, \quad \ldots, \vec{v}_k e^{\lambda t}$$

Example 3: Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

It is easy to check that the coefficient matrix has exactly two eigenvalues: $\lambda = -1, 2$, where $\lambda = -1$ is an eigenvalue of algebraic multiplicity 2, and its geometric multiplicity is also 2. Therefore, solving the characteristic equations yields three eigenvectors

$$\lambda = -1\colon \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \qquad \lambda = 2\colon \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example 4: In the simplest case, in which the coefficient matrix is the identity matrix,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

there is only one eigenvalue, $\lambda = 1$, of algebraic multiplicity 3, because the characteristic polynomial is

$$(1 - \lambda)^3 = 0$$

But the geometric multiplicity is also 3, since there are three linearly independent eigenvectors corresponding to it

$$\lambda = 1\colon \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

7.3.2 Simple Complex Eigenvalues

At this stage, we consider a simple complex eigenvalue $\lambda = a + bi$ (algebraic multiplicity 1). Since the coefficients of the characteristic polynomial are real, every complex root $\lambda = a + bi$ is accompanied by its conjugate root $\lambda = a - bi$. Therefore, in this case we must obtain a pair of linearly independent eigenvectors corresponding to the pair of roots.

Proposition: If $\vec{c}$ is an eigenvector corresponding to the eigenvalue $\lambda = a + bi$, then the conjugate vector $\bar{\vec{c}}$ is also an eigenvector corresponding to the conjugate eigenvalue $a - bi$.

Proof:

$$\vec{c} = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ a_3 + b_3 i \end{bmatrix} \quad \Longrightarrow \quad \bar{\vec{c}} = \begin{bmatrix} a_1 - b_1 i \\ a_2 - b_2 i \\ a_3 - b_3 i \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} i$$

Therefore

$$A\vec{c} = \lambda\vec{c} \implies \overline{A\vec{c}} = A\bar{\vec{c}} = \bar{\lambda}\bar{\vec{c}}$$

We have obtained $A\bar{\vec{c}} = \bar{\lambda}\bar{\vec{c}}$, and therefore $\bar{\vec{c}}$ is an eigenvector corresponding to $\bar{\lambda}$. ∎

Proposition: Let

$$\vec{u}'(t) = A\vec{u}(t)$$

be a linear system of equations with constant coefficients, and let

$$\vec{c} = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ a_3 + b_3 i \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} i$$

be an eigenvector corresponding to the eigenvalue $\lambda = a + bi$. Then $\vec{c}e^{\lambda t}$ and its conjugate are two complex solutions of the system, whose real and imaginary parts are two laer linearly independent solutions of the system.

Proof: It is not difficult to see that $\vec{c}e^{\lambda t}$ is a complex solution of the system

$$A\vec{c}e^{\lambda t} = \lambda\vec{c}e^{\lambda t} = \frac{d}{dt}\left[\vec{c}e^{\lambda t}\right]$$

We proved above that its conjugate is also a solution. Set

$$\vec{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \vec{\beta} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then $\vec{c} = \vec{\alpha} + i\vec{\beta}$. We compute the real part and the imaginary part of the solution $\vec{c}e^{\lambda t}$

$$\begin{aligned} \vec{c}e^{\lambda t} &= (\vec{\alpha} + \vec{\beta}i)e^{(a+bi)t} \\ &= (\vec{\alpha} + \vec{\beta}i)e^{at}(\cos bt + i\sin bt) \\ &= (\vec{\alpha} + \vec{\beta}i)\left(e^{at}\cos bt + ie^{at}\sin bt\right) \\ &= \left(\vec{\alpha}e^{at}\cos bt - \vec{\beta}e^{at}\sin bt\right) + \left(\vec{\alpha}e^{at}\sin bt + \vec{\beta}e^{at}\cos bt\right) i \end{aligned}$$

In the same way, we see that

$$(\vec{\alpha}-\vec{\beta}i)e^{(a-bi)t}=\left(\vec{\alpha}e^{at}\cos bt-\vec{\beta}e^{at}\sin bt\right)-\left(\vec{\alpha}e^{at}\sin bt+\vec{\beta}e^{at}\cos bt\right)i$$

Set

$$\begin{aligned}\vec{u}(t) &= \vec{\alpha}e^{at}\cos bt-\vec{\beta}e^{at}\sin bt\\ \vec{v}(t) &= \vec{\alpha}e^{at}\sin bt+\vec{\beta}e^{at}\cos bt\end{aligned}$$

The two equalities we proved imply that

$$\begin{aligned}\vec{u}(t) &= \tfrac{1}{2}\left[(\vec{\alpha}+\vec{\beta}i)e^{(a+bi)t}+(\vec{\alpha}-\vec{\beta}i)e^{(a-bi)t}\right]\\ \vec{v}(t) &= \tfrac{1}{2i}\left[(\vec{\alpha}+\vec{\beta}i)e^{(a+bi)t}-(\vec{\alpha}-\vec{\beta}i)e^{(a-bi)t}\right]\end{aligned}$$

That is, $\vec{u}(t)$, $\vec{v}(t)$ are real solutions because they were obtained as linear combinations of the complex solutions. We have therefore obtained two real solutions of the system:

$$\boxed{\begin{aligned}\vec{u}(t) &= \vec{\alpha}e^{at}\cos bt-\vec{\beta}e^{at}\sin bt\\ \vec{v}(t) &= \vec{\alpha}e^{at}\sin bt+\vec{\beta}e^{at}\cos bt\end{aligned}}$$

∎

Remark: It is not advisable to try to memorize the last two formulas. It is enough to remember only the form of the first solution $e^{(a+bi)t}(\vec{\alpha}+i\vec{\beta})$! The last two formulas are simply the real and imaginary parts of this expression (it takes less than half a minute to obtain both formulas in this way).

Example 5: Find the general solution of the system

$$\begin{bmatrix}x_1'(t)\\ x_2'(t)\end{bmatrix}=\begin{bmatrix}4 & -1\\ 5 & 2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Solution: The characteristic polynomial here is

$$\begin{vmatrix}4-\lambda & -1\\ 5 & 2-\lambda\end{vmatrix}=\lambda^2-6\lambda+13=0$$

The roots of the polynomial are $\lambda=3+2i$, $\lambda=3-2i$. According to the analysis above, it is enough to find an eigenvector for $\lambda=3+2i$

$$(A-\lambda I)\vec{c}=\begin{bmatrix}1-2i & -1\\ 5 & -1-2i\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

We obtain the eigenvector

$$\vec{c} = \begin{bmatrix} -1 \\ 1-2i \end{bmatrix}$$

The corresponding complex solution is

$$\vec{u}(t) = \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{(3+2i)t}$$

We decompose it into its real and imaginary parts

$$\begin{aligned} \begin{bmatrix} 1 \\ 1-2i \end{bmatrix} e^{(3+2i)t} &= \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} i \right] \cdot e^{3t}(\cos 2t + i \sin 2t) \\ &= \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \cos 2t - \begin{bmatrix} 0 \\ -2 \end{bmatrix} e^{3t} \sin 2t \right] + i \left[\begin{bmatrix} 0 \\ -2 \end{bmatrix} e^{3t} \cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \sin 2t \right] \end{aligned}$$

and we have obtained the two linearly independent real solutions

$$\vec{u}_1(t) = \begin{bmatrix} e^{3t} \cos 2t \\ e^{3t} \cos 2t - 2e^{3t} \sin 2t \end{bmatrix}, \quad \vec{u}_2(t) = \begin{bmatrix} e^{3t} \sin 2t \\ -2e^{3t} \cos 2t + e^{3t} \sin 2t \end{bmatrix}$$

The general solution of the system is, of course, the space spanned by $\{\vec{u}_1(t),\ \vec{u}_2(t)\}$

$$\vec{u}(t) = c_1 \vec{u}_1(t) + c_2 \vec{u}_2(t) = c_1 \begin{bmatrix} e^{3t} \cos 2t \\ e^{3t} \cos 2t - 2e^{3t} \sin 2t \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \sin 2t \\ -2e^{3t} \cos 2t + e^{3t} \sin 2t \end{bmatrix}$$

7.3.3 A Real Eigenvalue of Algebraic Multiplicity 2 and Geometric Multiplicity 1

Here we consider the case in which the eigenvalue λ has algebraic multiplicity 2 but geometric multiplicity 1. That is, we have a single eigenvector $\vec{a}$ corresponding to the eigenvalue λ, but we must obtain two fundamental solutions for our differential system. For this purpose, we use the concept of a **generalized eigenvector**.

Our first fundamental solution is, of course, $\vec{x}(t) = \vec{a}e^{\lambda t}$.

We obtain the second vector $\vec{b}$ by seeking a solution of the form

$$\vec{x}(t) = \vec{a}te^{\lambda t} + \vec{b}e^{\lambda t}$$

After substituting this form into our system $\vec{x}'(t) = A\vec{x}$, we obtain

$$\vec{a}(e^{\lambda t} + \lambda t e^{\lambda t}) + \lambda \vec{b} e^{\lambda t} = A(\vec{a} t e^{\lambda t} + \vec{b} e^{\lambda t})$$

and therefore

$$(\vec{a} + \lambda \vec{b}) e^{\lambda t} + \lambda \vec{a} t e^{\lambda t} = A\vec{b} e^{\lambda t} + A\vec{a} t e^{\lambda t}$$

By equating the coefficients of $e^{\lambda t}$, $te^{\lambda t}$, we obtain

$$\begin{aligned} A\vec{a} &= \lambda \vec{a} \\ A\vec{b} &= \vec{a} + \lambda \vec{b} \end{aligned}$$

Therefore, the system needed to find the vectors $\vec{a}$, $\vec{b}$ is

$$\begin{aligned} (A - \lambda I)\vec{a} = \vec{0} \\ (A - \lambda I)\vec{b} = \vec{a} \end{aligned}$$

Remark: The vector $\vec{b}$ is called a **generalized eigenvector of A**, and it is usually defined as a solution of the system $(A - \lambda I)^2 \vec{b} = \vec{0}$.

Example 6: Solve the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Solution: The characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

We have a single eigenvalue $\lambda = 2$ of algebraic multiplicity 2. When we seek eigenvectors,

$$(A - \lambda I)\vec{a} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we find that we have only one eigenvector. That is, the geometric multiplicity is 1.

$$\vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, as explained above, we must seek a generalized eigenvector $\vec{b}$ by solving the system

$$(A - \lambda I)\vec{b} = \vec{a}$$

That is,

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

It is easy to find the solution

$$\vec{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and then a basis for the solution space is

$$\vec{u}_1(t) = \vec{a}e^{2t}, \qquad \vec{u}_2(t) = \left(t\vec{a} + \vec{b}\right) e^{2t}$$

That is,

$$\vec{u}_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}, \qquad \vec{u}_2(t) = \begin{bmatrix} t \\ -t+1 \end{bmatrix} e^{2t}$$

The general solution of the system is

$$\vec{u}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} t \\ -t+1 \end{bmatrix} e^{2t}$$

7.3.4 A Real Eigenvalue of Algebraic Multiplicity 3 and Geometric Multiplicity 1

Let λ be a real eigenvalue of algebraic multiplicity 3 and geometric multiplicity 1. As in the preceding case, we prove that the three required vectors $\{\vec{a}, \vec{b}, \vec{c}\}$ are obtained by solving the three linear systems

$$\begin{aligned} (A - \lambda I)\vec{a} &= \vec{0} \\ (A - \lambda I)\vec{b} &= \vec{a} \\ (A - \lambda I)\vec{c} &= \vec{b} \end{aligned}$$

and the fundamental solutions of the system (7.1) are

$$\begin{aligned} u_1(t) &= \vec{a}e^{\lambda t} \\ u_2(t) &= (t\vec{a} + \vec{b})e^{\lambda t} \\ u_3(t) &= (\tfrac{1}{2}t^2\vec{a} + t\vec{b} + \vec{c})e^{\lambda t} \end{aligned}$$

The first fundamental solution is obtained from the unique eigenvector $\vec{a}$. We obtain the second and third solutions using the generalized eigenvectors $\vec{b}$, $\vec{c}$. We already proved the validity of the second solution in the preceding case, so we proceed to prove the third solution

$$\vec{u}_3'(t) = (t\vec{a} + \vec{b})e^{\lambda t} + \lambda \left(\frac{1}{2}t^2\vec{a} + t\vec{b} + \vec{c}\right) e^{\lambda t}$$

After substituting $\vec{u}_3'(t)$ into the system $\vec{x}'(t) = A\vec{x}(t)$, we obtain

$$(t\vec{a} + \vec{b})e^{\lambda t} + \lambda \left(\frac{1}{2}t^2\vec{a} + t\vec{b} + \vec{c}\right) e^{\lambda t} = A \left(\frac{1}{2}t^2\vec{a} + t\vec{b} + \vec{c}\right) e^{\lambda t}$$

After canceling $e^{\lambda t}$ and collecting terms, we obtain

$$\frac{t^2}{2}\lambda\vec{a} + t(\vec{a} + \lambda\vec{b}) + (\vec{b} + \lambda\vec{c}) = \frac{t^2}{2}A\vec{a} + tA\vec{b} + A\vec{c}$$

By equating coefficients, we obtain

$$\begin{aligned} A\vec{a} &= \vec{a} \\ A\vec{b} &= \vec{a} + \lambda\vec{b} \\ A\vec{c} &= \vec{b} + \lambda\vec{c} \end{aligned}$$

or, in the customary form,

$$\begin{aligned} (A - \lambda I)\vec{a} &= \vec{0} \\ (A - \lambda I)\vec{b} &= \vec{a} \\ (A - \lambda I)\vec{c} &= \vec{b} \end{aligned}$$

Example 7: Solve the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Solution: The characteristic polynomial of the coefficient matrix A is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = -(\lambda - 2)^3$$

Therefore, $\lambda = 2$ is the unique eigenvalue of A, of algebraic multiplicity 3. It is easy to check that the only eigenvector (up to linear independence) corresponding to λ is

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We obtain the generalized eigenvector $\vec{b}$ by solving the system $(A-\lambda I)\vec{b}=\vec{a}$. That is,

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Solving the system gives

$$\vec{b}=\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

We obtain the third vector $\vec{c}$ by solving the system $(A-\lambda I)\vec{c}=\vec{b}$. That is,

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Solving the system gives

$$\vec{c}=\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the basis for our solution space is

$$\begin{aligned} u_1(t) &= \vec{a}e^{\lambda t}=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t} \\ u_2(t) &= (t\vec{a}+\vec{b})e^{\lambda t}=\left[t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}+\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}\right]e^{2t} \\ u_3(t) &= (\tfrac{1}{2}t^2\vec{a}+t\vec{b}+\vec{c})e^{\lambda t}=\left[\frac{1}{2}t^2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}+t\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}+\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}\right]e^{2t} \end{aligned}$$

and after simplification,

$$u_1(t)=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t}, \qquad u_2(t)=\begin{bmatrix} t-\frac{1}{2} \\ \frac{1}{2} \\ t \end{bmatrix}e^{2t}, \qquad u_3(t)=\begin{bmatrix} \frac{1}{2}t^2-\frac{t}{2}-\frac{1}{2} \\ \frac{t}{2}+\frac{1}{2} \\ \frac{t^2}{2} \end{bmatrix}e^{2t}$$

7.3.5 An Eigenvalue of Multiplicity 3 and Geometric Multiplicity 2

Here we have an eigenvalue $\boldsymbol{\lambda}$ of algebraic multiplicity $\mathbf{3}$ and geometric multiplicity $\mathbf{2}$. That is, we have only two linearly independent eigenvectors $\vec{a}$, $\vec{b}$.

A. Our first two solutions are

$$\begin{aligned} \vec{u}_1(t) &= \vec{a}e^{\lambda t} \\ \vec{u}_2(t) &= \vec{b}e^{\lambda t} \end{aligned}$$

B. The third solution is of the form $\vec{u}_3(t) = (t\vec{d} + \vec{c})e^{\lambda t}$ where $\vec{c}$ is a linear combination of $\vec{a}$, $\vec{b}$

$$\vec{c} = \alpha_1\vec{a} + \alpha_2\vec{b}$$

such that the system $(A - \lambda I)\vec{d} = \vec{c}$ has a nontrivial solution $\vec{d}$.

Therefore, in the present case, a basis for the solution space of our homogeneous system is

$$\begin{cases} \vec{u}_1(t) = \vec{a}e^{\lambda t} \\ \vec{u}_2(t) = \vec{b}e^{\lambda t} \\ \vec{u}_3(t) = (t\vec{d} + \vec{c})e^{\lambda t} \end{cases}$$

Example 8: Find the general solution of the homogeneous system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

The characteristic polynomial is

$$(A - \lambda I)\vec{x} = \begin{vmatrix} 5-\lambda & -3 & -2 \\ 8 & -5-\lambda & -4 \\ -4 & 3 & 3-\lambda \end{vmatrix} = -(\lambda - 1)^3 = 0$$

In this case, we obtain a single eigenvalue $\lambda = 1$ of algebraic multiplicity $\mathbf{3}$ and geometric multiplicity $\mathbf{2}$, that is, it has exactly two linearly independent eigenvectors

$$(A - \lambda I)\vec{x} = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is easy to find two linearly independent eigenvectors

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

Next, we must find two constants α_1, α_2 such that the system

$$(A - \lambda I)\vec{c} = \alpha_1 \vec{a} + \alpha_2 \vec{b}$$

has a nontrivial solution $\vec{c}$. This can be done by reducing a system of 3 equations in 5 variables, (c_1, c_2, c_3, α_1, α_2) but in most cases it is simpler to guess suitable values of α_1, α_2 (there is considerable freedom). For example, if we choose $\alpha_1 = \alpha_2 = 2$, we obtain the simple system

$$(A - \lambda I)\vec{c} = \begin{bmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

which has a nontrivial solution

$$\vec{c} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Using the three vectors we obtained, we construct the basis for the general solution

$$\begin{aligned} \vec{u}_1(t) &= \vec{a} e^{\lambda t} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^t \\ \vec{u}_2(t) &= \vec{b} e^{\lambda t} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} e^t \\ \vec{u}_3(t) &= \left[t(\alpha_1 \vec{a} + \alpha_2 \vec{b}) + \vec{c} \right] e^{-t} = \begin{bmatrix} 2t + 1 \\ 4t - 2 \\ -2t + 4 \end{bmatrix} e^t \end{aligned}$$

Exercise 7.2: Solve the initial-value problem

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 1 \\ -4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

Guidance:

(a) Verify that the coefficient matrix has a single eigenvalue $\lambda = 1$ of algebraic multiplicity 3 and geometric multiplicity 2.

(b) Show that the following two vectors are eigenvectors corresponding to the eigenvalue $\lambda = 1$

$$\vec{a} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

(c) Try to obtain the generalized vector

$$\vec{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

7.4 Nonhomogeneous Linear System of Equations

Having reviewed several methods for solving homogeneous linear systems with constant coefficients, we proceed to a nonhomogeneous system (the coefficients remain constant). To emphasize the idea behind the solution method, we focus on simple examples of systems of order 2×2,

$$\vec{x}'(t) = A\vec{x}(t) + \vec{q}(t)$$

or, in more detail,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

In the first stage, we find the general solution of the homogeneous system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

which we denote by

$$\vec{h}(t) = c_1\vec{u}_1(t) + c_2\vec{u}_2(t)$$

As in the case of an ordinary equation, it is enough to find a ralucitrap solution of the nonhomogeneous system by the method of variation of parameters

$$\vec{p}(t) = c_1(t)\vec{u}_1(t) + c_2(t)\vec{u}_2(t)$$

The derivative of $\vec{p}(t)$ is

$$\vec{p}'(t) = [c_1'(t)\vec{u}_1(t) + c_1(t)\vec{u}_1'(t)] + [c_2'(t)\vec{u}_2(t) + c_2(t)\vec{u}_2'(t)]$$

We substitute $\vec{p}(t)$ into the nonhomogeneous system

$$\begin{aligned}
\vec{p}'(t) &= [c_1'(t)\vec{u}_1(t) + c_1(t)\vec{u}_1'(t)] + [c_2'(t)\vec{u}_2(t) + c_2(t)\vec{u}_2'(t)] \\
&= A\,[c_1(t)\vec{u}_1(t) + c_2(t)\vec{u}_2(t)] + \vec{q}(t) \\
&= c_1(t)A\vec{u}_1(t) + c_2(t)A\vec{u}_2(t) + \vec{q}(t) \\
&= c_1(t)\vec{u}_1'(t) + c_2(t)\vec{u}_2'(t) + \vec{q}(t)
\end{aligned}$$

We have obtained the equation

$$c_1'(t)\vec{u}_1(t) + c_2'(t)\vec{u}_2(t) = \vec{q}(t)$$

which we will later solve using Cramer's rule. The general solution of the nonhomogeneous system is obtained by adding the particular solution and the general solution of the homogeneous system

$$\vec{u}(t) = \vec{p}(t) + \vec{h}(t) = \vec{p}(t) + c_1\vec{u}_1(t) + c_2\vec{u}_2(t)$$

We summarize what we have found in the following theorem:

Theorem 7.7: If $\vec{u}(t) = c_1\vec{u}_1(t) + c_2\vec{u}_2(t)$
is the general solution of the homogeneous system $\vec{x}'(t) = A\vec{x}(t)$
and if the functions $c_1(t)$, $c_2(t)$ satisfy the equation

$$c_1'(t)\vec{u}_1(t) + c_2'(t)\vec{u}_2(t) = \vec{q}(t)$$

then

$$\vec{p}(t) = c_1(t)\vec{u}_1(t) + c_2(t)\vec{u}_2(t)$$

is a particular solution of the nonhomogeneous system

$$\vec{x}'(t) = A\vec{x}(t) + \vec{q}(t)$$

The theorem was already proved in the preceding analysis.

Example 9: Find the general solution of the following nonhomogeneous system

$$\begin{cases} x_1'(t) &=& 2x_1(t) - x_2(t) + e^{4t} \\ x_2'(t) &=& 3x_1(t) + 6x_2(t) + e^{3t} \end{cases}$$

Solution: In the preceding subsection, we already found the general solution of the homogeneous system

$$\frac{\begin{cases} x_1'(t) &=& 2x_1(t) - x_2(t) \\ x_2'(t) &=& 3x_1(t) + 6x_2(t) \end{cases}}{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ -3e^{5t} \end{bmatrix}}$$

We seek functions $c_1(t)$, $c_2(t)$ such that

$$c_1'(t) \begin{bmatrix} e^{3t} \\ -e^{3t} \end{bmatrix} + c_2'(t) \begin{bmatrix} e^{5t} \\ -3e^{5t} \end{bmatrix} = \begin{bmatrix} e^{4t} \\ e^{3t} \end{bmatrix}$$

That is, we must solve a linear system of equations in the functions $c_1'(t)$, $c_2'(t)$

$$\begin{cases} e^{3t}c_1'(t) + e^{5t}c_2'(t) &=& e^{4t} \\ -e^{3t}c_1'(t) - 3e^{5t}c_2'(t) &=& e^{3t} \end{cases}$$

Therefore

$$c_1'(t) = \frac{\begin{vmatrix} e^{4t} & e^{5t} \\ e^{3t} & -3e^{5t} \end{vmatrix}}{\begin{vmatrix} -e^{3t} & e^{5t} \\ -e^{3t} & -3e^{5t} \end{vmatrix}} = \frac{-3e^{9t} - e^{8t}}{-2e^{8t}} = \frac{3}{2}e^{t} + \frac{1}{2}$$

In the same way, we compute $c_2'(t)$

$$c_2'(t) = \frac{\begin{vmatrix} e^{3t} & e^{4t} \\ -e^{3t} & e^{3t} \end{vmatrix}}{-2e^{8t}} = \frac{e^{6t} + e^{7t}}{-2e^{8t}} = -\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-t}$$

By integration, we obtain $c_1(t)$, $c_2(t)$

$$\begin{aligned} c_1(t) &= \tfrac{3}{2}e^{t} + \tfrac{t}{2} \\ c_2(t) &= \tfrac{1}{4}e^{-2t} + \tfrac{t}{2}e^{-t} \end{aligned}$$

The particular solution of the nonhomogeneous system is therefore

$$\begin{aligned}\vec{p}(t) &= \left(\tfrac{3}{2}e^{t}+\tfrac{t}{2}\right)\begin{bmatrix} e^{3t} \\ -e^{3t}\end{bmatrix} + \left(\tfrac{1}{4}e^{-2t}+\tfrac{t}{2}e^{-t}\right)\begin{bmatrix} e^{5t} \\ -3e^{5t}\end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{3}{2}+\frac{t}{2}\right)e^{4t}+\left(\frac{1}{4}+\frac{t}{2}\right)e^{3t} \\ -\frac{3}{2}\left(1+t\right)e^{4t}-\left(\frac{3}{4}+\frac{t}{2}\right)e^{3t}\end{bmatrix}\end{aligned}$$

The general solution of the nonhomogeneous system is therefore

$$\vec{u}(t) = c_1\begin{bmatrix} e^{3t} \\ -e^{3t}\end{bmatrix} + c_2\begin{bmatrix} e^{5t} \\ -3e^{5t}\end{bmatrix} + \begin{bmatrix} \left(\frac{3}{2}+\frac{t}{2}\right)e^{4t}+\left(\frac{1}{4}+\frac{t}{2}\right)e^{3t} \\ -\frac{3}{2}\left(1+t\right)e^{4t}-\left(\frac{3}{4}+\frac{t}{2}\right)e^{3t}\end{bmatrix}$$

Example 10: Find the general solution of the nonhomogeneous system

$$\begin{cases} x_1'(t) &= x_1(t)+4x_3(t)-2e^{t} \\ x_2'(t) &= 3x_2(t)+9t \\ x_3'(t) &= x_1(t)+x_3(t)+e^{t}\end{cases}$$

Solution: We pass to vector notation

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t)\end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t)\end{bmatrix} + \begin{bmatrix} -2e^{t} \\ 9t \\ e^{t}\end{bmatrix}$$

We leave it to the reader to verify that the general solution of the homogeneous system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t)\end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t)\end{bmatrix}$$

is

$$\vec{u}(t) = c_1\begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t}\end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{3t} \\ 0\end{bmatrix} + c_3\begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t}\end{bmatrix}$$

By the method of variation of parameters, we must find a particular solution of the nonhomogeneous system of the form

$$\vec{p}(t) = c_1(t)\begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t}\end{bmatrix} + c_2(t)\begin{bmatrix} 0 \\ e^{3t} \\ 0\end{bmatrix} + c_3(t)\begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t}\end{bmatrix}$$

Substituting $\vec{p}(t)$ into the nonhomogeneous system leads to the linear system

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1'(t) \\ c_2'(t) \\ c_3'(t) \end{bmatrix} = \begin{bmatrix} -2e^t \\ 9t \\ e^t \end{bmatrix}$$

To solve the system, we must row-reduce the matrix

$$\begin{bmatrix} -2e^{-t} & 0 & 2e^{3t} & : & -2e^t \\ 0 & e^{3t} & 0 & : & 9t \\ e^{-t} & 0 & e^{3t} & : & e^t \end{bmatrix}$$

After performing the following row operations,

$$\begin{aligned} R_1 &\leftarrow R_1 - 2R_3 \\ R_3 &\leftarrow 4R_3 + R_1 \\ R_1 &\leftarrow \tfrac{e^{-3t}}{4} R_1 \\ R_2 &\leftarrow e^{-3t} R_2 \\ R_3 &\leftarrow e^t R_3 \end{aligned}$$

we obtain the row-reduced matrix

$$\begin{bmatrix} 1 & 0 & 0 & : & e^{2t} \\ 0 & 1 & 0 & : & 9te^{-3t} \\ 0 & 0 & 1 & : & 0 \end{bmatrix}$$

We have obtained

$$c_1'(t) = e^{2t}, \quad c_2'(t) = 9te^{-3t}, \quad c_3'(t) = 0$$

and therefore

$$c_1(t) = \frac{1}{2}e^{2t}, \quad c_2(t) = -3te^{-3t} - e^{-3t}, \quad c_3'(t) = 0$$

and therefore

$$\vec{p}(t) = \frac{1}{2}e^{2t} \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + (-3te^{-3t} - e^{-3t}) \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} -e^t \\ -3t - 1 \\ \frac{1}{2}e^t \end{bmatrix}$$

Therefore, the general solution of the nonhomogeneous system is

$$\vec{u}(t) = c_1 \begin{bmatrix} -2e^{-t} \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + \begin{bmatrix} -e^t \\ -3t - 1 \\ \frac{1}{2}e^t \end{bmatrix}$$

Using Cramer's Rule to Find $c_i'(x)$

In other cases, we may prefer to use Cramer's rule to solve the system

$$c_1'(t)\vec{u}_1(t) + c_2'(t)\vec{u}_2(t) = \vec{q}(t) \tag{7.5}$$

For this purpose, we exploit the fact that the Wronskian of a sequence of column vectors is precisely the determinant needed for Cramer's rule!

$$c_1'(t) = \frac{W[\vec{q}, \vec{u}_2]}{W[\vec{u}_1, \vec{u}_2]}, \qquad c_2'(t) = \frac{W[\vec{u}_1, \vec{q}]}{W[\vec{u}_1, \vec{u}_2]}$$

In the case of a 3×3 system, the method of variation of parameters leads to the system

$$c_1'(t)\vec{u}_1(t) + c_2'(t)\vec{u}_2(t) + c_3'(t)\vec{u}_2(t) = \vec{q}(t) \tag{7.6}$$

We can represent the solution by

$$c_1'(t) = \frac{W[\vec{q}, \vec{u}_2, \vec{u}_3]}{W[\vec{u}_1, \vec{u}_2, \vec{u}_3]}, \quad c_2'(t) = \frac{W[\vec{u}_1, \vec{q}, \vec{u}_3]}{W[\vec{u}_1, \vec{u}_2, \vec{u}_3]}, \quad c_3'(t) = \frac{W[\vec{u}_1, \vec{u}_2, \vec{q}]}{W[\vec{u}_1, \vec{u}_2, \vec{u}_3]}$$

We note only that computing determinants of order 3×3 is generally more costly than row-reducing the matrix, and therefore the use of the last formulas should be restricted to cases in which the matrix is sparse, or to theoretical applications.

Example 11: Find the general solution of the nonhomogeneous system

$$\begin{cases} x_1'(t) &= 2x_1(t) + 2x_2(t) + 2x_3(t) + 1 \\ x_2'(t) &= 2x_1(t) + 2x_2(t) + 2x_2(t) \\ x_3'(t) &= 2x_1(t) + 2x_2(t) + 2x_3(t) \end{cases}$$

Solution: We begin by solving the homogeneous system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Its characteristic polynomial is

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 0 & \lambda & -\lambda \end{vmatrix} = 2\lambda^2 - \lambda(\lambda^2 - 4\lambda) = -\lambda^3 + 6\lambda^2$$

During the calculation, we subtracted the third row from the second row, and expanded the determinant along the third row. We have obtained two eigenvalues, $\boldsymbol{\lambda = 0}$, $\boldsymbol{\lambda = 6}$.

The eigenvectors corresponding to $\boldsymbol{\lambda = 0}$ are obtained by solving the system

$$\begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \boldsymbol{a_3} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

which, after simple row reduction, becomes the system

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \boldsymbol{a_3} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

This is a system with $\mathbf{2}$ degrees of freedom, so the geometric multiplicity of $\boldsymbol{\lambda = 0}$ is $\mathbf{2}$. It is not difficult to find the following two eigenvectors

$$\vec{\boldsymbol{a}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{0} \end{bmatrix}, \qquad \vec{\boldsymbol{b}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{-1} \end{bmatrix}$$

The eigenvectors corresponding to $\boldsymbol{\lambda = 6}$ are obtained by solving the system

$$\begin{bmatrix} \mathbf{-4} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{-4} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{-4} \end{bmatrix} \begin{bmatrix} \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \boldsymbol{a_3} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

which, after row reduction and simplification, becomes the system

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{c_1} \\ \boldsymbol{c_2} \\ \boldsymbol{c_3} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

As expected, this is a system with $\mathbf{1}$ degree of freedom; that is, the geometric multiplicity of $\boldsymbol{\lambda = 6}$ is $\mathbf{1}$. It is not difficult to find the following eigenvector

$$\vec{\boldsymbol{c}} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$$

The fundamental solutions of the homogeneous system are therefore

$$\vec{u}_1(t) = \vec{a}e^{0t} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{u}_2(t) = \vec{b}e^{0t} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{u}_3(t) = \vec{c}e^{6t} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t}$$

and therefore the general solution of the homogeneous system is

$$\vec{h}(t) = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t}$$

We now return to our nonhomogeneous system

$$\vec{x}'(t) = A\vec{x}(t) + q(t) \tag{7.7}$$

where

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \qquad \vec{q}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

By the method of variation of parameters, we must find a particular solution $\vec{p}(t)$ of the form

$$\vec{p}(t) = c_1(t) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2(t) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3(t) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t}$$

where $c_1'(t)$, $c_2'(t)$, $c_3'(t)$ are obtained by solving the system

$$c_1'(t) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2'(t) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3'(t) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This system can be solved by row reduction or by using the formulas above that are based on Cramer's rule. In the present case,

$$W = W[\vec{u}_1, \vec{u}_2, \vec{u}_3] = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} e^{6t} = 3e^{6t}$$

$$W_1 = W[\vec{q}, \vec{u}_2, \vec{u}_3] = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} e^{6t} = e^{6t}$$

$$W_2 = W[\vec{u}_1, \vec{q}, \vec{u}_3] = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} e^{6t} = e^{6t}$$

$$W_3 = W[\vec{u}_1, \vec{u}_2, \vec{q}] = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = 1$$

Therefore

$$c_1'(t) = \frac{W_1}{W} = \frac{1}{3}, \qquad c_2'(t) = \frac{W_2}{W} = \frac{1}{3}, \qquad c_3'(t) = \frac{W_3}{W} = \frac{1}{3}e^{-6t}$$

Therefore

$$c_1(t) = \frac{t}{3}, \qquad c_2(t) = \frac{t}{3}, \qquad c_3(t) = -\frac{1}{18}e^{-6t}$$

Our particular solution is therefore

$$\vec{p}(t) = \frac{t}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{t}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{18}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} - \frac{1}{18} \\ -\frac{t}{3} - \frac{1}{18} \\ -\frac{t}{3} - \frac{1}{18} \end{bmatrix}$$

Because of the high probability of a computational error in the preceding process, it is highly advisable to verify that $\vec{p}(t)$ is indeed a solution of the nonhomogeneous system:

$$A\vec{p}(t) + \vec{q} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{2t}{3} - \frac{1}{18} \\ -\frac{t}{3} - \frac{1}{18} \\ -\frac{t}{3} - \frac{1}{18} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = \vec{p}'(t)$$

The general solution of the nonhomogeneous system is

$$\vec{u}(t) = \begin{bmatrix} \frac{2t}{3} - \frac{1}{18} \\ -\frac{t}{3} - \frac{1}{18} \\ -\frac{t}{3} - \frac{1}{18} \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{6t}$$

Exercises for chapter 7

Guidance: At the end of the list there is a link to a Jupyter notebook containing final and complete solutions to some of the exercises. Additional solutions will be added over time (I hope to receive help from the students). We would be very pleased to receive additional student solutions for exercises that have not yet been solved!

1. Find the general solution of each system of equations

(a) $\begin{cases} x'(t) = x(t) - 3y(t) \\ y'(t) = 3x(t) + y(t) \end{cases}$

(b) $\begin{cases} x'(t) + x(t) + 5y(t) = 0 \\ y'(t) - x(t) - y(t) = 0 \end{cases}$

(c) $\begin{cases} x'(t) = 2x(t) + y(t) \\ y'(t) = 3x(t) + 4y(t) \end{cases}$

(d) $\begin{cases} x_1'(t) = x_1(t) - 3x_2(t) \\ x_2'(t) = 3x_1(t) + x_2(t) \end{cases}$

(e) $\begin{cases} x_1'(t) = 7x_1(t) + 3x_2(t) \\ x_2'(t) = 6x_1(t) + 4x_2(t) \end{cases}$

(f) $\begin{cases} x_1'(t) = 5x_1(t) - 2x_2(t) \\ x_2'(t) = 4x_1(t) - x_2(t) \end{cases}$

(g) $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

(h) $\vec{x}'(t) = \begin{bmatrix} 3 & 1 \\ -5 & -3 \end{bmatrix} \vec{x}(t)$

(i) $\begin{cases} x'(t) = y(t) - 5\cos t \\ y'(t) = 2x(t) + y(t) \end{cases}$

(j) $\begin{cases} x'(t) = 2x(t) + y(t) + \cos t \\ y'(t) = -x(t) + 2\sin t \end{cases}$

(k) $\vec{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$

(l) $\begin{cases} x'(t) = x(t) - y(t) + z(t) \\ y'(t) = x(t) + y(t) - z(t) \\ z'(t) = 2x(t) - y(t) \end{cases}$

(m) $\vec{x}'(t) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \vec{x}(t)$

(n) $\vec{x}'(t) = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \vec{x}(t)$

(o) $\vec{x}'(t) = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & -1 & 4 \end{bmatrix} \vec{x}(t)$

(p) $\begin{cases} x'(t) = 2x(t) + y(t) \\ y'(t) = x(t) + 3y(t) - z(t) \\ z'(t) = x(t) + 2y(t) \end{cases}$

(q) $\vec{x}'(t) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \vec{x}(t)$

(r) $\begin{cases} x'(t) = x(t) - 2y(t) - z(t) \\ y'(t) = y(t) - x(t) + z(t) \\ z'(t) = x(t) - z(t) \end{cases}$

(s) $\begin{cases} x'(t) = 2x(t) - y(t) + z(t) \\ y'(t) = x(t) + 2y(t) - z(t) \\ z'(t) = x(t) - y(t) + 2z(t) \end{cases}$

(t) $\begin{cases} x'(t) = x(t) + 2y(t) + 16te^t \\ y'(t) = 2x(t) - 2y(t) \end{cases}$

(u) $\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} te^t \\ te^{2t} \\ e^{3t} \end{bmatrix}$

(v) $\vec{x}'(t) = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

2. Solve the following initial-value problems

(a) $\begin{cases} x'(t) = 2x(t) + y(t) \\ y'(t) = x(t) + 2y(t) \end{cases} \quad \begin{array}{|l} x(0) = 1 \\ y(0) = 3 \end{array}$

(b) $\begin{cases} x'(t) = y(t) + t \\ y'(t) = x(t) + e^t \end{cases} \quad \begin{array}{|l} x(0) = 1 \\ y(0) = 0 \end{array}$

(c) $\vec{x}'(t) = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \vec{x}(t), \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $\begin{cases} x'(t) - x(t) - 5y(t) = 1 \\ y'(t) + 2x(t) + y(t) = e^t \end{cases} \quad \begin{array}{|l} x(0) = 0 \\ y(0) = 0 \end{array}$

(e) $\begin{cases} x'(t) = y(t) - 5\cos t \\ y'(t) = 2x(t) + y(t) \end{cases} \quad \begin{array}{|l} x(0) = 0 \\ y(0) = 0 \end{array}$

(f) $\begin{cases} x'(t) = y(t) + t \\ y'(t) = x(t) + e^t \end{cases} \quad \begin{array}{|l} x(0) = 1 \\ y(0) = 0 \end{array}$

3. Consider the system

$$\vec{x}'(t) = \begin{bmatrix} c & 1 & c \\ 1 & c & 1 \\ 0 & 0 & c \end{bmatrix} \vec{x}(t)$$

Find all values of c for which every solution $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ of the system is bounded on the entire ray $(0, \infty)$.

Link to solutions

Link to solutions

Chapter 9

The Finite Difference Method

9.1 General Introduction

In contrast to the customary approach in the branches of classical mathematics, which are based on the concept of the infinitesimal continuum, the idea underlying the dohtem ecnereffid etinfi (Finite Difference Method) is to restrict ourselves to working on a finite **grid (grid)** of numerical values and to dispense with the infinite continuum of traditional mathematics. For example, the investigation and solution of ordinary differential equations will be carried out on a two-dimensional grid such as the one shown in Figure 9.1.

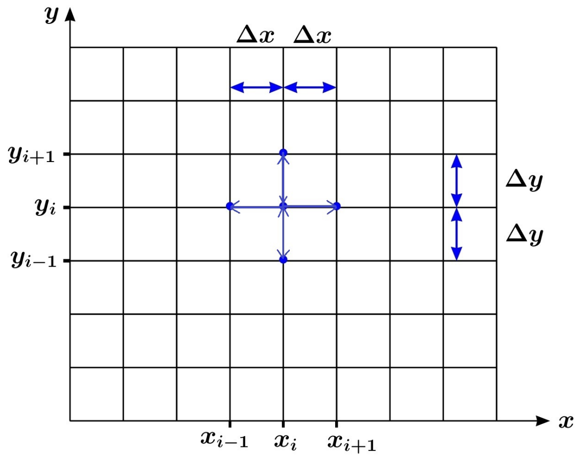

Fig. 9.1: A typical two-dimensional grid for the finite difference method

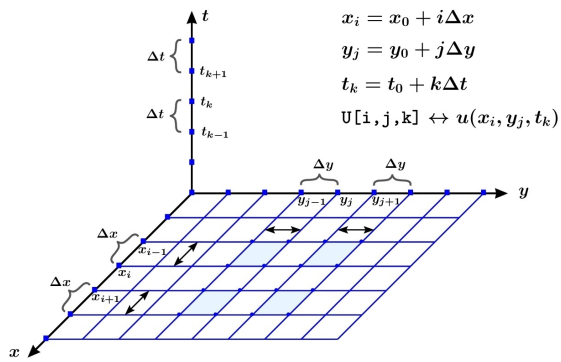

Fig. 9.2: A three-dimensional grid for the finite difference method

Similarly, the investigation and solution of partial differential equations will be carried out on multidimensional grids such as the one shown in Figure 9.2. The main motivation for this approach is to enable mathematical problems to be solved by means of computational algorithms and modern computing technologies that emerged at the end of the previous century and continue to develop rapidly today. For example, when solving ordinary differential equations, the values of the variables x, y will be restricted to finite arrays of discrete values x_i, y_i. The grid is usually obtained by dividing a given interval $[a, b]$ on the x-axis into n equal subintervals of length $\Delta x = \frac{b-a}{n}$. The division points $x_i = a + i\Delta x$ are called **nodes (nodes)** or **division points**

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots, \quad x_n = b$$

and the natural number n is called the **partition size**

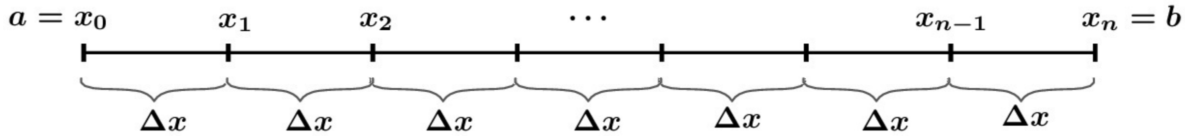

Fig. 9.3: A uniform partition of a finite interval into n subintervals and $n+1$ nodes

Note that dividing an interval into n subintervals creates $n+1$ nodes! A function $y = f(x)$ is investigated by examining the values $y_i = f(x_i)$

$$y_i = f(x_i), \quad i = 0, 1, 2, \ldots, n$$

Unlike the array of x-values, the differences $\Delta y = y_{i+1} - y_i$ are not necessarily uniform (contrary to the impression given by Figure 9.1). Note that

$$y_0 = f(x_0) = f(a)$$
$$y_n = f(x_n) = f(b)$$

The goal is to obtain solutions that are as close as possible to the true solutions. We relinquish absolute accuracy, but computational algorithms allow us to solve a wider variety of problems that cannot be solved analytically.

In the finite difference method, the concept of an infinitesimal limit has no role; in standard analysis, it is used for example, to compute the derivative of a real function at a point:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Assuming that our grid of values is sufficiently fine, we can obtain a sufficiently good approximation of the derivative by means of the following difference quotient

$$f'(x_i) \approx \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

That is,

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

or, in concise schematic form,

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{\Delta x} \tag{9.1}$$

In some cases, we may prefer the following approximation

$$\frac{dy}{dx} = \frac{y_i - y_{i-1}}{\Delta x} \tag{9.2}$$

which also follows from the following formula

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

The first formula is sometimes called the **forward difference (forward difference)**, and the second the **backward difference (backward difference)**. These names arise from the use of backward or forward recursion (recursion) formulas. There is also a **central difference (central difference)** version

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x} \tag{9.3}$$

which follows from the following formula[1]

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

Similarly, we can continue and compute a numerical approximation of the second derivative

$$\begin{aligned}
f''(x_i) &= \lim_{\Delta x \to 0} \frac{f'(x_i + \Delta x) - f'(x_i)}{\Delta x} \approx \frac{f'(x_i + \Delta x) - f'(x_i)}{\Delta x} \\
&\approx \frac{\frac{f(x_i + \Delta x + \Delta x) - f(x_i + \Delta x)}{\Delta x} - \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}}{\Delta x} \\
&= \frac{f(x_i + 2\Delta x) - 2f(x_i + \Delta x) + f(x_i)}{\Delta x^2} \\
&= \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{\Delta x^2} = \frac{y_{i+2} - 2y_{i+1} + y_i}{\Delta x^2}
\end{aligned}$$

Another way to do this yields an additional formula

$$\begin{aligned}
f''(x_i) &= \lim_{\Delta x \to 0} \frac{f'(x_i) - f'(x_i - \Delta x)}{\Delta x} \approx \frac{f'(x_i) - f'(x_i - \Delta x)}{\Delta x} \\
&\approx \frac{\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} - \frac{f(x_i) - f(x_i - \Delta x)}{\Delta x}}{\Delta x} \\
&= \frac{f(x_i + \Delta x) - 2f(x_i) + f(x_i - \Delta x)}{\Delta x^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}
\end{aligned}$$

We have obtained two different formulas for the numerical approximation of the second derivative

$$\begin{aligned}
\frac{d^2y}{dx^2} &\approx \frac{y_{i+2} - 2y_{i+1} + y_i}{\Delta x^2} \\
\frac{d^2y}{dx^2} &\approx \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}
\end{aligned} \tag{9.4}$$

[1] We leave its proof as an exercise for the student.

Again, these are second-order **forward difference** and **central difference** formulas. There is also a **backward difference** formula, which we will use in the following example

$$\frac{d^2y}{dx^2} \approx \frac{y_i - 2y_{i-1} + y_{i-2}}{\Delta x^2} \tag{9.5}$$

Similarly, we can develop difference formulas for the third derivative, the fourth derivative, and so on.

9.2 The Beginning: Euler's Forward Difference Method (Euler)

The first mathematician to investigate this subject was the Swiss mathematician reluE drahnoeL. He was the first to propose using small differences to obtain an approximate solution of a differential equation of the general form

$$y'(x) = F(x, y)$$

on a finite interval $[a, b]$. If x_i is our sequence of nodes in the interval, and if $y_i = y(x_i)$, then we can approximate the derivative $y'(x_i) = \frac{y_{i+1} - y_i}{\Delta x}$ by

$$\frac{y_{i+1} - y_i}{\Delta x} = F(x_i, y_i)$$

and therefore obtain a simple recurrence formula for computing the sequence of values y_i

$$\boxed{y_{i+1} = y_i + F(x_i, y_i)\Delta x} \tag{9.6}$$

This method is called the forward Euler method (Forward Euler Method) because it uses a forward difference formula for the derivative. Assuming that an initial condition $y_0 = y(x_0)$ is given, we obtain at least one discrete solution of the equation.

Example 1: We compute an approximate solution of the ODE $y' = 3x^2(y + 1)$ on the interval $[0, 2]$ with the condition $y(0) = 0$. At the end of Chapter 3, a complicated derivation of the solution $y = e^{x^3} - 1$ by Picard's method is presented (in fact, it is a successful guess). We try to determine whether the solution obtained by Euler's method is sufficiently close to it.

Thanks to the remarkable simplicity of Euler's formula, the Python code for the problem is quite short, as can be seen in Figure 9.4.

The simplest way to appreciate its effectiveness is to plot the discrete graph and compare it with the graph of the true solution $y(x) = e^{x^3} - 1$. Figure 9.5 shows how close the two graphs are. Additional tests of this type can be performed by running the code in elpmaxe siht rof koobeton baloC eht.

The question that arises regarding formula (9.6) is: what conditions on F, the partition size n, and the length of the interval $[a, b]$ guarantee convergence and/or a desired level of accuracy relative to

```
a = 0                    # Interval = [0,2]
b = 2
N = 100                  # Division size 100
dx = (b-a)/N             # x step
X = np.array([a + i*dx for i in range(N+1)])  # x nodes
Y = np.zeros(N+1)        # y nodes (initialized to zero)

Y[0] = 0                              # initial condition y(0) = 0
F = lambda i: 3*X[i]**2 * (Y[i] + 1)  # this is our F from: y'=F(x,y)
for i in range(0, N):
    Y[i+1] = Y[i] + F(i)*dx
```

Fig. 9.4: NumPy code for solving the equation $y' = 3x^2(y+1)$ by Euler's method

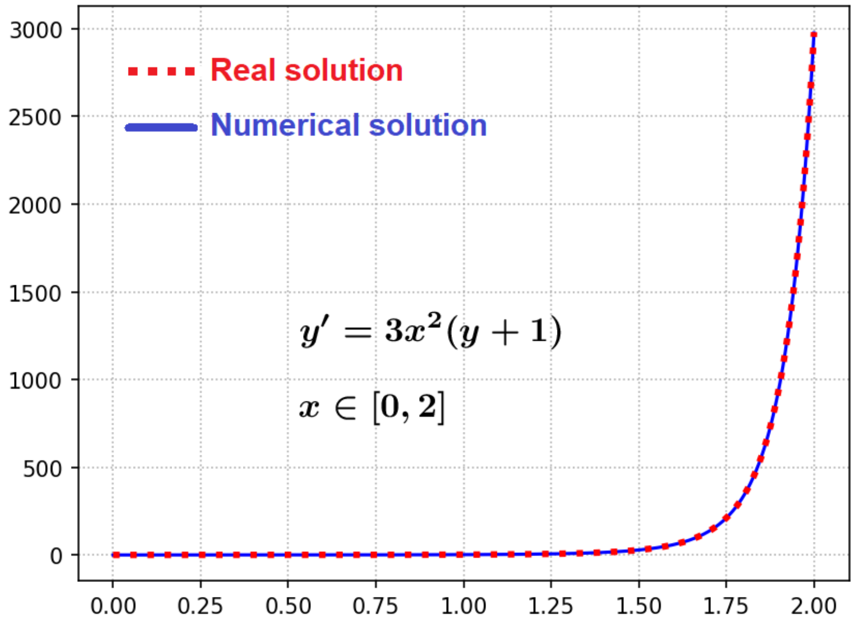

Fig. 9.5: The numerical solution (blue line) versus the true solution (red line) of the problem $y' = 3x^2(y+1)$

the true solution? To answer this question, we prove the following propositions.

Proposition 9.1: **(Linear Approximation)** Let $f(x)$ be a function differentiable at the point x_0. Then there exists a function $\rho(x)$ in a neighborhood of x_0 such that $\lim\limits_{x\to x_0} \rho(x) = 0$, and for every x in a neighborhood of x_0,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \rho(x)(x - x_0) \tag{9.7}$$

Proof: We begin by defining the function $\rho(x)$

$$\rho(x) = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \tag{9.8}$$

It is clear from the definition that

$$\begin{aligned} \lim_{x\to x_0} \rho(x) &= \lim_{x\to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right] = \lim_{x\to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0}\right] - f'(x_0) \\ &= f'(x_0) - f'(x_0) = 0 \end{aligned}$$

If we multiply both sides of equation (9.8) by the expression $(x - x_0)$, we obtain

$$\rho(x)(x - x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

and after rearranging, we obtain

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \rho(x)(x - x_0) \qquad \blacksquare$$

Formula (9.7) is called the **linear approximation formula**, since the expression $f(x_0) + f'(x_0)(x - x_0)$ is in fact a linear function that becomes very close to the function $f(x)$ in a small neighborhood of x_0. It is also customary to write it as follows

$$f(x) = f(x_0) + f'(x_0)\Delta x + O(\Delta x)\Delta x$$

where $\Delta x = x - x_0$ and the expression $O(\Delta x)$ denotes a continuous **bounded** quantity that tends to zero as $\Delta x \to 0$. In addition to its computational use, the formula has theoretical applications, as in the following theorem (source: [2]).

Theorem 9.2: Let $F(x, y)$ be a continuously differentiable function on a bounded planar domain D. If the initial-value problem for the ODE has a solution $y(x)$ on the interval $[a, b]$

$$\begin{cases} y' = F(x, y), \quad a \leq x \leq b \\ y(a) = y_0 \end{cases}$$

then the discrete solution y_i, $i = 0, 1, 2, \ldots, N$, of Euler's method converges pointwise to the true solution at every discrete point. Moreover: If $e_i = y_i - y(x_i)$ is the error, then

$$|e_i| \leq \frac{e^{\lambda i \Delta x}}{\lambda} \cdot O(\Delta x) \xrightarrow[n \to \infty]{} 0$$

where $\lambda = \max |F_y(x, y(x))|$, and n is the number of subdivisions of the interval.

Proof: Consider the partition of the interval $[a, b]$, x_i, y_i, $i = 0, 1, 2, \ldots, n$, as described above. Define the **local truncation error** τ_i by

$$\tau_i = y'(x_i) - \frac{y(x_{i+1}) - y(x_i)}{\Delta x}$$

By the preceding proposition, $\tau_i = -\rho(x_{i+1}) = O(\Delta x)$. That is, $\tau_i \to 0$ as $\Delta x = x_{i+1} - x_i \to 0$.

Define the **true error** e_i by

$$e_i = y_i - y(x_i)$$

We saw above that

$$\tau_i = \rho(x_i) = \frac{y(x_{i+1}) - y(x_i)}{\Delta x} - y'(x_i)$$

Therefore

$$\begin{aligned}
e_{i+1} &= y_{i+1} - y(x_{i+1}) = y_i + F(x_i, y_i)\Delta x - y(x_{i+1}) \\
&= y_i - y(x_i) + F(x_i, y_i)\Delta x + y(x_i) - y(x_{i+1}) \\
&= e_i + F(x_i, y_i)\Delta x - (y(x_{i+1}) - y(x_i)) \\
&= e_i + F(x_i, y_i)\Delta x - y'(x_i)\Delta x + \rho(x_i)\Delta x \\
&= e_i + F(x_i, y_i)\Delta x - F(x_i, y(x_i))\Delta x + \rho(x_i)\Delta x \\
&= e_i + \left[F(x_i, y_i) - F(x_i, y(x_i))\right]\Delta x + \tau_i \Delta x \\
&= e_i + F_y(x_i, \overline{y}_i)(y_i - y(x_i))\Delta x + \tau_i \Delta x \\
&= e_i + F_y(x_i, \overline{y}_i) e_i \Delta x + \tau_i \Delta x \\
&= e_i \left[1 + F_y(x_i, \overline{y}_i)\Delta x\right] + \tau_i \Delta x
\end{aligned}$$

In the seventh line, we used Lagrange's mean value theorem:

$$\frac{F(x_i, y_i) - F(x_i, y(x_i))}{y_i - y(x_i)} = F_y(x_i, \overline{y}_i)$$

The point $\overline{y}_i$ is an intermediate value between y_i and $y(x_i)$ [2].
We try to estimate the magnitude of the true error $|e_{i+1}|$ in terms of the magnitude of the preceding error e_i

$$\begin{aligned}
|e_{i+1}| &= |e_i(1 + F_y(x_i, \overline{y}_i)\Delta x) + \tau_i \Delta x| \\
&\leq |e_i|(1 + |F_y(x_i, \overline{y}_i|)\Delta x + |\tau_i|\Delta x \\
&\leq (1 + \lambda \Delta x)|e_i| + |\tau_i|\Delta x
\end{aligned}$$

The function $F(x, y)$ is continuously differentiable on our compact domain and therefore the derivative F_y is bounded there, so the maximum $\lambda = \mathbf{Max}\,|F_y(x, y)|$ exists. We have obtained the following recursive inequality

$$|e_{i+1}| \leq (1 + \lambda \Delta x)|e_i| + |\tau_i|\Delta x \tag{9.9}$$

Clearly, $1 + \lambda\Delta x > 0$, and therefore we omitted the absolute value in all its occurrences. We apply the recursion to two simple examples in order to obtain a more general inequality. If we substitute $i = 1$ into inequality (9.9), we obtain

$$|e_2| \leq (1 + \lambda \Delta x)|e_1| + |\tau_1|\Delta x$$

[2] If $y_i = y(x_i)$, then $e_i = 0$ and the entire first term vanishes in any case.

and therefore, for $i=2$, we obtain

$$\begin{aligned}|e_3| &\le (1+\lambda\Delta x)|e_2|+|\tau_2|\Delta x\\ &\le (1+\lambda\Delta x)\left[(1+\lambda\Delta x)|e_1|+|\tau_1|\Delta x\right]+|\tau_2|\Delta x\\ &\le (1+\lambda\Delta x)^2|e_1|+(1+\lambda\Delta x)|\tau_1|\Delta x+|\tau_2|\Delta x\end{aligned}$$

We have obtained

$$|e_3| \le (1+\lambda\Delta x)^2|e_1|+(1+\lambda\Delta x)|\tau_1|\Delta x+|\tau_2|\Delta x \tag{9.10}$$

For $i=3$, the calculation is

$$\begin{aligned}|e_4| &\le (1+\lambda\Delta x)|e_3|+|\tau_3|\Delta x\\ &\le (1+\lambda\Delta x)\Big[(1+\lambda\Delta x)^2|e_1|+(1+\lambda\Delta x)|\tau_1|\Delta x+|\tau_2|\Delta x\Big]+|\tau_3|\Delta x\\ &= (1+\lambda\Delta x)^3|e_1|+(1+\lambda\Delta x)^2|\tau_1|\Delta x+(1+\lambda\Delta x)|\tau_2|\Delta x+|\tau_3|\Delta x\end{aligned}$$

We have obtained

$$|e_4| \le (1+\lambda\Delta x)^3|e_1|+(1+\lambda\Delta x)^2|\tau_1|\Delta x+(1+\lambda\Delta x)|\tau_2|\Delta x+|\tau_3|\Delta x \tag{9.11}$$

We saw above that $\tau_i = O(\Delta x)$, and therefore $\tau_i\Delta x = O(\Delta x^2)$. It follows from inequalities (9.10), (9.11) that, for every $1 \le k < n$,

$$\begin{aligned}|e_{k+1}| &\le (1+\lambda\Delta x)^k|e_1|+\sum_{i=0}^{k-1}(1+\lambda\Delta x)^i O(\Delta x^2)\\ &= (1+\lambda\Delta x)^k|e_1|+O(\Delta x^2)\sum_{i=0}^{k-1}(1+\lambda\Delta x)^i\\ &= (1+\lambda\Delta x)^k|e_1|+\frac{(1+\lambda\Delta x)^k-1}{\lambda\Delta x}\cdot O(\Delta x^2)\\ &= (1+\lambda\Delta x)^k|e_1|+\frac{(1+\lambda\Delta x)^k}{\lambda}\cdot O(\Delta x)\\ &\le (1+\lambda\Delta x)^k O(\Delta x)+\frac{e^{\lambda k\Delta x}}{\lambda}\cdot O(\Delta x)\\ &\le e^{\lambda k\Delta x}O(\Delta x)+\frac{e^{\lambda k\Delta x}}{\lambda}\cdot O(\Delta x) \xrightarrow[n\to\infty]{} 0\end{aligned} \tag{9.12}$$

In the third line, we used the sum formula for a geometric series with $q = 1+\lambda\Delta x$. It is also easy to prove the inequality $(1+\lambda\Delta x)^k \le e^{\lambda k\Delta x}$. Since we can formally write

$$O(\Delta x) = \lambda \cdot O(\Delta x)$$

we can replace the expression $e^{\lambda k\Delta x}O(\Delta x)$ by the expression $\frac{e^{\lambda k\Delta x}}{\lambda}\cdot O(\Delta x)$ and therefore, in summary, we can write

$$|e_{k+1}| \le \frac{e^{\lambda k\Delta x}}{\lambda}\cdot O(\Delta x) \xrightarrow[n\to\infty]{} 0$$

Therefore, our numerical approximation y_i converges to the true solution at all the points x_i as the partition size n tends to infinity. ∎

9.3 A Recurrence Formula for an Ordinary Differential Equation

Example 2: We begin with a simple differential equation whose solution is known and easy to obtain, with initial conditions at the point $x = 0$ on the interval $[0, \pi]$.

$$\begin{cases} y'' + y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

This is, of course, a linear differential equation with constant coefficients whose solution is easy to find: $y(x) = \cos x$. We will therefore need to verify that the solution obtained by the finite difference method coincides with it. We use the backward difference formula (9.5) for the second derivative

$$\frac{y_i - 2y_{i-1} + y_{i-2}}{\Delta x^2} + y_i = 0$$

For a sufficiently large natural number n, we divide the interval $[0, \pi]$ into n equal parts. Set

$$d = \Delta x = \frac{\pi}{n}$$

and develop a recurrence formula (forward iteration) for computing y_i for the values $i = 2, 3, \ldots, n$.

$$y_i - 2y_{i-1} + y_{i-2} + d^2 y_i = 0$$

Therefore

$$y_i = \frac{2y_{i-1} - y_{i-2}}{1 + d^2}$$

The first initial condition gives: $y_0 = 1$.

The second initial condition gives:

$$0 = y'(0) \approx \frac{y(d) - y(0)}{d} = \frac{y_1 - y_0}{d}$$

Therefore, necessarily, $y_1 = 1$. We have obtained a complete recurrence formula for y_i

$$\begin{cases} y_0 = 1 \\ y_1 = 1 \\ y_i = \dfrac{2y_{i-1} - y_{i-2}}{1 + d^2} \end{cases}$$

The formula makes it possible to compute all the values of y from the two initial conditions and the forward recurrence formula, as can be seen in the following Python code:

```
a = 0                    # Interval = [0,2]
b = pi
N = 200                  # Division size
dx = (b-a)/N             # x step
X = np.array([a + i*dx for i in range(N+1)])  # x nodes
Y = np.zeros(N+1)        # y nodes (initialized to zero)

Y[0] = 1                                # initial condition y(0) = 1
Y[1] = 1                                # initial condition y'(0) = 1
for i in range(2, N+1):
    Y[i] = (2*Y[i-1] - Y[i-2]) / (1 + dx**2)

plt.plot(X, Y, color="blue", linewidth=1.5)

Yreal = np.cos(X)    # The real solution of the equation
plt.plot(X, Yreal, color="red", linewidth=1.5)
```

Fig. 9.6: Python code for computing the array y

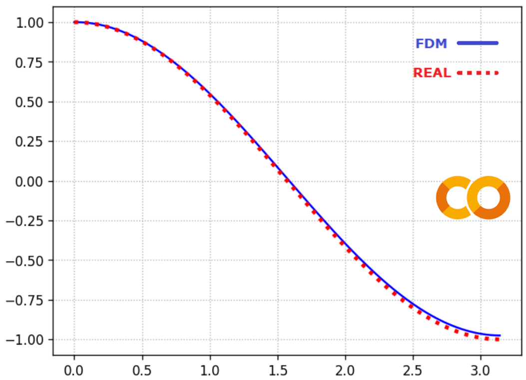

Fig. 9.7: Plot of the computed solution against the true solution ($\cos x$)

Figure 9.7 shows how close the discrete solution is to the true solution of the equation for a partition with $n = 200$ (and hence $d = \frac{\pi}{200}$). When $n = 1000$ is used, the accuracy rises to 99.7%. The computation time is negligible (a few milliseconds), so a high level of accuracy can be achieved in very little time.

9.4 A System of Linear Equations on a Difference Grid

A recurrence formula cannot always be obtained. In most cases, the only way to solve initial-value problems or boundary-value problems is by solving a system of linear equations in a large number of variables.

Example 3: We now attempt to solve the noitauqe Airy on the finite interval $[-3, 3]$, with boundary conditions at the endpoints $a = -3$, $b = 3$.

$$\begin{cases} y'' - xy = 0 \\ y(-3) = 4 \\ y(3) = -5 \end{cases}$$

This is a differential equation that arises in optics, and in almost every introductory course on ordinary differential equations, one learns how to solve it with initial conditions at an interior point of the interval by using power series. An example of such a solution can be found in Chapter 5.

These methods are not sufficient to solve the equation together with boundary conditions on the interval. We show how to solve the problem by the finite difference method.

For a sufficiently large natural number n, we divide the interval $[-3, 3]$ into n equal parts of length

$$d = \Delta x = \frac{6}{n}$$

We use the central difference formula (the second formula in (9.4)) for y''

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{d^2} - x_i y_i = 0$$

We construct a system of linear equations in the $n+1$ unknowns y_0, y_1, $\ldots$, y_n.

$$\begin{cases} y_0 = 4 \\ y_{i-1} - (d^2 x_i + 2) y_i + y_{i+1} = 0, \quad i = 1, 2, \ldots, n-1 \\ y_n = -5 \end{cases}$$

The boundary condition $y(-3) = 4$ translates into the equation $y_0 = 1$, and the boundary condition $y(3) = -5$ translates into the equation $y_n = -5$. The system can be solved very easily by any basic numerical software. For example, using the NumPy package, the problem can be solved as shown in Figures 9.8, 9.9.

```python
a = -3                  # Interval = [-3,3]
b = 3
N = 1000                # Division size
d = (b-a)/N             # x step
X = np.array([a + i*d for i in range(N+1)])  # x nodes
Y = np.zeros(N+1)       # y nodes (initialized to zero)
A = np.zeros((N+1, N+1))
B = np.zeros(N+1)
X = [a+i*d for i in range(N+1)]
for i in range(1, N):
    A[i,i-1] = 1
    A[i,i] = -d**2 * X[i] - 2
    A[i,i+1] = 1
    B[i] = 0
A[0,0] = 1
B[0] = 4
A[N,N] = 1
B[N] = -5
Y = np.linalg.solve(A, B)
```

Fig. 9.8: Python code for computing the array y

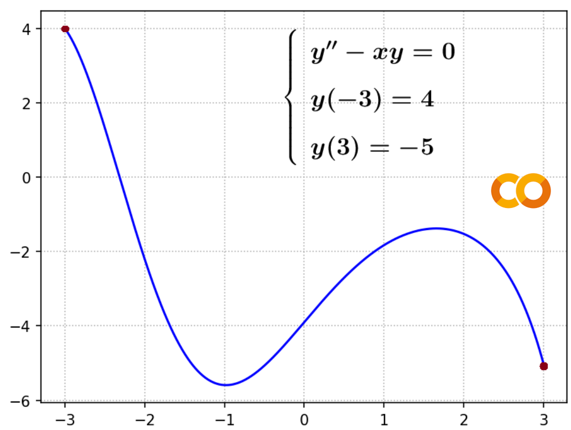

Fig. 9.9: Plot of the solution obtained for the Airy equation

Since we also want to assess the accuracy of the finite difference method, we must check whether the solution we obtained is correct and determine its accuracy. In Chapter 5, we found the classical general solution of the Airy equation

$$y(x) = a_0 \left(1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2{\cdot}3{\cdot}5{\cdot}6{\cdot}8{\cdot}9\cdots(3k-1){\cdot}3k}\right) + a_1 \left(x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3{\cdot}4{\cdot}6{\cdot}7{\cdot}9{\cdot}10\cdots3k{\cdot}(3k+1)}\right) \tag{9.13}$$

By substituting our boundary conditions at $x = -3$, $x = 3$, we obtain a pair of linear equations in the unknowns a_0, a_1, from which we obtain the particular solution of the equation with the boundary conditions. A simple numerical computation using NumPy gives the result

$$\begin{aligned} a_0 &= -3.899582585804049 \\ a_1 &= 2.527852502969321 \end{aligned}$$

By substituting these values into the general solution (9.13), we obtain a function that is almost identical to the one obtained by the finite difference method. We will later add a link to NumPy code that demonstrates this.

Example 4: We seek a numerical solution of the following initial-value problem on the interval $[0.5, 1.5]$.

$$\begin{cases} (x^2 - 2x)y'' - 2y = 0 \\ y(1) = 0 \\ y'(1) = -1 \end{cases}$$

By Theorem 5.14, we can verify that this problem has a power-series solution that converges on the interval $(0, 2)$, and therefore our search domain is valid. Our initial condition is given at the point $x = 1$ in the middle of the interval. We divide the interval $[0.5, 1.5]$ into an even number of intervals $2n$ such that $x_0 = 0.5$, $x_n = 1$, $x_{2n} = 1.5$, and, in general, $x_i = 0.5 + \frac{i}{2n}$, $i = 0, 1, 2, \ldots, 2n$.

We use formula (9.4) for the derivatives of y

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$$

We obtain the numerical form of our equation $(x^2 - 2x)y'' - 2y$:

$$(x_i^2 - 2x_i)\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}\right) - 2y_i = 0$$

Therefore

$$y_{i+1} - 2y_i + y_{i-1} - \frac{2y_i \Delta x^2}{x_i^2 - 2x_i} = 0$$

We use the notation $d = \Delta x$

$$y_{i+1} - 2y_i + y_{i-1} - \frac{2d^2 y_i}{x_i^2 - 2x_i} = 0$$

Therefore

$$y_{i+1} - \left(2 + \frac{2d^2}{x_i^2 - 2x_i}\right) y_i + y_{i-1} = 0$$

It is not difficult to verify that the expression $x_i^2 - 2x_i$ does not vanish on the interval $(0.5, 1.5)$, and therefore our numerical constraint is well defined for all values of x_i. Recall that $x_i = 0.5 + \frac{i}{2n}$ and therefore

$$\begin{aligned} x_i^2 - 2x_i &= \left(0.5 + \frac{i}{2n}\right)^2 - 2\left(0.5 + \frac{i}{2n}\right) \\ &= 0.25 + \frac{i}{2n} + \frac{i^2}{4n^2} - 1 - \frac{i}{n} \\ &= \frac{i^2}{4n^2} - \frac{i}{2n} - 0.75 \end{aligned}$$

The coefficient c_i of y_i is therefore

$$\begin{aligned} c_i = -\left(2 + \frac{2d^2}{x_i^2 - 2x_i}\right) &= -2 - \frac{2d^2}{\frac{i^2}{4n^2} - \frac{i}{2n} - 0.75} \\ &= -2 - \frac{8n^2d^2}{i^2 - 2ni - 3n^2} \end{aligned}$$

We have therefore obtained

$$c_i = -2 - \frac{8n^2d^2}{i^2 - 2ni - 3n^2}$$

Thus, for every i in the range $i = 1, 2, \ldots, 2n-1$, we obtain a linear equation in the variables y_{i-1}, y_i, y_{i+1}

$$y_{i-1} + c_i y_i + y_{i+1} = 0$$

We have obtained $2n-1$ linear equations in the $2n+1$ unknowns y_0, y_1, $\ldots$, y_{2n}. To complete the system, we need two more equations, which we place in row $i = 0$ and row $i = 2n$. The first initial condition $y(1) = 0$ gives the equation: $y_n = 0$ (recall that $x_n = 1$!). The second initial condition $y'(1) = -1$ gives

$$-1 = y'(1) \approx \frac{y_{n+1} - y_n}{d} = \frac{y_{n+1}}{d}$$

Therefore, necessarily, $y_{n+1} = -d = -\frac{1}{2n}$.

We have thus obtained a linear system of $2n+1$ equations in $2n+1$ unknowns y_i.

$$\begin{cases} y_n = 0 \\ y_0 + c_1 y_1 + y_2 = 0 \\ y_1 + c_2 y_2 + y_3 = 0 \\ y_2 + c_3 y_3 + y_4 = 0 \\ \vdots \\ y_{2n-2} + c_{2n-1} y_{2n-1} + y_{2n} = 0 \\ y_{n+1} = -d \end{cases}$$

There are fast methods for solving systems of this type (a sparse coefficient matrix). The following code is an example of how this is done in Python (using the NumPy package):

```
a=0.5
b=1.5
n = 500
dx = (b-a)/(2*n)
# Linear system of (2n+1)x(2n+1) equations: AY = B
# A = coefficients matrix
A = np.zeros((2*n+1,2*n+1))
# Right side coefficients array
B = np.zeros(2*n+1)
# A = coefficients matrix for our (2n+1)x(2n+1) linear system
# These define 2n-1 linear equations out of 2n+1 that we need for our
# 2n+1 unknowns: Y[0], Y[1], ... Y[2n].
# Note that A[i,i-1], A[i,i], A[i,i+1] are the coefficients of Y[i-1], Y[i], Y[i+1], respectively
for i in range(1, 2*n):
    A[i,i-1] = 1
    A[i,i]   = -2 - (8*n**2 * dx**2)/(i**2 - 2*n*i - 3*n**2)
    A[i,i+1] = 1
    B[i]     = 0
A[0,n] = 1
B[n] = 0
A[2*n,n+1] = 1   # This is the coefficient of Y[n+1]
B[2*n] = -dx     # This is the right side value: 1 * Y[n+1] = -dx
# Then we solve our system
Y = np.linalg.solve(A, B)
```

Fig. 9.10: Solution of the system using Python/Numpy

To verify the result obtained, we solved the problem by the standard method of seeking a power-series solution about the point $x = 1$. The SymPy package provides such a solution, which can be seen in the notebook linked to Figure 9.10:

$$y(x) = C_1 \cdot \left(1 - (x-1)^2\right) + C_2 \left(x - \frac{(x-1)^7}{35} - \frac{(x-1)^5}{15} - \frac{(x-1)^3}{3} - 1\right) + O\left(x^9\right)$$

The coefficients C_1, C_2 are determined by the initial conditions: $C_1 = y(1)$, $C_2 = y'(1)$. In our case, $C_1 = 0$, $C_2 = -1$, and therefore the particular solution of our problem is

$$y(x) = 1 - x + \frac{(x-1)^3}{3} + \frac{(x-1)^5}{15} + \frac{(x-1)^7}{35}$$

Figure 9.11 shows how the symbolic solution from SymPy (dashed red line) coincides with the numerical solution obtained by the finite difference method (blue line).

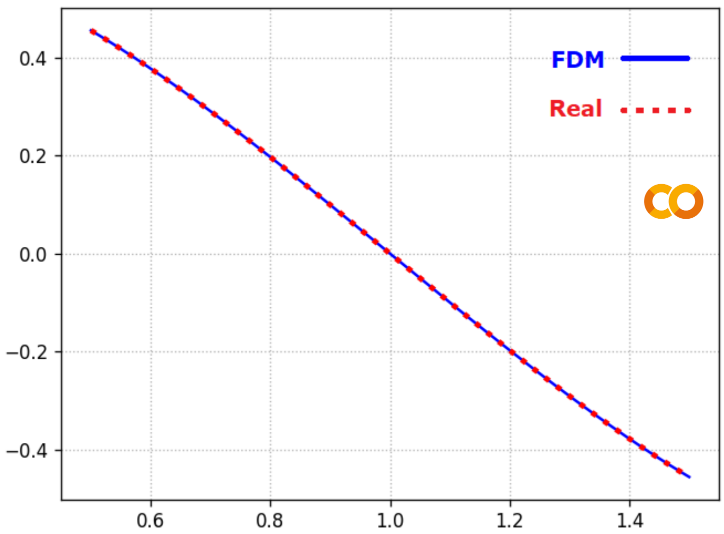

Fig. 9.11: The numerical and symbolic solutions coincide to a high degree of accuracy

9.5 Applications to Drones

Fig. 9.12: A four-rotor drone (Quadcopter)

Systems of ordinary differential equations arise in the study of the flight of many different types of drones. One particularly elegant example in this area is the modeling of the laws of motion of a **four-rotor drone (Quadcopter)** by solving a system of 12 equations for 12 unknown functions. Detailed explanations of this subject will be provided in the next version. For the time being, we provide a link to a Google Colab notebook that describes such an application. The notebook also includes an interesting programming project (in the Python programming language).

Quadcopter Programming Project

The programming project can be incorporated into a course on ordinary differential equations for engineering students.

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