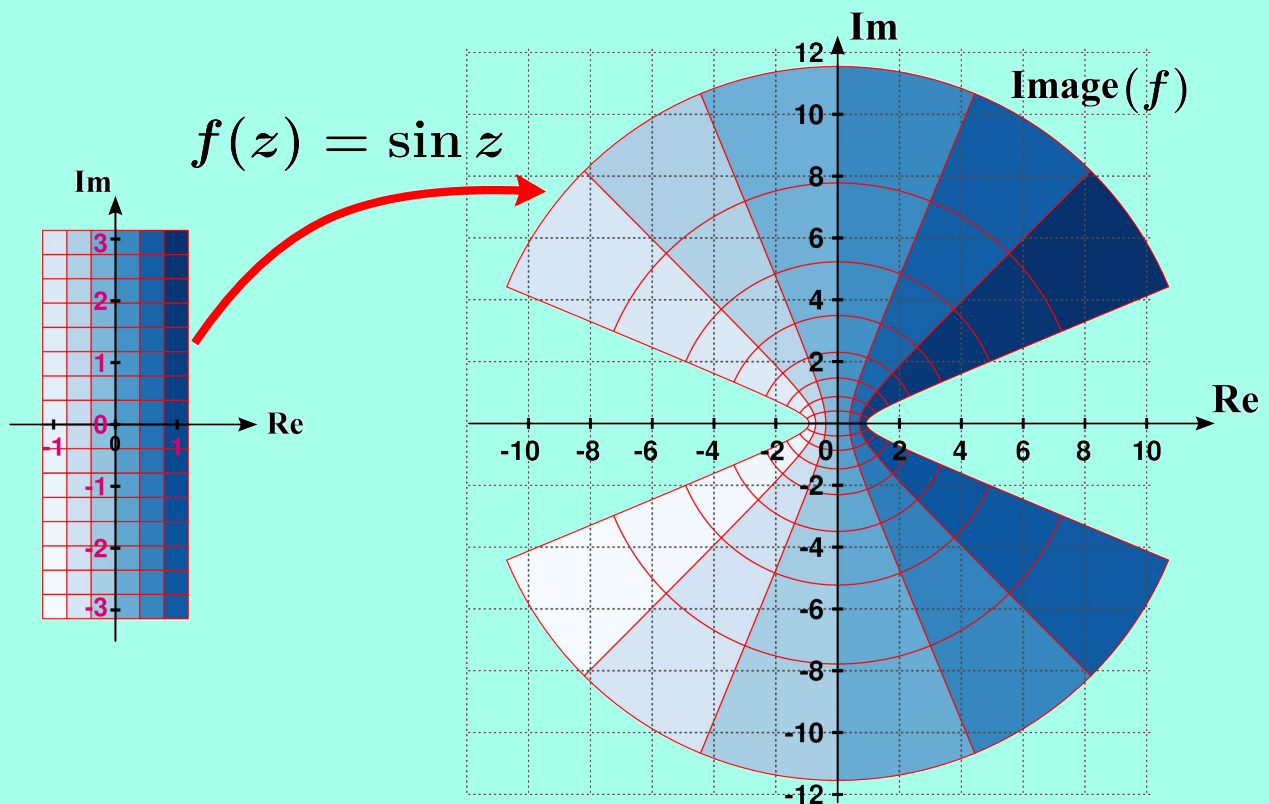

Complex Functions

Course



Link to most updated version: https://samyzaf.com/technion/cfit/cf_eng.pdf

Introduction

This booklet is a translation of our [Hebrew booklet](#) (which came out on May 2021) into English using the  ChatGPT artificial-intelligence model (version 5.6 Sol). Minor adjustments were made, but the vast majority of the work was carried out by , in a matter of 2 or 3 hours, saving us months of hard work! We are very thankful for the Technion Mathematics department for sponsoring the  ChatGPT subscription which enabled us to do this work. And of course, we are very grateful to [OpenAI Group PBC](#) for creating this miraculous AI tool which saved us months of hard work. See more details note on this process, after the introduction.

Since this is the first English edition, we would be pleased to receive feedback from readers. As far as my close examination goes, my impression is that  has done a superb translation job, and I do not anticipate to find many errors.

This booklet covers most of the material in an academic course on complex functions, as it is usually taught at institutions of higher education in the State of Israel. Its purpose was to help Israeli students, wherever they may be, study complex functions in a variety of academic settings (and outside them as well). After being translated to English, we hope it will be useful to students all over the world. It is based on extensive experience teaching similar courses at the Technion and at universities. It was written at the height of the COVID-19 pandemic, out of a desire to help students study the subject from home and bridge gaps created by the cancellation of in-person lectures and the difficulties of learning the material through Zoom. A special effort was made to present the material clearly and concisely (through “tweets”).

Special thanks are due to Dr. Daniel Rabayev of the Technion Faculty of Mathematics. His excellent lecture notes for the course Complex Functions and Integral Transforms (104221), and his unique and innovative approach to this subject helped us greatly, inspired us, and helped shape the spirit of this booklet. Many thanks also to all the students in the Winter 2018–2019 semester for their error corrections and helpful comments on the early drafts of several chapters of the booklet that were used in that course.

The format of the booklet is designed primarily for convenient reading on tablets and various other electronic devices available today (thereby perhaps saving a few trees on planet Earth). The large typeface, hyperlinks, and concision should also make reading easier and save time. We would be very pleased to receive comments, corrections, suggestions for improvement, and solutions to exercises (which are currently available for part of the exercises, although we intend to add more in future versions of the booklet).

The only effective way to learn mathematics is to study the material repeatedly while solving a large number of exercises. A set of exercises appropriate to the topics of each chapter is included at the end of that chapter. A special effort was made to collect and compose interesting exercises spanning a broad range of difficulty. Many of the exercises are accompanied by an emoji indicating the nature and type of their difficulty. A table explaining the meaning of these symbols appears at the end of the booklet. Most of the difficult exercises are accompanied by guidance that includes hints and outlines of possible solutions, and in many cases final answers are also provided. It is advisable to attempt the exercises first without this assistance and only afterward consult the guidance. We would be very pleased to receive comments, corrections, and suggestions for additional exercises (our email address appears at the bottom of every page). Full solutions to exercises will be gratefully received, and if enough are submitted, they will be published in a dedicated solutions booklet.

The present booklet is available on several different websites, but it is updated frequently. Not all the websites are updated at the same rate, and some contain older versions of the booklet. The most recent version of the booklet can be downloaded by clicking the following link:

 [to the latest version of the Complex Functions booklet](#)

We very much hope that the material is presented in a way that will enable lecturers and teaching assistants to teach the course as effectively as possible, and we wish every student who uses the booklet every success in their studies.

Samy Zafrany, September 2026

note on the English translation

The English edition of this booklet was prepared with the assistance of  (version 5.6 Sol). An initial experiment used the rendered PDF pages as input and produced fluent English, but reconstructing the original typesetting would have required considerable manual work. A second and more effective workflow supplied GPT with the raw Hebrew $\text{\LaTeX}$ source. The model was instructed to translate only the Hebrew natural-language text, including comments, captions, hints, and exercises, while leaving existing English and all $\text{\LaTeX}$ commands, environments, mathematics, labels, references, URLs, indentation, and line endings unchanged. The chapters were processed separately so that the terminology and mathematical prose could be reviewed carefully. After each chapter, automated comparisons checked the line count and the ordered sequence of $\text{\LaTeX}$ commands, mathematical fragments, hyperlinks, internal references, text encoding, and any Hebrew text that might remain. This method preserved the original formatting while producing a fully editable English $\text{\LaTeX}$ source, and it avoided many weeks of manual reformatting. Machine-generated language can still contain subtle linguistic or mathematical errors, so the resulting text requires careful human review by readers familiar with the subject. The author remains responsible for its mathematical and linguistic content; comments, corrections, and suggestions are warmly welcomed. The experience suggests that other authors can adapt this workflow to technical books and lecture notes, provided that they keep reliable backups of the original sources and combine machine translation with systematic validation and expert proofreading.



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Chapter 1

Complex Numbers

1.1 A History of the Number System

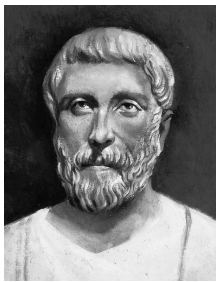


Fig. 1.1: Pythagoras
570BC–495BC

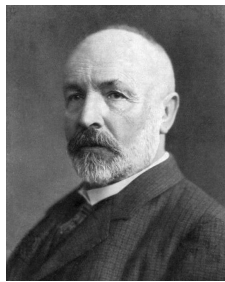


Fig. 1.2: Georg Cantor
1845–1918

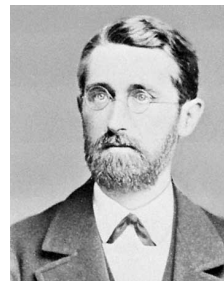


Fig. 1.3: Richard Dedekind
1831–1916



Fig. 1.4: Bernard Bolzano
1781–1848

- The natural numbers and the rational numbers (fractions) were already known to the ancient Egyptians and Babylonians, who used them for many purposes, including the construction of the pyramids. These numbers received a more scientific representation and treatment in Greek writings from the sixth century BCE onward.
- Irrational numbers were discovered by [Pythagoras](#) in the sixth century BCE, but they were not granted the same status as rational numbers and were regarded as mysterious and incomprehensible. Only about 2,500 years later did the mathematicians Georg Cantor, Richard Dedekind, and Bernard Bolzano give a mathematical definition of an irrational number and incorporate irrational numbers into the real number system on an equal footing.
- The concept of zero appeared around the fifth century in the Indian and Arabic cultures of that period and is considered one of the most important inventions in human history. It entered European use after the Muslim conquest of Spain, around the eleventh century, and was received there with great difficulty and suspicion; some religious authorities even regarded it as a satanic number. Its eventual symbol, an empty circle, was established by the Persian mathematician Muhammad ibn Musa al-Khwarizmi in the ninth century. Al-Khwarizmi needed zero for the



Fig. 1.5: Muhammad ibn Musa al-Khwarizmi 780–850



Fig. 1.6: Gerolamo Cardano 1501–1576



Fig. 1.7: Hero of Alexandria 10–70



Fig. 1.8: Diophantus of Alexandria approx. 210–280

creation of algebra, and especially for the development of algorithms for multiplying and dividing integers and rational numbers. The word **algorithm** derives from a corruption of his name.

- **Negative numbers.** The Greek mathematician Diophantus, who lived in the third century CE and is credited with founding number theory, regarded the equation $4x + 20 = 0$ as an absurd equation with no solution. Nevertheless, negative numbers appeared at about the same time in Chinese mathematical literature. Chinese mathematicians knew the elementary rules for adding and subtracting positive and negative numbers, and they could solve systems of equations having negative solutions. In seventh-century India, negative numbers were used to represent debt.
- The idea passed to Muslim mathematicians in the ninth century, but initially saw little significant use. Al-Khwarizmi, for example, did not mention negative numbers in his book on algebra. Only in the twelfth century did Muslim mathematicians begin to formulate arithmetic rules for signed quantities, such as $(a \pm b)(c \pm d)$, and even to accept negative roots of polynomials.
- European mathematicians resisted negative numbers until the seventeenth century. Even during the eighteenth century it was still common to ignore negative solutions of equations as meaningless. The first mathematician to integrate negative numbers as an essential part of a mathematical system was Gottfried Wilhelm Leibniz (1646–1716). After the publication of his work on differential and integral calculus, the use of negative numbers throughout science became natural and widespread.
- The “imaginary” number i received that name in the seventeenth century as an expression of contempt and hostility toward the idea. It was regarded as a fictitious and useless object, sometimes almost as a joke, just as irrational numbers, negative numbers, and zero had been before it.
- Although there is evidence that the idea of complex numbers was raised by the engineer and mathematician Hero of Alexandria (approximately 10–70 CE), their discovery is usually attributed to the Italian mathematician Gerolamo Cardano (1501–1576). Most mathematicians of Cardano’s time, and for many years afterward, did not recognize complex numbers as legitimate numbers.
- In his work on roots of cubic polynomials of the form

$$ax^3 + bx + c = 0,$$



Fig. 1.9: Gottfried Wilhelm Leibniz 1646–1716

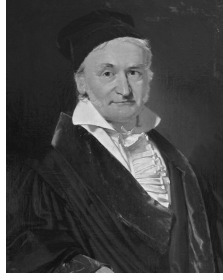


Fig. 1.10: Carl Friedrich Gauss 1777–1855



Fig. 1.11: Leonhard Euler 1707–1783



Fig. 1.12: Caspar Wessel 1745–1818

Cardano found that some roots could be obtained from the formula

$$x = \sqrt[3]{\frac{-c}{2a} + \sqrt{\frac{c^2}{4a^2} + \frac{b^3}{27a^3}}} + \sqrt[3]{\frac{-c}{2a} - \sqrt{\frac{c^2}{4a^2} + \frac{b^3}{27a^3}}}$$

In some cases the expression under the inner square root was negative. Thus, without accepting the “imaginary” number i , it was impossible to reach the roots of the polynomial even when all those roots were real: the imaginary components canceled only in the final calculation.

➤ For example, one root of the equation

$$x^3 - 15x - 4 = 0,$$

whose three roots are real, is given by

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Without using i , the real root cannot be recovered from this formula. With $\sqrt{-121} = 11i$ we obtain

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} = 4.$$

The other two roots are $-2 \pm \sqrt{3}$. Cardano’s discovery meant that without accepting complex numbers one could not solve higher-degree polynomials even in cases where every root was real.

- Further examples of this kind eventually made it clear that mathematics could not continue to develop without accepting complex numbers as first-class numbers. This view gathered momentum through the mathematical work of Leonhard Euler (1707–1783) and Carl Friedrich Gauss (1777–1855). The important representation of complex numbers as points in a two-dimensional plane was developed by the Belgian mathematician and cartographer Caspar Wessel (1745–1818).
- It was later discovered that many physical problems in electricity, electromagnetism, and quantum theory cannot be solved without complex numbers. The “imaginary” number i , for example, appears in one of the fundamental equations of quantum theory; without it, the laws of that theory cannot even be formulated.

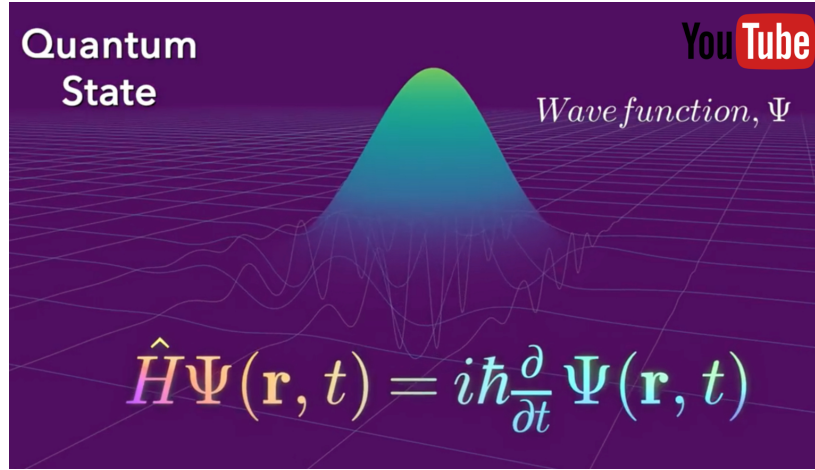


Fig. 1.13: A 3D surface-plot Schrodinger-equation quantum wave function visualization

- As a result of these discoveries, the disparaging name became less and less representative, while the standing of complex numbers in mathematics and physics steadily increased. Today complex numbers are regarded as every bit as legitimate as real numbers, despite the unfortunate historical terminology.
- Leonhard Euler defined the imaginary number i as a root of the equation $x^2 + 1 = 0$ and introduced the familiar notation¹

$$i = \sqrt{-1}$$

- After adding a new number to the number system, we must also add all numbers obtained by multiplying it by and adding it to the existing numbers. The resulting set is the set of complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

Yet this innocent-looking definition is only where the trouble begins. The set is closed under addition, subtraction, multiplication, and division, but mathematics also contains exponentiation and root extraction:

- ➡ How should a^b be defined when a and b are complex numbers? What, for example, are the values of 2^i and i^i ?
- ➡ How should the trigonometric and logarithmic functions be defined?
- ➡ Does the equation $x^2 + i = 0$ have a solution? If not, must another imaginary number be invented to solve it, followed perhaps by still more new numbers for further equations?
- In general, after complex numbers $a + bi$ are admitted, many subjects have to be rebuilt from the foundations upward. We must reconsider polynomials with complex coefficients, exponential equations, and trigonometric equations whose arguments and coefficients may now be complex.

¹ There is a small philosophical problem with this formula: -1 has two square roots, so which one is i ? It turns out not to matter. The two choices produce the same algebraic structure and cannot be intrinsically distinguished. Once one root has been denoted by i , the other is $-i$.

Is the seemingly innocent act of introducing i a recipe for disaster?

- After many years of investigation, all these questions received satisfactory answers. Adding complex numbers does not complicate mathematics; it simplifies and unifies many of its areas. Complex numbers are sufficient to answer all the questions above, and no sequence of ever-new imaginary numbers is needed. Moreover, complex numbers make it possible to solve real problems that could not otherwise be solved. This strongly supports the conclusion that they form a natural, correct, and necessary extension of the real number system.

1.2 The Field of Complex Numbers: Basic Properties

- Syntactically, the complex numbers may be defined as all expressions $a + bi$, where $a, b \in \mathbb{R}$, with addition and multiplication given by

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi) \cdot (c + di) &= (ac - bd) + (ad + bc)i\end{aligned}$$

- The multiplication formula follows immediately from the fundamental identity

$$i^2 = -1$$

- In many contexts it is useful to represent the complex numbers by the two-dimensional vector space

$$\mathbb{R}^2 = \{ (a, b) : a, b \in \mathbb{R} \}$$

which is also the standard algebraic model of the Euclidean plane. Vector addition is the usual one; if we add the following multiplication, we obtain a vector model of the complex numbers:

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc)\end{aligned}$$

This is only one possible model. It is called the **complex plane** and was first presented by Caspar Wessel in 1799. Throughout the course we shall therefore sometimes treat a complex number as a vector in $\mathbb{R}^2$.

- Another important geometric model, which we shall meet later, is [the Riemann Sphere](#).
- A third important model represents $a + bi$ by the square matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. These matrices also represent linear transformations of the real plane. In this model every complex number can be



interpreted as a linear map that contracts, stretches, and rotates the plane. For example,

$$i \longleftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is rotation by 90° counterclockwise:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

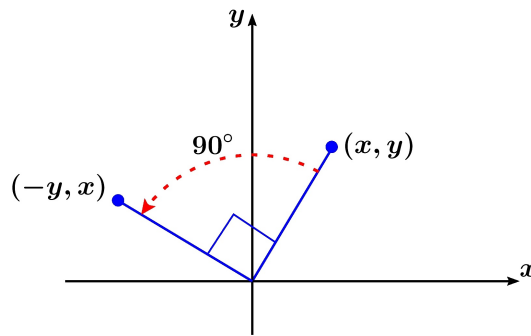


Fig. 1.14: The linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (-y, x)$, a 90° counterclockwise rotation of the real plane

- ◉ Besides helping us visualize and understand complex numbers, these geometric models have many consequences for plane and spatial geometry, some of which we shall encounter later. Algebraically, the complex numbers form a **field** under the addition and multiplication defined above. We recall the definition.

i **Definition 1.1:** A **field** is an algebraic structure consisting of a set F and two binary operations, called addition and multiplication and denoted by $+$ and $\cdot$. The set contains at least two distinguished elements, 0 and 1 , and for all $a, b, c \in F$ satisfies:

1. **Additive identity:** $a + 0 = a$.
2. **Multiplicative identity:** $a \cdot 1 = a$.
3. **Commutativity:** $a + b = b + a$ and $ab = ba$.
4. **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
5. **Additive inverses:** for each a there is $b \in F$ such that $a + b = 0$.
This element is denoted $-a$.
6. **Multiplicative inverses:** if $a \neq 0$, there is $b \in F$ such that $ab = 1$.
This element is denoted a^{-1} .
7. **Distributivity:** $a(b + c) = ab + ac$.



i Theorem 1.1: The set $\mathbb{C}$ of complex numbers is a field.

✓ **Proof:** The zero and identity are the real numbers 0 and 1 . The first four field axioms follow directly from the definitions and are left to the reader. The additive inverse of $a + bi$ is plainly $-a - bi$.

For the multiplicative inverse, let $a + bi \neq 0$. We seek a complex number $x + yi$ satisfying

$$(a + bi)(x + yi) = (ax - by) + (ay + bx)i = 1 + 0i.$$

This is equivalent to the real linear system

$$\begin{cases} ax - by = 1 \\ bx + ay = 0 \end{cases}$$

Its coefficient determinant is

$$\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 + b^2 > 0$$

The determinant is nonzero because $a + bi \neq 0$, so the system has a unique solution. Cramer's rule gives

$$x = \frac{\begin{vmatrix} 1 & -b \\ 0 & a \end{vmatrix}}{\begin{vmatrix} a & -b \\ b & a \end{vmatrix}} = \frac{a}{a^2 + b^2} \quad y = \frac{\begin{vmatrix} a & 1 \\ b & 0 \end{vmatrix}}{\begin{vmatrix} a & -b \\ b & a \end{vmatrix}} = -\frac{b}{a^2 + b^2}$$

Consequently,

$$(a + bi)^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i \quad (1.1)$$

The distributive law follows immediately from the definitions. Thus all seven axioms hold.

➤ We may now use the standard name **the field of complex numbers** for $\mathbb{C}$. If $z = a + bi$, then a is called the **real part** of z and b its **imaginary part**:

$$\operatorname{Re}(z) = a, \quad \operatorname{Im}(z) = b.$$

For $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned} \operatorname{Re}(z_1 + z_2) &= \operatorname{Re}(z_1) + \operatorname{Re}(z_2) \\ \operatorname{Im}(z_1 + z_2) &= \operatorname{Im}(z_1) + \operatorname{Im}(z_2) \end{aligned}$$



The **absolute value**, or **modulus**, of $z = a + bi$ is defined by

$$|z| = \sqrt{a^2 + b^2}$$

In many calculations it is more convenient to use the equivalent identity

$$|z|^2 = a^2 + b^2 \quad (1.2)$$

➤ The **complex conjugate** of $z = a + bi$ is

$$\bar{z} = a - bi$$

Conjugation is especially useful in the technique of “multiplying by the conjugate.” It allows us to write (1.1) more compactly:

$$z^{-1} = \frac{\bar{z}}{|z|^2}, \quad z \neq 0 \quad (1.3)$$

For $z, w \in \mathbb{C}$ with $w \neq 0$, complex division is defined by

$$\frac{z}{w} = z \cdot w^{-1}$$

The fundamental conjugation identity is

$$z\bar{z} = |z|^2$$

Multiplying by the conjugate gives a practical division formula. If $z = a + bi$ and $w = c + di$, then

$$\begin{aligned} \frac{z}{w} &= \frac{z \cdot \bar{w}}{|w|^2} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i \end{aligned}$$



Thus

$$\frac{z}{w} = \frac{z \cdot \overline{w}}{|w|^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i \quad (1.4)$$

➤ The next theorem collects twelve elementary identities that will be used to build more complicated formulas and prove further results.

i Theorem 1.2: The 12 Fundamental Identities

- | | |
|--|---|
| (1) $ \overline{z} = z $ | (7) $\operatorname{Im}(z) = (z - \overline{z})/2i$ |
| (2) $z\overline{z} = z ^2$ | (8) $ z_1 z_2 = z_1 z_2 $ |
| (3) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ | (9) $ \operatorname{Re}(z) \leq z $ |
| (4) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ | (10) $ \operatorname{Im}(z) \leq z $ |
| (5) $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$ | (11) $\frac{ z_1 }{ z_2 } = \frac{ z_1 }{ z_2 }$ |
| (6) $\operatorname{Re}(z) = (z + \overline{z})/2$ | (12) $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$ |

In (11) and (12), $z_2 \neq 0$.

♥ **Proof:** We prove identities (5), (8), (9), and (11); the rest are left as exercises. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

For (5),

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i} \\ &= (x_1 x_2 - y_1 y_2) - (x_1 y_2 + x_2 y_1)i \\ &= x_1 x_2 - x_1 y_2 i - y_1 y_2 - y_1 x_2 i \\ &= x_1(x_2 - y_2 i) - y_1(x_2 - iy_2)i \\ &= (x_1 - y_1 i)(x_2 - y_2 i) = \overline{z_1} \overline{z_2} \end{aligned}$$

For (8), it is convenient to square the modulus and use (5):

$$|z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = z_1 z_2 \overline{z_1} \overline{z_2} = z_1 \overline{z_1} z_2 \overline{z_2} = |z_1|^2 |z_2|^2$$

Both sides are nonnegative, so $|z_1 z_2| = |z_1| |z_2|$.

For (9), if $z = x + yi$, then

$$|\operatorname{Re}(z)| = |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |z|$$



Finally, to prove (11), we use (1), (5), and (8),

$$\left| \frac{z_1}{z_2} \right| = \left| \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} \right| = \left| \frac{z_1 \cdot \overline{z_2}}{|z_2|^2} \right| = \frac{|z_1 \cdot \overline{z_2}|}{|z_2|^2} = \frac{|z_1| \cdot |\overline{z_2}|}{|z_2|^2} = \frac{|z_1| \cdot |z_2|}{|z_2|^2} = \frac{|z_1|}{|z_2|}$$

➤ To prove the next theorem, We use the immediate identity $\overline{\overline{z}} = z$.

i Theorem 1.3: (Triangle inequalities) For all $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned} |z_1 + z_2| &\leq |z_1| + |z_2| \\ |z_1 - z_2| &\geq ||z_1| - |z_2|| \end{aligned}$$

✓ **Proof:** As before, we work with the square of the modulus:

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= |z_1|^2 + z_1 \overline{z_2} + \overline{z_1} z_2 + |z_2|^2 \\ &= |z_1|^2 + 2 \operatorname{Re}(z_1 \overline{z_2}) + |z_2|^2 \\ &\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= (|z_1| + |z_2|)^2. \end{aligned}$$

Taking nonnegative square roots proves the first inequality.

The second follows at once from the first:

$$\begin{aligned} |z_1| &= |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2| \\ |z_1| &= |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|, \end{aligned}$$

so

$$|z_1| - |z_2| \leq |z_1 - z_2|.$$

Interchanging z_1 and z_2 yields the symmetric absolute-value form.

➤ The first picture represents z_1 , z_2 , and $z_1 + z_2$ as the sides of a triangle, so the first inequality is the familiar geometric fact that the sum of two side lengths is at least the third side. The second picture gives a corresponding geometric interpretation of the reverse inequality.



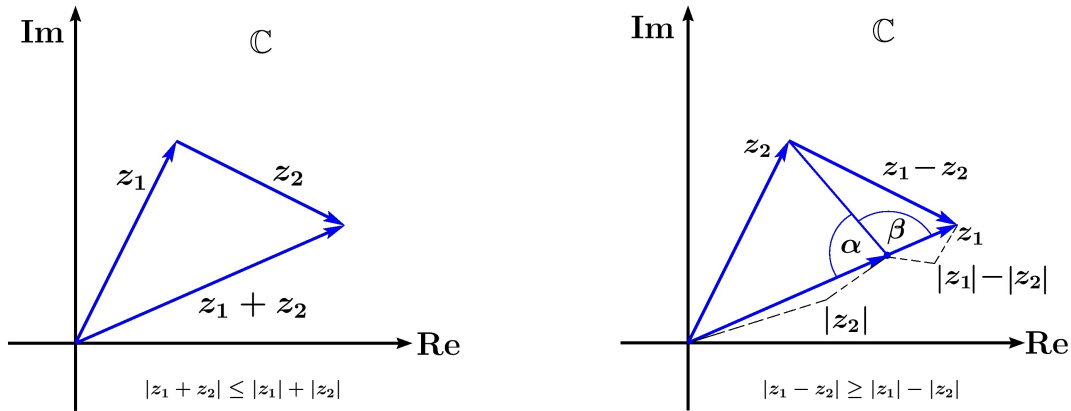


Fig. 1.15: Vector representations of the triangle inequality and the reverse triangle inequality in the $\mathbb{R}^2$ model of the complex numbers

1.3 Polar Representation of a Complex Number

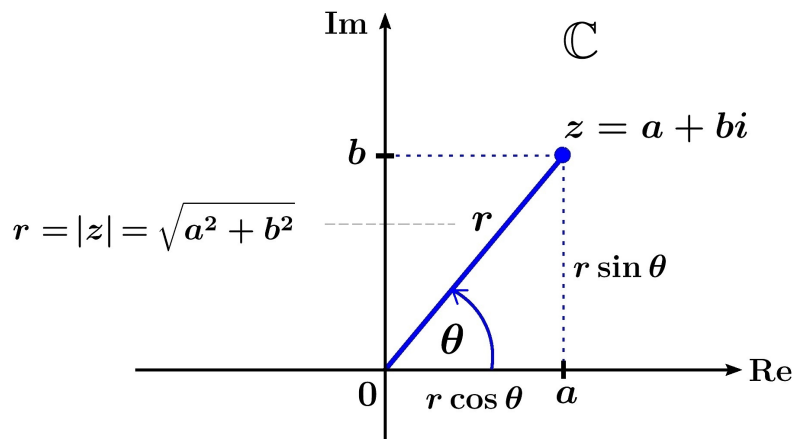


Fig. 1.16: The complex number $z = a + bi$ described by its modulus $r = |z|$ and argument θ , with $z = r(\cos \theta + i \sin \theta)$

- The polar representation of complex numbers came into significant use through the work of the Swiss mathematician [Leonhard Euler](#) (1701-1783), in connection with power-series expansions of the function e^{iz} .
- As we see in figure 1.16, the two principal quantities in the polar representation are the modulus r and the argument θ , the directed angle from the positive real axis to the radius vector of the point representing z .
- The modulus is

$$r = |z| = \sqrt{a^2 + b^2}$$

It is the distance from the Cartesian point (a, b) representing $z = a + bi$ to the origin $(0, 0)$. We shall also use r for the geometric line segment from the origin to (a, b) when the meaning is clear.

➤ As we see in figure [1.16](#)

$$\begin{aligned} a &= r \cos \theta \\ b &= r \sin \theta \end{aligned}$$

Hence the **polar representation** of a complex number is

$$z = a + bi = r(\cos \theta + i \sin \theta) \quad (1.5)$$

The pair (r, θ) is called a system of **polar coordinates** for z . The nonnegative real number r is the **modulus**, and the angle θ is the **argument** of z .

➤ Unlike the Cartesian representation (a, b) , the polar representation is not unique. If θ is an argument, then adding any integral multiple $2n\pi$ gives another representation of the same number:

$$a + bi = r(\cos \theta + i \sin \theta) = r[\cos(\theta + 2n\pi) + i \sin(\theta + 2n\pi)].$$

For example,

$$\begin{aligned} i &= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \\ &= \cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2} \\ &= \cos \frac{9\pi}{2} + i \sin \frac{9\pi}{2} \\ &= \cos \frac{(2k+1)\pi}{2} + i \sin \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z} \end{aligned}$$

This nonuniqueness sometimes creates difficulties, but it also has important uses in solving problems that cannot be understood through Cartesian form alone.

A Preliminary Formula for the Argument

➤ For $z = x + iy$, a preliminary expression for the possible argument values is

$$\theta \in \arg(z) = \begin{cases} \arctan \frac{y}{x} + 2n\pi, & n \in \mathbb{Z}, \quad x \neq 0 \\ \frac{\pi}{2} + 2n\pi, & n \in \mathbb{Z}, \quad x = 0, y > 0 \\ \frac{3\pi}{2} + 2n\pi, & n \in \mathbb{Z}, \quad x = 0, y < 0 \\ \text{Undefined} & x = 0, y = 0 \end{cases} \quad (1.6)$$

The notation $\arg(z)$ is called the **argument function**, but it is actually multivalued and therefore is not a function in the strict sense: every nonzero z , $\arg(z)$ has infinitely many values. As Section [1.4](#) will explain, the first line of (1.6) does not yet contain enough information to choose the correct quadrant when $x < 0$.

➤ To obtain a single-valued assignment, the argument must be restricted to an interval such as

$[0, 2\pi)$ or $(-\pi, \pi]$. More generally, any interval of the form $[\alpha, \alpha + 2\pi)$ or $(\alpha, \alpha + 2\pi]$ will do. Under such a restriction, each nonzero complex number has one and only one angle in the interval.

- Most textbooks use $(-\pi, \pi]$ and define the single-valued function

$$\text{Arg}(\cdot) : \mathbb{C} \setminus \{0\} \longrightarrow (-\pi, \pi],$$

called the **principal branch of the argument**. The multivalued argument can then be expressed through the principal branch as

$$\arg(z) = \{ \text{Arg}(z) + 2n\pi : n \in \mathbb{Z} \} \quad (1.7)$$

Euler's Formula

- In 1740 Euler proved that for every real number θ ,

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (1.8)$$

Euler discovered a deep relationship among the Taylor expansions of the complex functions e^z , $\sin z$, and $\cos z$, which we shall study later in the course.

- At this stage we do not yet have the tools needed to define the complex exponential e^z . For the time being, therefore, $e^{i\theta}$ will be used only as notation for the complex number $\cos \theta + i \sin \theta$. Another common notation is

$$\text{cis}(\theta) = \cos \theta + i \sin \theta$$

This text will prefer $e^{i\theta}$, although the notation $\text{cis}(\theta)$ may appear in older exercise sheets and examinations.

- The most conspicuous property of this expression is that its modulus equals **1** for every θ :

$$|e^{i\theta}| = |\cos \theta + i \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1.$$

$$|e^{i\theta}| = |\cos \theta + i \sin \theta| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

- The unit circle is central to understanding the structure of $\mathbb{C}$ and has many applications. In a sense, it is the essence of $\mathbb{C}$, since every nonzero complex number is a positive real multiple of a point on the unit circle: $z = re^{i\theta}$.

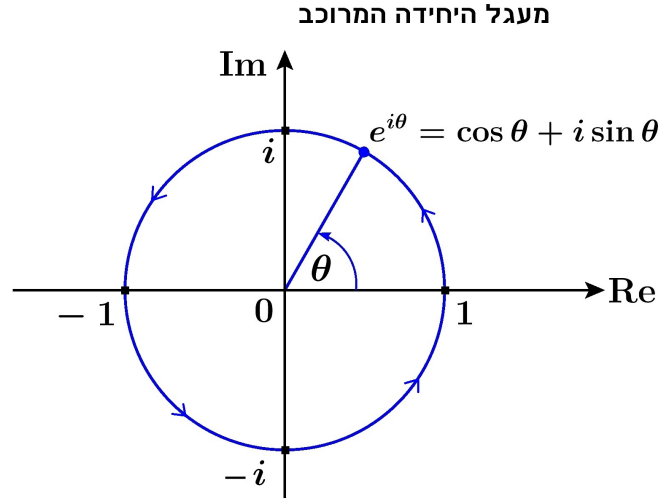


Fig. 1.17: The complex unit circle, consisting of all numbers $\cos \theta + i \sin \theta$ with $\theta \in \mathbb{R}$

► The four axis points give

$$e^{0i} = 1, \quad e^{\pi i/2} = i, \quad e^{\pi i} = -1, \quad e^{3\pi i/2} = -i.$$

Further useful identities are

$$\begin{aligned} e^{\pi i/4} &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, & e^{3\pi i/4} &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \\ e^{5\pi i/4} &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, & e^{7\pi i/4} &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i. \end{aligned} \tag{1.9}$$

Likewise,

$$\begin{aligned} e^{\pi i/6} &= \frac{\sqrt{3}}{2} + \frac{1}{2}i, & e^{\pi i/3} &= \frac{1}{2} + \frac{\sqrt{3}}{2}i, \\ e^{2\pi i/3} &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i, & e^{5\pi i/6} &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \\ e^{7\pi i/6} &= -\frac{\sqrt{3}}{2} - \frac{1}{2}i, & e^{4\pi i/3} &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \\ e^{5\pi i/3} &= \frac{1}{2} - \frac{\sqrt{3}}{2}i, & e^{11\pi i/6} &= \frac{\sqrt{3}}{2} - \frac{1}{2}i. \end{aligned} \tag{1.10}$$

Polar Multiplication

► Let

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$

be two complex numbers in polar form. Then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}. \tag{1.11}$$

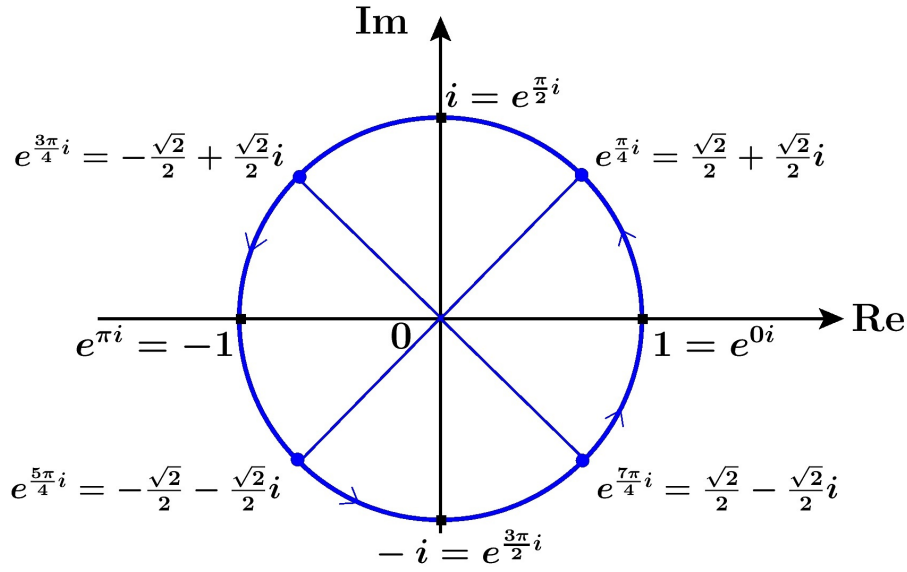


Fig. 1.18: Canonical points at multiples of $\pi/4$, on the complex unit circle

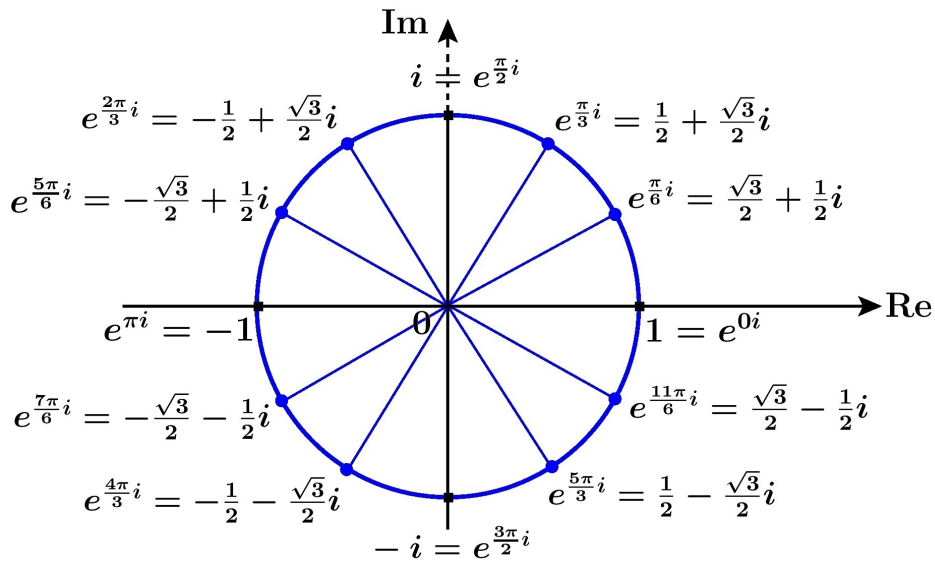


Fig. 1.19: Canonical points at multiples of $\pi/6$, and $\pi/3$, on the complex unit circle

Thus the polar representation of a product is obtained by multiplying the moduli and adding the arguments. Equivalently, multiplication by $re^{i\theta}$ scales by r and rotates counterclockwise through the angle θ .

➤ Indeed, using the trigonometric addition formulas,

$$\begin{aligned}
 z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2) \\
 &= r_1 r_2 \left[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \right. \\
 &\quad \left. + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) \right] \\
 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\
 &= r_1 r_2 e^{i(\theta_1 + \theta_2)}.
 \end{aligned}$$

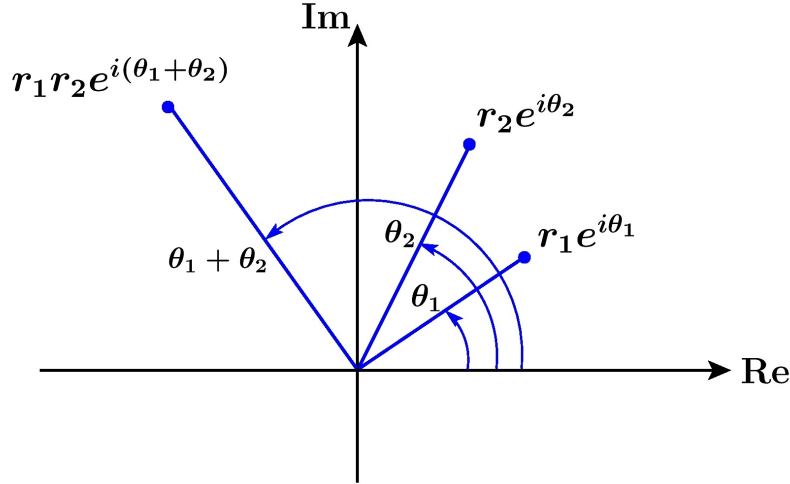


Fig. 1.20: The polar representation of a product: moduli are multiplied and arguments are added

i Proposition 1.4: Multiplication of a complex number z by $e^{i\alpha}$ rotates z counterclockwise through the angle α about the origin. In other words, if $z = re^{i\theta}$, then

$$e^{i\alpha} z = re^{i(\theta+\alpha)}. \quad (1.12)$$

De Moivre's Formula

➤ For every natural number n and every real angle θ ,

$$(e^{i\theta})^n = e^{in\theta}, \quad (1.13)$$

or equivalently,

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta). \quad (1.14)$$

- Recall that $e^{i\theta}$ is currently only convenient notation for $\cos \theta + i \sin \theta$, so the ordinary laws of exponents cannot yet be assumed. De Moivre's formula is therefore a substantive assertion: the n th power of a point on the unit circle is another point on the unit circle whose angle has been multiplied by n .
- The proof is by induction. For $n = 1$ the result is immediate. Suppose $(e^{i\theta})^n = e^{in\theta}$. By the polar multiplication formula,

$$(e^{i\theta})^{n+1} = e^{i\theta}(e^{i\theta})^n = e^{i\theta}e^{in\theta} = e^{i(n+1)\theta}.$$

This completes the induction.

➤ Corollary: For every $r \geq 0$, real θ , and natural number n ,

$$(re^{i\theta})^n = r^n e^{in\theta}. \quad (1.15)$$

Polar Division

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, with $z_2 \neq 0$. Then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}. \quad (1.16)$$

Thus division by $r e^{i\theta}$ divides the modulus by r and rotates clockwise through the angle θ . To verify the formula, multiply its right-hand side by z_2 :

$$z_2 \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} = r_2 e^{i\theta_2} \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} = r_1 e^{i\theta_1} = z_1.$$

Division by z_2 gives (1.16).

Roots of Unity

Proposition 1.5: For every positive natural number n , the equation

$$z^n = 1$$

has exactly n distinct roots, given by

$$z_k = e^{2k\pi i/n} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, 2, \dots, n-1. \quad (1.17)$$

They are called the *n th roots of unity*.

Proof: Write $z = r e^{i\theta}$. The equation $z^n = 1$ becomes

$$r^n e^{in\theta} = 1.$$

Taking moduli gives $r^n = 1$. Since r is a nonnegative real number, $r = 1$. It remains to solve

$$e^{in\theta} = 1.$$

Because $1 = e^{2k\pi i}$ for every $k \in \mathbb{Z}$, we require

$$e^{in\theta} = e^{2k\pi i}, \quad k \in \mathbb{Z},$$

and hence

$$\theta = \frac{2k\pi}{n}, \quad k \in \mathbb{Z}.$$

Modulo 2π , the distinct values are

$$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \dots, \frac{2(n-1)\pi}{n}. \quad (1.18)$$



Every other integer k gives a value already represented in this list. Thus there are exactly n roots.

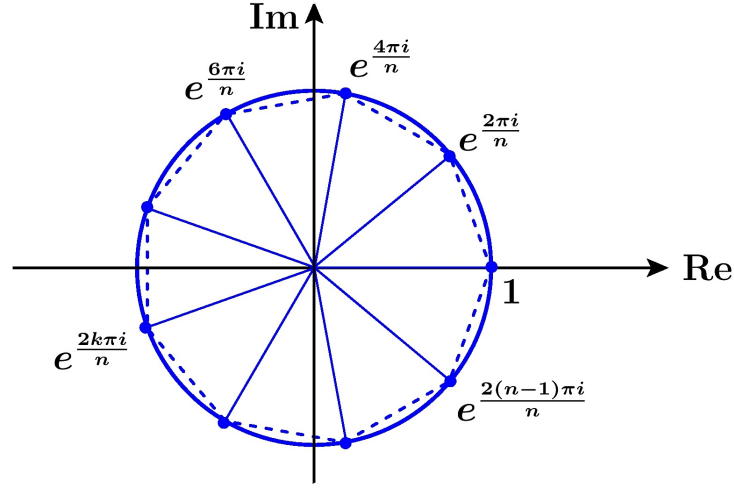


Fig. 1.21

Roots of a Nonzero Complex Number

i Proposition 1.6: For every $z \in \mathbb{C} \setminus \{0\}$, the equation

$$u^n = z$$

has exactly n distinct roots. If $z = re^{i\theta}$, they are

$$u_k = \sqrt[n]{r} e^{i(\theta+2k\pi)/n}, \quad k = 0, 1, 2, \dots, n-1. \quad (1.19)$$

✓ Proof: Let $u = qe^{i\alpha}$ be a root. By De Moivre's formula,

$$u^n = q^n e^{in\alpha} = re^{i\theta} = re^{i(\theta+2k\pi)}, \quad k \in \mathbb{Z}.$$

Since q and r are positive real numbers, $q = \sqrt[n]{r}$, and

$$\alpha = \frac{\theta + 2k\pi}{n}, \quad k \in \mathbb{Z}.$$

Periodicity leaves exactly the n distinct values corresponding to $k = 0, 1, \dots, n-1$, and therefore exactly the n roots listed in [1.19](#).

1.4 The Argument Function and Its Principal Branch

- We did not dwell on this topic during the lecture because it is time-consuming and can be confusing. Here we complete the missing details and deepen our understanding of the argument function.
- The argument $\theta \in \arg(z)$ is supposed to provide all directed angles between the positive real axis and the radius vector of $z = x + iy$. There are infinitely many such angles because adding an integral multiple of 2π leaves z at the same point. Is there a convenient formula for $\arg(z)$?
- To answer this, we seek a single-valued branch, denoted by $\mathbf{Arg}$, that returns angles in one cycle, such as $[0, 2\pi)$ or $(-\pi, \pi]$. The first idea that comes to mind is the inverse tangent:

$$\mathbf{Arg}(z) = \arctan \frac{y}{x}. \quad (\text{Bad idea!})$$

This formula works in the first quadrant, where $x > 0$ and $y \geq 0$, but it is undefined on the imaginary axis because $x = 0$.

- If $z = x + iy$ lies in the second quadrant, where $x < 0$ and $y \geq 0$, then $\arctan(y/x)$ returns an angle in $(-\pi/2, 0]$, whereas a second-quadrant angle should lie in $[\pi/2, \pi]$. The third quadrant also receives the wrong angle, since the one-variable arctangent cannot distinguish antipodal points: it returns the same value for (x, y) and $(-x, -y)$. Consequently, values returned for third-quadrant points actually belong to the first quadrant. In the fourth quadrant, arctangent returns values in $(-\pi/2, 0]$. Even if one were to define it on the imaginary axis, this attempted branch would be discontinuous across several coordinate rays.

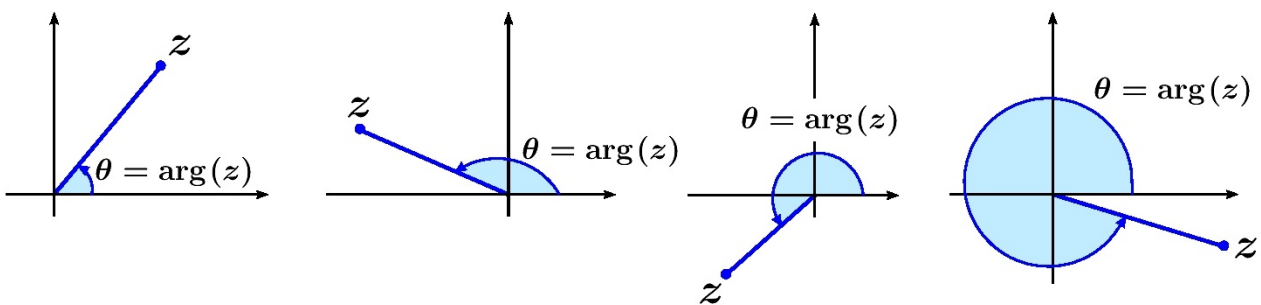


Fig. 1.22: The argument of z is the directed angle between the positive real axis and the radius vector of z

- To overcome these defects, IBM introduced a new function called **arctan2**, usually abbreviated **atan2**, in the FORTRAN programming language in the early 1950s. Its definition was carefully designed to provide a complete basis for a single-valued branch of the argument that is continuous on as much of the plane as possible. The function

$$\mathbf{atan2} : \mathbb{R}^2 \setminus \{(0, 0)\} \longrightarrow (-\pi, \pi]$$

is defined by

$$\text{atan2}(y, x) = \begin{cases} \arctan(y/x), & x > 0, \\ \arctan(y/x) + \pi, & x < 0 \text{ and } y \geq 0, \\ \arctan(y/x) - \pi, & x < 0 \text{ and } y < 0, \\ \pi/2, & x = 0 \text{ and } y > 0, \\ -\pi/2, & x = 0 \text{ and } y < 0, \\ \text{undefined}, & x = 0 \text{ and } y = 0. \end{cases}$$

Unlike **arctan**, the function **atan2** takes two arguments: the coordinates of a point in the plane, in the order y, x .

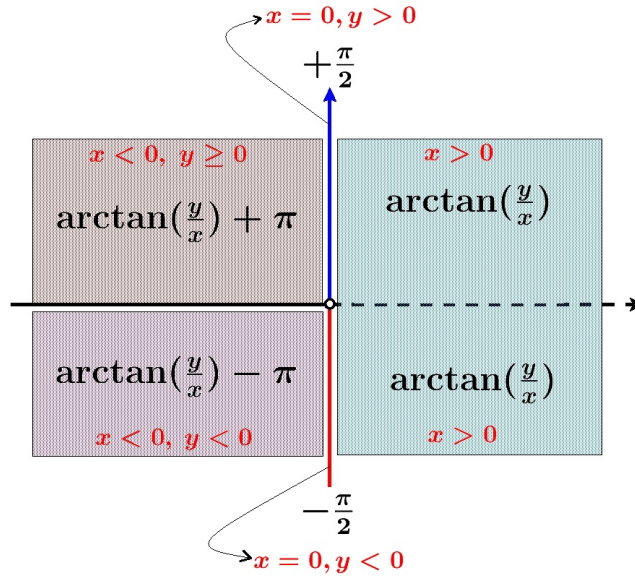


Fig. 1.23: The five regions in the definition of **atan2**: the open right half-plane; the closed upper-left quadrant; the open lower-left quadrant; the positive imaginary axis; and the negative imaginary axis.

- The two constraints behind this definition are that all values lie in $(-\pi, \pi]$ and that the function be continuous on the largest possible region. The function is not continuous along the negative real axis and is not defined at the origin.
- Although the definition may look artificial, a heat map of

$$\text{atan2} : \mathbb{R}^2 \setminus \{(0, 0)\} \longrightarrow (-\pi, \pi]$$

reveals an almost continuous and natural picture. Each point (x, y) is colored according to the value **atan2**(y, x). In a grayscale map, the values vary from white near $-\pi$ to black near π .

- The heat map shows that **atan2** is continuous throughout the plane except on the nonpositive real ray $(-\infty, 0]$, where its value drops from π to $-\pi$; at the endpoint 0 it is undefined. We can now define the principal branch of the argument formally by

$$\text{Arg}(z) = \text{Arg}(x + iy) = \text{atan2}(y, x). \quad (1.20)$$

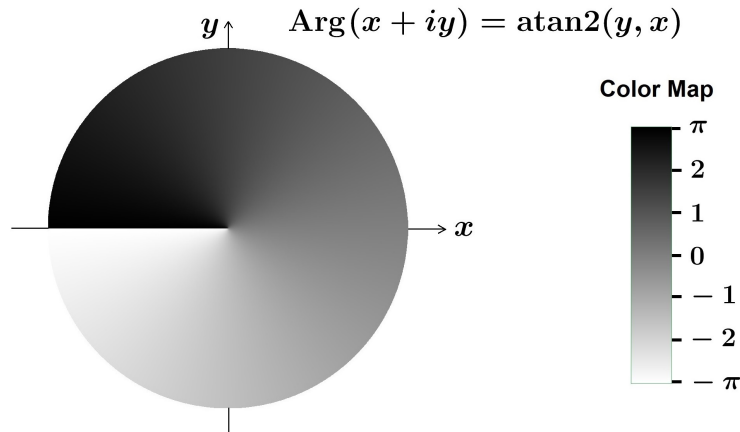


Fig. 1.24: A heat map of $\text{atan2}(y, x)$, with each point colored by the corresponding argument value.

The multivalued argument is

$$\arg(z) = \text{Arg}(z) + 2n\pi, \quad n \in \mathbb{Z}, \quad (1.21)$$

meaning the set of all values $\{\text{Arg}(z) + 2n\pi : n \in \mathbb{Z}\}$. Thus every nonzero z has infinitely many arguments.

- The set obtained by removing the ray $(-\infty, 0]$ from the complex plane is called the **slit plane** and is written

$$\mathbb{C} \setminus (-\infty, 0].$$

The continuity of atan2 on this slit plane follows directly from the two-variable definition of continuity studied in multivariable calculus.

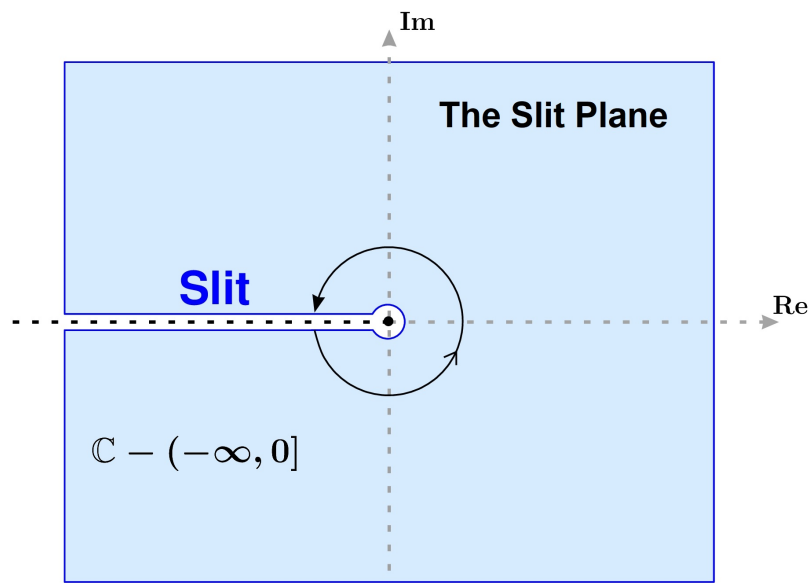


Fig. 1.25: The complex plane with a slit along the nonpositive real axis

1.5 Links to Online Resources

The following is a selection of online materials about complex numbers and complex functions. It is only a drop in the ocean of information available online; its purpose is to suggest the kinds of resources one can search for and find. Skillful use of search engines can be very helpful in learning any course and is worth practicing frequently. Artificial-intelligence agents such as  ChatGPT can also provide supplementary information and help with problem solving. These AI agents can also be used for digesting the text in this book and provide further summaries and analysis.

The original chapter links to resources bearing the following titles:

 [When are imaginary numbers used in real life? What practical applications do they have?](#)

 [What are the applications of complex numbers?](#)

 [What are some real life applications of complex numbers in engineering and practical life?](#)

 [Do imaginary numbers have any practical applications?](#)

 [An Introduction to Complex Analysis for Engineers](#)

 Chat GPT

 [Using Harmonic functions in the animation industry \(Pixar\)](#)

 [The Real World Uses of Imaginary Numbers](#)

 [Imaginary Numbers Are Real \[Part 1: Introduction\]](#)

 [Complex Numbers: AC Circuit Application](#)

 [Divergence and curl: The language of Maxwell's equations, fluid flow, and more](#)

 [L1.3 Necessity of complex numbers.](#)

 [4 APPLICATIONS OF COMPLEX NUMBERS IN REAL DAILY LIFE](#)

 [Complex number fundamentals — Lockdown math ep. 3](#)

 [How Imaginary Numbers Were Invented](#)

Exercises for chapter 1

Guidance: The following link links to a Google notebook containing final and complete solutions to some of these exercises: [link to Google Colab notebook](#). Further solutions may be added over time, and students are invited to contribute solutions to exercises not yet covered.

1. Simplify the following expressions:

$$(a) \frac{1+2i}{3-4i} + \frac{2-i}{5i} \quad (b) \frac{10i}{(1-i)(2-i)(3-i)} \quad (c) (1-i)^4$$

$$(d) (\sqrt{3}+i)^{38} \quad (e) \sqrt[4]{1-i\sqrt{3}} \quad (f) \left(\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}}\right)^4$$



Answer: (a) $-\frac{2}{5}$ (b) -1 (c) -4

2. Write the following expression in the form $x + iy$:

$$\frac{a + ib}{a - ib} - \frac{a - ib}{a + ib}.$$



3. Solve the following equations:

(a) $z^2 + \sqrt{32}z - 6i = 0$ (b) $z^2 = \bar{z}$ (c) $z^2 = -|z|$
 (d) $4z^2 + 8|z|^2 - 3 = 0$ (e) $z^3 = 2iz$ (f) $z^n = \bar{z}$
 (g) $z^2 = |z|^2 - 4$ (h) $z^2 = i(|z|^2 - 4)$

Hint: For part (f), use $z = re^{i\theta}$. It is easy to obtain $r = 0$ or $r = 1$; then solve $e^{i(n+1)\theta} = 1$.

4. Find a Cartesian description of the solution set of each equation:

(a) $\operatorname{Re}(z^2) = \operatorname{Im}(z^2)$ (b) $\operatorname{Im}(z^2)/z^2 = -i$ (c) $\arg\left(\frac{z}{z-1}\right) = \frac{\pi}{2}$

Hint: A Cartesian solution is an algebraic expression in x and y describing all points $z = x + iy$ that satisfy the equation. For example, the solution of (a) is $y = (-1 \pm \sqrt{2})x$; the solution of (b) is $y = \pm x$; and the solution of (c) is $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$, with $y < 0$.

5. The complex number $z = 2 + i$ is a solution of

$$z^4 - 6z^3 + 16z^2 - 22z + k = 0,$$

where k is an integer. Find all the other solutions and the corresponding value of k .



6. Prove that for any two nonzero complex numbers z_1, z_2 ,

$$\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) \quad (1.9)$$

Remark: The function $\arg$ is multivalued, so the difference on the right has infinitely many values. Strictly speaking, prove that the set of values of $\arg(z_1/z_2)$ equals the set of all differences between values in $\arg(z_1)$ and values in $\arg(z_2)$.

7. Use (1.9) to calculate the angles of the triangle formed by three distinct, noncollinear points z_1, z_2, z_3 .

Hint: Translate the triangle so that one vertex is at the origin. For example, translating z_1 to the origin gives the congruent triangle $\{0, z_2 - z_1, z_3 - z_1\}$.



8. Let z_1, z_2 be nonzero, linearly independent complex numbers.

(a) Prove that the four points $\{0, z_1, z_2, z_1 + z_2\}$ form a parallelogram whose area is

$$|\operatorname{Im}(\bar{z}_1 z_2)|.$$



(b) Prove that the angle θ between the vectors z_1, z_2 is given by $\theta \in \arg((z_1/z_2))$.

Hint: Recall that $\mathbb{C}$ can be represented by the real vector space $\mathbb{R}^2$. Two complex numbers z_1, z_2 are linearly dependent if $z_2 = rz_1$ for some $r \in \mathbb{R}$. A complex number $z = x + iy$ may also be represented by $(x, y, 0) \in \mathbb{R}^3$, allowing use of the parallelogram-area formula $\text{Area} = |z_1 \times z_2|$ from multivariable calculus. For part (b), consult Exercises 6 and 7.

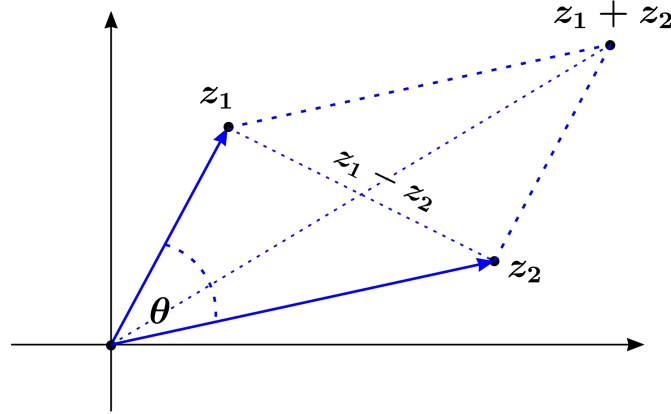


Fig. 1.26: The parallelogram formed by $\{0, z_1, z_2, z_1 + z_2\}$.

9. Prove the parallelogram identity

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Hint: First review Exercise 8 and Figure 1.26.



10. Prove that for any n complex numbers $z_1, z_2, \dots, z_n \in \mathbb{C}$,

$$\begin{aligned} |z_1 z_2 \cdots z_n| &= |z_1| |z_2| \cdots |z_n| \\ \overline{z_1 z_2 \cdots z_n} &= \overline{z_1} \overline{z_2} \cdots \overline{z_n} \end{aligned} \tag{1.10}$$

Hint: Use mathematical induction. The base case is $n = 2$.



11. Prove that for every $z \in \mathbb{C}$ and every natural number n , $|z^n| = |z|^n$, $\overline{z^n} = \overline{z}^n$.

Hint: This follows immediately from Exercise 10.



12. Prove that for any n complex numbers $z_1, z_2, \dots, z_n \in \mathbb{C}$, the generalized triangle inequality holds:

$$\left| \sum_{k=1}^n z_k \right| \leq \sum_{k=1}^n |z_k| \tag{1.11}$$

13. Prove that every complex number z satisfies $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2}|z|$.



14. Prove that for every real x , $\left| \frac{x-i}{x+i} \right| = 1$.



15. Suppose $z = e^{2\pi ki/n}$ is an n th root of unity and the integers k, n are relatively prime. Prove that

$$\{z^j : j = 0, 1, 2, \dots, n-1\}$$

is the complete set of n th roots of unity.

Hint: Two integers are relatively prime if their only common divisor is 1. You may use the elementary number-theoretic fact that, when k and n are relatively prime, the residues $jk \pmod{n}$ for $j = 0, 1, \dots, n-1$ are precisely $0, 1, \dots, n-1$.



16. Prove that for every $z \neq 1$,

$$1 + z + z^2 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

Hint: Let $S = 1 + z + \dots + z^n$. Compute $S - zS$ and solve for S .

17. Using the previous exercise, prove that for every angle $0 < \theta < 2\pi$,

$$1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin \frac{(2n+1)\theta}{2}}{2 \sin \frac{\theta}{2}}$$



18. Prove that for every $n \geq 2$, the sum of all n th roots of unity is 0.

Hint: Use Exercises 15 and 16.

19. Prove that the product of all n th roots of unity is $(-1)^{n+1}$.



20. Find the polar form $re^{i\theta}$ of $\frac{(1-i)^{23}}{(\sqrt{3}-i)^{13}}$.



21. Let $P(z)$ be a polynomial with real coefficients. Prove that for every $z \in \mathbb{C}$, $\overline{P(z)} = P(\bar{z})$.



22. Let $Q(z)$ be a rational function with real coefficients. Prove that, wherever it is defined, $\overline{Q(z)} = Q(\bar{z})$.



23. Let $P(z)$ be a polynomial all of whose coefficients are real. Prove that if z is a root of P , then $\bar{z}$ is also a root of P .

24. Compute $(1+i)^n + (1-i)^n$ for every natural number n .



25. Let $z_1, z_2 \in \mathbb{C} - \{0\}$. Prove that $\arg(z_1) = \arg(z_2)$ if and only if there exists a real number $r > 0$ such that $z_1 = rz_2$.

Hint: Review Exercises 6 and 8.

26. Let z_1, z_2, z_3 be three distinct points on the unit circle $|z| = 1$. Prove that if $z_1 + z_2 + z_3 = 0$, then z_1, z_2, z_3 are the vertices of an equilateral triangle.



27. Let z_1, z_2, z_3, z_4 be four distinct points in the complex plane. Prove that they are the vertices of a nondegenerate parallelogram if and only if $z_1 + z_3 = z_2 + z_4$ and $\operatorname{Im}\left(\frac{z_1 - z_3}{z_1 - z_2}\right) \neq 0$.

Hint: The second condition prevents degeneracy by requiring the angle between two adjacent sides to be nonzero. Use Exercises 6 and 8.

28. Let z_1, z_2, z_3, z_4 be four distinct points in the complex plane. Find the simplest possible algebraic condition equivalent to their being
- (a) the vertices of a square;
 - (b) vertices of a rectangle.



29. If z_1, z_2 are two vertices of a square, find the other two vertices. Take all possibilities into account.

30. Given three distinct points z_1, z_2, z_3 on a circle in the complex plane, find a formula for the center and radius of the circle.



31. Prove or disprove by counterexample: for all $z, z_1, z_2 \in \mathbb{C} - \{0\}$,

$$\begin{aligned}\operatorname{Arg}(z_1 z_2) &= \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \\ \operatorname{Arg}(z_1 / z_2) &= \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) \\ \operatorname{Arg}(\bar{z}) &= -\operatorname{Arg}(z)\end{aligned}$$

32. Prove that three complex numbers $z_1, z_2, z_3 \in \mathbb{C}$ are the vertices of an equilateral triangle if and only if $z_1 + \alpha z_2 + \alpha^2 z_3 = 0$, where α is a nonreal root of $\alpha^3 = 1$.

Hint: Review Exercises 6 and 8.

33. Suppose that z_1, z_2, z_3 are three distinct complex numbers and

$$|z_1| = |z_2| = |z_3|$$

Prove that

$$\arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \frac{1}{2} \arg\left(\frac{z_2}{z_1}\right)$$

Hint: Review Exercises 6 and 8.



34. Let z_1, z_2, z_3 be three distinct points in the complex plane, and let $\theta \in \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$. Prove that

$$|z_3 - z_2|^2 = |z_2 - z_1|^2 + |z_3 - z_1|^2 - 2|z_2 - z_1||z_3 - z_1|\cos\theta$$

Hint: Consider the triangle determined by z_1, z_2, z_3 and recall the law of cosines. Review Exercises 6 and 8.

35. Let $z_1 = (1 + \cos\theta + i\sin\theta)^n$ and $z_2 = 2^n \cos^n \frac{\theta}{2}$. Prove that $|z_1| = |z_2|$ for every angle



θ and every natural number n .

Hint: Put $1 + \cos \theta + i \sin \theta$ into polar form $re^{i\alpha}$.

36. Prove that an argument function can also be written in the following compact form:

$$\arg(x + iy) = \begin{cases} \arctan \frac{y}{x} + \frac{\pi}{2}[1 - \text{sign}(x)], & x \neq 0 \\ \frac{\pi}{2} \text{sign}(y), & x = 0 \text{ and } y \neq 0 \\ \text{Undefined}, & x = 0 \text{ and } y = 0 \end{cases} \quad (1.12)$$

where the real $\text{sign}(x)$ function is defined by

$$\text{sign}(x) = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$



37. Define the set of matrices

$$\mathbb{F} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

(a) Prove that $\mathbb{F}$ is simultaneously a real vector subspace of $\mathbb{R}^{2 \times 2}$ and a field under the standard operations of matrix addition and multiplication. The zero of the field is the zero matrix, and its multiplicative identity is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Verify all seven field axioms from Definition 1.1.

(b) Find a matrix $J \in \mathbb{F}$ such that $J^2 = -I$.

(c) Prove that the field $\mathbb{F}$ is isomorphic to the field $\mathbb{C}$. That is, construct a bijection $T : \mathbb{C} \rightarrow \mathbb{F}$ such that for all $z_1, z_2 \in \mathbb{C}$,

$$T(z_1 + z_2) = T(z_1) + T(z_2), \quad T(z_1 z_2) = T(z_1)T(z_2).$$



[Link to Solutions](#)



Chapter 2

Sets of Complex Numbers

2.1 Elementary Sets in the Complex Plane

We present a list of common elementary point sets in the plane that we will encounter throughout the course.

Straight Line

- A straight line in the plane is characterized by a point $z_0 = x_0 + y_0i$ lying on it, and by a direction vector $v = a + bi \in \mathbb{C}$. As recalled from the first chapter, a complex number may represent a point in the two-dimensional plane as well as a vector in the vector space $\mathbb{R}^2$!
- Let t be a real parameter. The set of all points $z = tv$, obtained by multiplying the vector $v = a + bi$ by the parameter t , is a straight line in the complex plane (see the line L_1 in Figure [2.1](#)). We may therefore write

$$L_1 = \{tv : t \in \mathbb{R}\}$$

The geometric meaning of the expression tv is a stretching or contraction of the vector v , and hence all the points obtained in this way form a straight line L_1 . As recalled from the first chapter, addition of complex numbers is equivalent to vector addition, and therefore the set of points $z = z_0 + tv$ is a translation of the line L_1 by the vector z_0 , as shown in Figure [2.1](#).

$$L_2 = \{z_0 + tv : t \in \mathbb{R}\}$$

The usual physical interpretation of v is a velocity vector (including magnitude and direction), and t is sometimes called the **time parameter**.



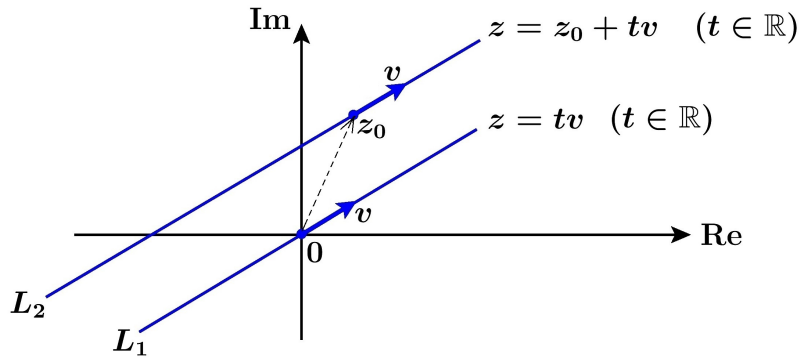


Fig. 2.1: The equation of the line L_1 is $z = tv$. The equation of the line L_2 is $z = z_0 + tv$. In both cases, v is a direction vector.

- ✦ The equation of the line can also be written in the familiar parametric form from analytic geometry

$$z = z_0 + tv = x_0 + y_0i + t(a + bi) = (x_0 + at) + (y_0 + bt)i$$

- ✦ **Example 1:** The equation of the line through the point $z_0 = 2 + 3i$ in the direction $v = 5 + 4i$ is obtained as follows:

$$z = z_0 + tv = 2 + 3i + t(5 + 4i) = (2 + 5t) + (3 + 4t)i$$

The final form is precisely the parametric representation of a straight line in the plane.

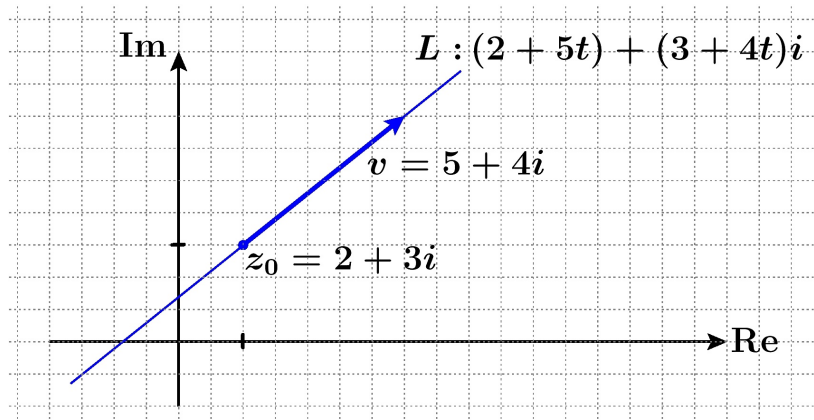


Fig. 2.2: The equation of the line $2 + 3i + t(5 + 4i) = (2 + 5t) + (3 + 4t)i$

Half-Plane

- Every straight line in the complex plane divides it into two half-planes. Figures 2.3 and 2.4 show suitable examples.

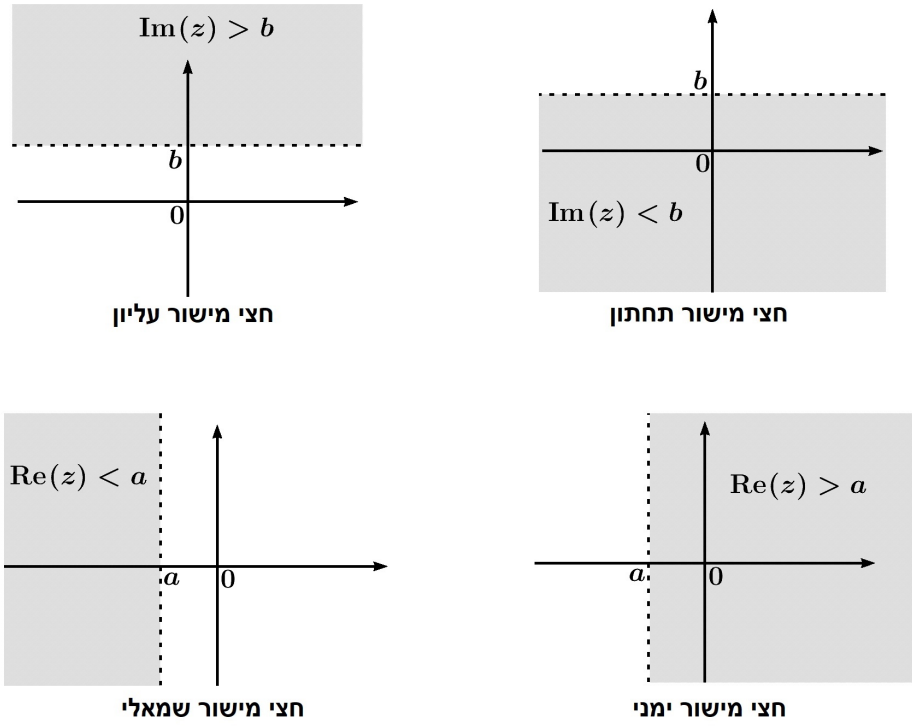


Fig. 2.3: Upper and lower half-planes. Right and left half-planes. The boundary is excluded (open).

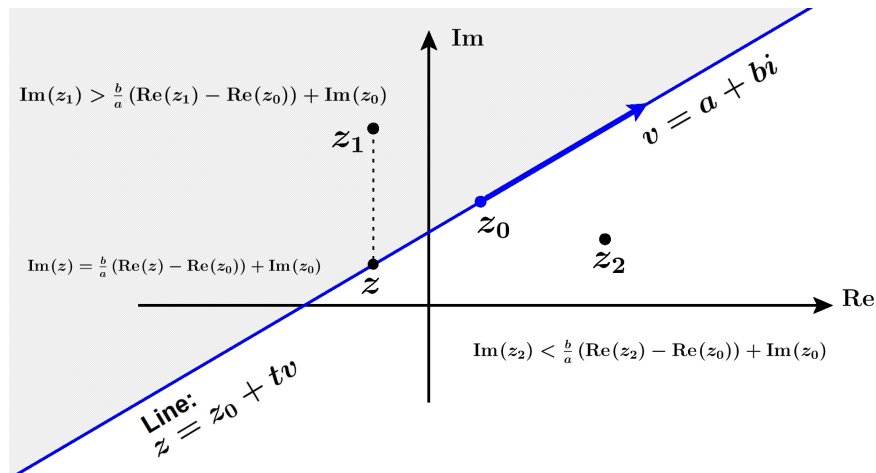


Fig. 2.4: The upper half-plane above the line $z = z_0 + tv$

- A straight line that is neither horizontal nor vertical can be characterized by the equation

$$\frac{y - y_0}{x - x_0} = \frac{b}{a} \quad (x \neq x_0)$$

where $v = a + bi$ is a direction vector of the line (and hence $\frac{b}{a}$ is the **slope** of the line), and $z_0 = x_0 + iy_0$ is a given point on the line, while $z = x + iy$ is a general point. Thus, the

general equation is

$$y = \frac{b}{a}(x - x_0) + y_0$$

In complex notation, we may write

$$\operatorname{Im}(z) = \frac{b}{a}(\operatorname{Re}(z) - \operatorname{Re}(z_0)) + \operatorname{Im}(z_0)$$

Figure 2.4 shows that for a point z_1 to lie above the line, it must satisfy

$$\operatorname{Im}(z_1) > \frac{b}{a}(\operatorname{Re}(z_1) - \operatorname{Re}(z_0)) + \operatorname{Im}(z_0)$$

For a point z_2 to lie below the line, it must satisfy

$$\operatorname{Im}(z_2) < \frac{b}{a}(\operatorname{Re}(z_2) - \operatorname{Re}(z_0)) + \operatorname{Im}(z_0)$$

Quadrant

Like the real plane, the complex plane is divided into four **quadrants**. See Figure 2.5.

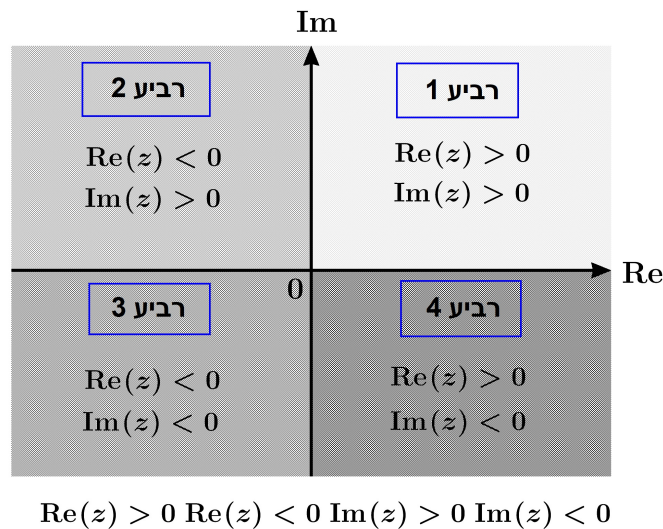


Fig. 2.5: Division of the complex plane into four quadrants: quadrant 1, quadrant 2, quadrant 3, and quadrant 4

Infinite Strip

- A vertical infinite strip is the set of all points in the complex plane whose real part lies between two values a and b .
- A horizontal infinite strip is the set of all points in the complex plane whose imaginary part lies between two values a and b .
- Figure 2.6 shows **open** strips, in the sense that they do not include their boundary lines. It

is customary to indicate this by a dashed line. **Closed** strips are obtained by replacing the inequality sign $<$ with $\leq$.

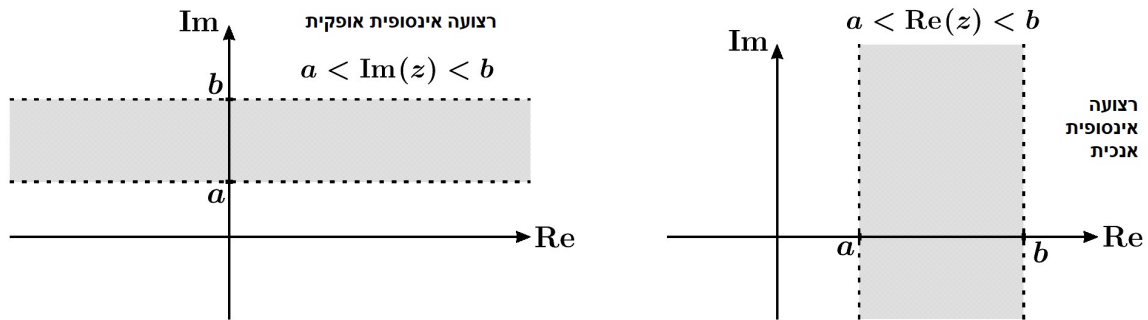


Fig. 2.6: Horizontal and vertical infinite strips (open - without the boundary of the region)

Rectangle

- A canonical rectangular region in the complex plane is characterized by lower and upper bounds on its real and imaginary parts.

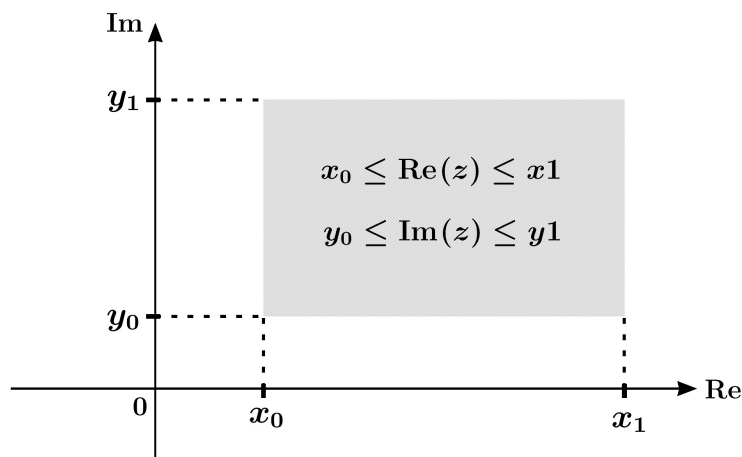


Fig. 2.7: A canonical rectangular region in the complex plane

The equation of a general rectangle in the complex plane is more complicated, and we leave it as an exercise for the student (see Exercise [27](#) in Chapter [??](#), and Exercise [5](#) at the end of the present chapter).

Disk and Circle

- A disk is the set of all points z whose distance from a given point z_0 is bounded by a positive real constant $r > 0$ (the **radius of the disk**).
- A circle is the set of all points z whose distance from a given point z_0 equals r (the **radius of the circle**).

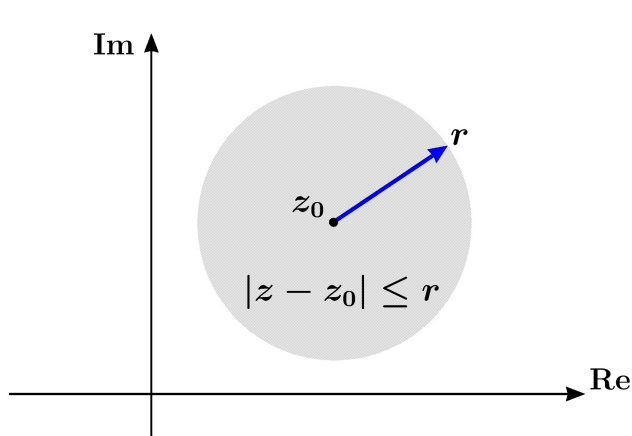


Fig. 2.8: Equation of a closed disk in the complex plane

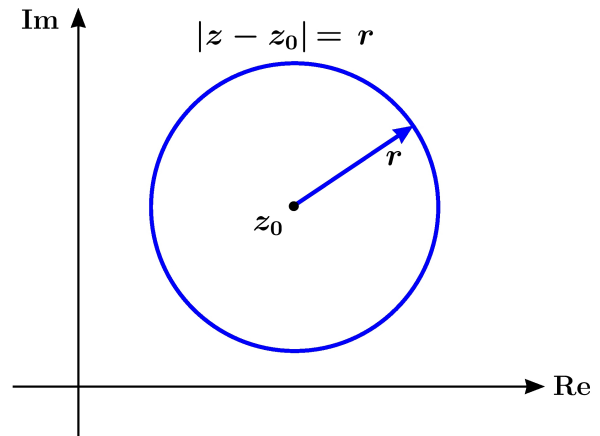


Fig. 2.9: Equation of a circle in the complex plane: $|z - z_0| = r$

Circular Sector

- A circular sector is the region enclosed by a circle and two rays emanating from the center of the circle.

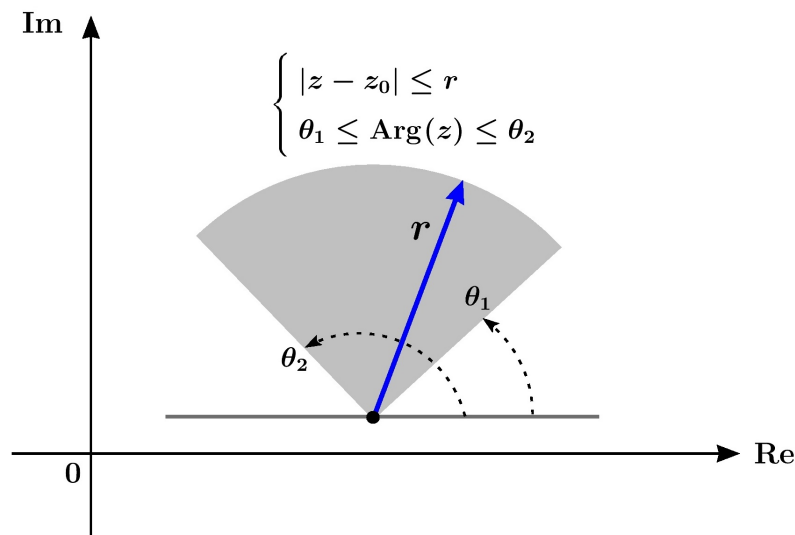


Fig. 2.10: Equation of a circular sector in the complex plane

Ellipse

- In the complex plane, the general equation of a filled ellipse has a simple and elegant form. An ellipse is characterized by two foci z_1 , z_2 , and a positive real constant $c > 0$. The boundary of the ellipse is the locus of all points in the plane for which the sum of the distances from the foci is c . Elementary geometric considerations show that the filled ellipse (including its interior) is characterized by the inequality in Figure 2.11.
- Later in the chapter we will also encounter a parametric representation of the boundary of the ellipse.

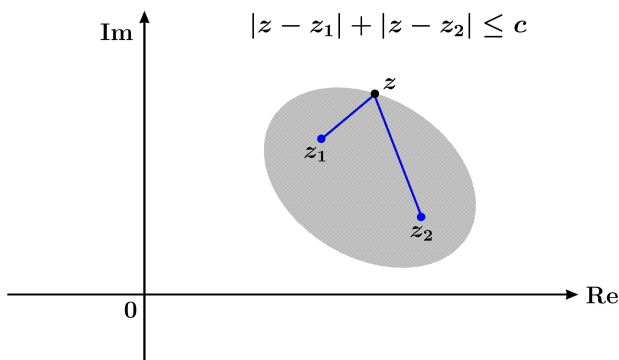


Fig. 2.11: Equation of a filled ellipse (including its interior)

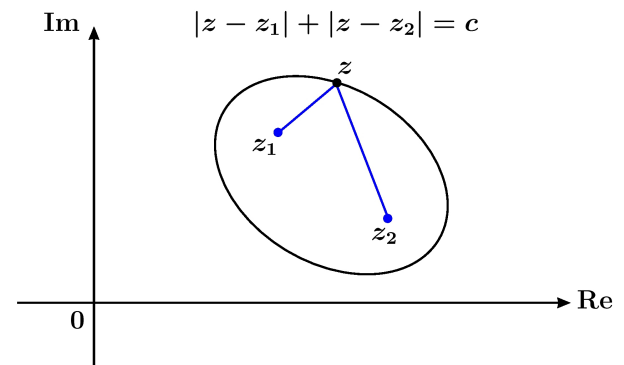


Fig. 2.12: Equation of an ellipse (boundary) in the complex plane

Punctured Plane

- A punctured plane is the complex plane after a “hole” has been drilled at the point z_0 with radius r , as shown in Figure 2.13.

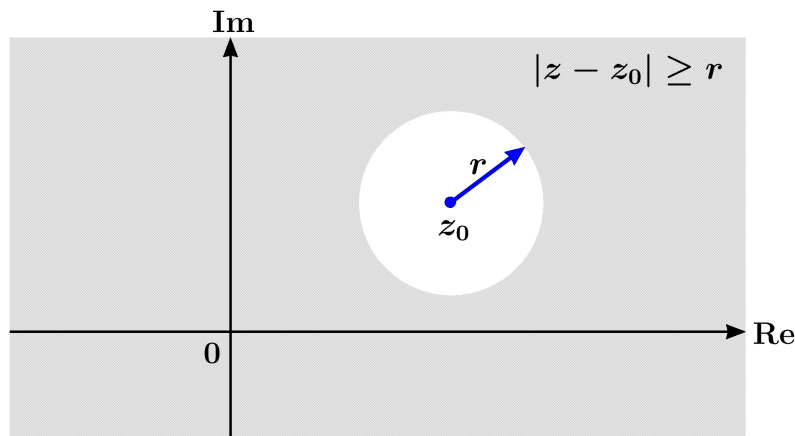


Fig. 2.13: Characterization of a punctured plane (the complement of a disk)

- One must of course distinguish between an **open punctured plane** $|z - z_0| > r$ and a **closed punctured plane** $|z - z_0| \geq r$. The same observation applies to all the sets presented in this survey, and we shall not repeat it in every case.

Annulus

- An annulus is the region enclosed between two concentric circles. In contexts encountered later in the course, it is sometimes useful to describe an annulus as the intersection of a disk $|z - z_0| \leq r_2$ and a punctured plane $|z - z_0| \geq r_1$.

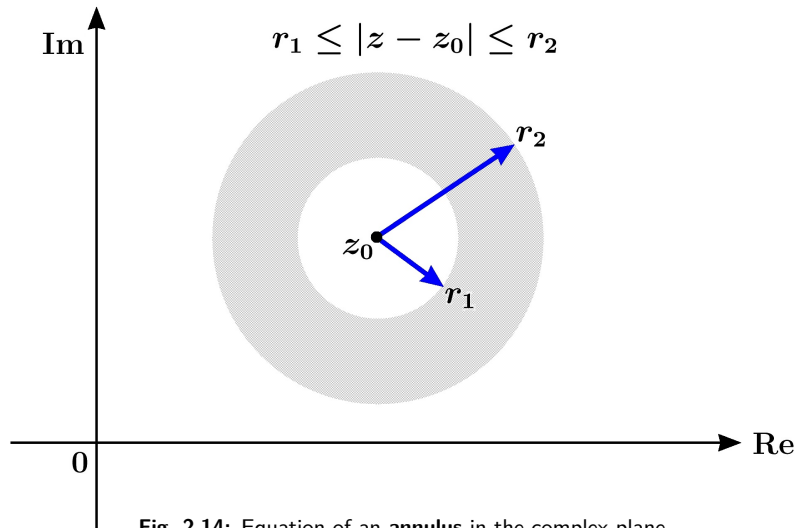


Fig. 2.14: Equation of an **annulus** in the complex plane

Ray

- A ray is simply half of a straight line and may be defined by a line equation in which the time parameter t is restricted to a real half-line by one of the following four conditions: $t \geq t_0$, $t > t_0$, $t \leq t_0$, $t < t_0$.
- ◆ **Example 2:** Figure 2.15 shows two rays emanating from the point $z_0 = 2 + 3i$. The lower ray travels in the direction of the vector $v = 2 - i$, and the upper ray in the direction $v = 3 + 2i$.

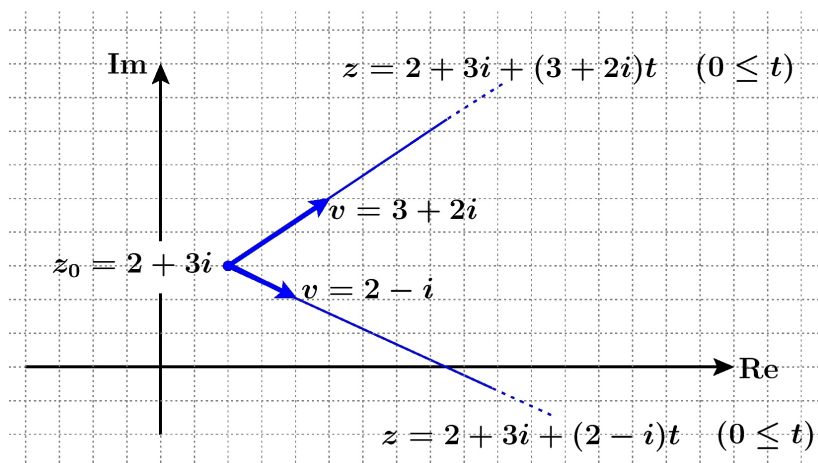


Fig. 2.15: Equations of two rays in the complex plane

Segment

- A segment in the complex plane can be described by a line equation $z = z_0 + tv$, in which the

time parameter t is restricted to a real interval of one of the following forms:

$$[t_0, t_1], \quad [t_0, t_1), \quad (t_0, t_1], \quad (t_0, t_1)$$

In the first case we obtain a **closed segment**, and in the fourth case an **open segment**. In the second and third cases we obtain a **half-open segment** (or **half-closed**). In the special case where the endpoints z_0, z_1 are given (see Figure 2.16), we may take the direction vector $v = z_1 - z_0$ and the initial point z_0 . Thus the equation of the (closed) segment is

$$\begin{cases} z = z_0 + t(z_1 - z_0) \\ 0 \leq t \leq 1 \end{cases}$$

Of course, we may choose the opposite vector $u = z_0 - z_1$ and obtain another equivalent equation. In fact, any multiple of the vector $v = z_1 - z_0$ will do, so there are infinitely many ways to represent the same line. As with the notation for intervals in the real number field, it is customary to denote a complex segment by $[z_0, z_1]$.

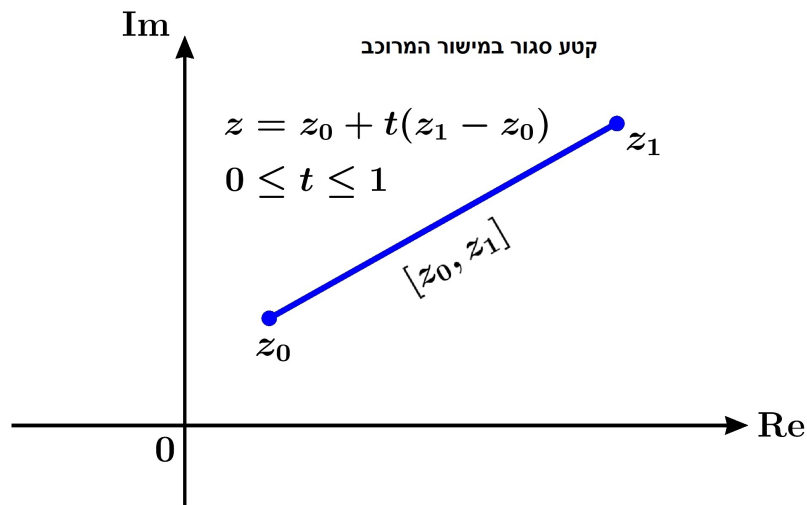


Fig. 2.16: Parametrization of a closed segment $[z_0, z_1]$ in the complex plane

Polyline

- A concatenation of segments $[z_0, z_1, z_2, \dots, z_{n-1}]$ is called a **polyline**. Note the abbreviated notation

$$[z_0, z_1, z_2, \dots, z_{n-1}] = [z_0, z_1] \cup [z_1, z_2] \cup \dots \cup [z_{n-2}, z_{n-1}]$$

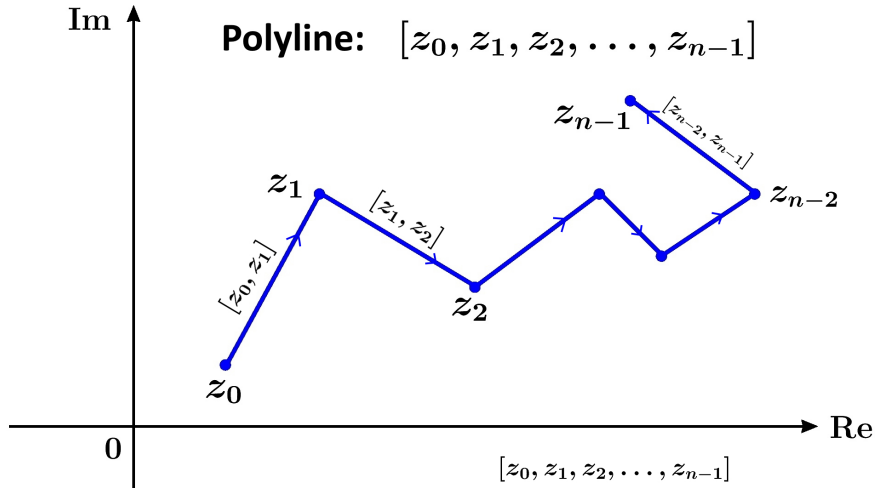


Fig. 2.17: Equation of a **polyline** $[z_0, z_1, z_2, \dots, z_{n-1}]$ in the complex plane

❖ **Example 3:** Figure 2.18 shows the polyline $[0, 1 + i, 2, 3 + i]$, which consists of three segments

$$[0, 1 + i, 2, 3 + i] = [0, 1 + i] \cup [1 + i, 2] \cup [2, 3 + i]$$

$$= \{t + ti : 0 \leq t \leq 1\} \cup \{(1 + t) + (1 - t)i : 0 \leq t \leq 1\} \cup \{(2 + t) + ti : 0 \leq t \leq 1\}$$

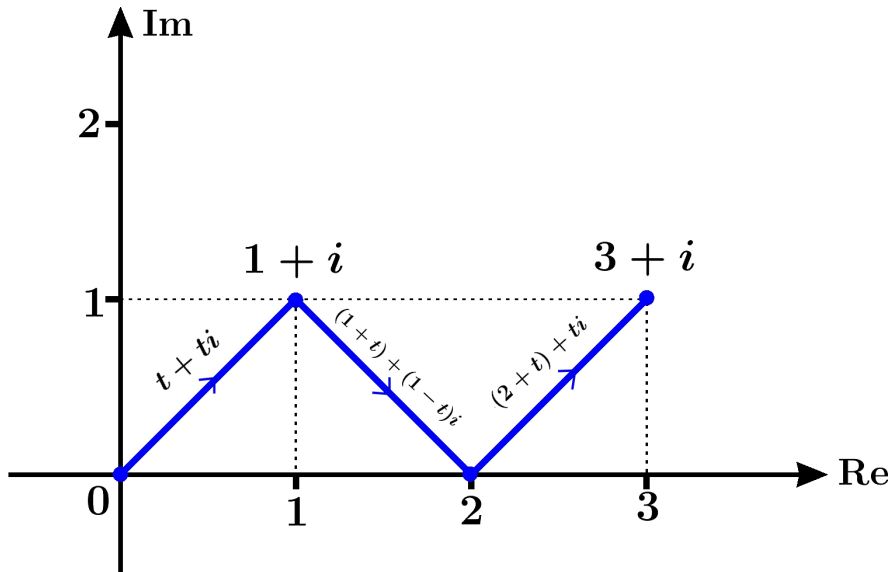


Fig. 2.18: The polyline $[0, 1 + i, 2, 3 + i]$, consisting of three closed segments

❖ A polyline whose initial point is also its terminal point is called a **closed polyline**, or simply a **polygon (Polygon)**.

❖ **Example 4:** The boundary of a rectangle is a polygon with four segments. A simple example of a rectangle is

$$[2 + i, 6 + i, 6 + 4i, 2 + 4i, 2 + i]$$

Figure 2.19 gives the parametric representations of each side of the rectangle.

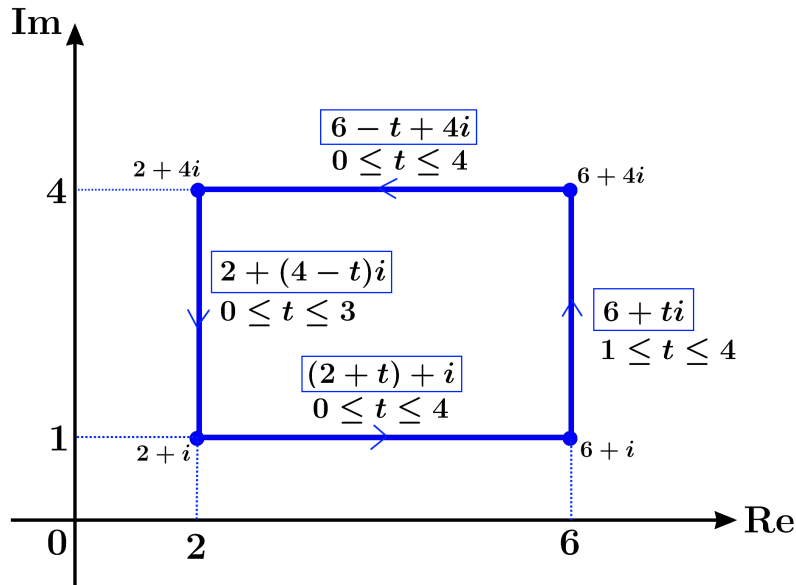


Fig. 2.19: Equations of the segments of the rectangle $[2+i, 6+i, 6+4i, 2+4i, 2+i]$

Paths

- Paths in the complex plane are a natural generalization of polylines to curves that are not necessarily straight. They have a parametric representation and are one-dimensional, but they also have a direction of travel. The formal definition of a **path** σ is given by a continuous function

$$\sigma: I \rightarrow \mathbb{C}$$

where $I \subseteq \mathbb{R}$ is a real interval (closed, open, half-open, finite, or infinite).

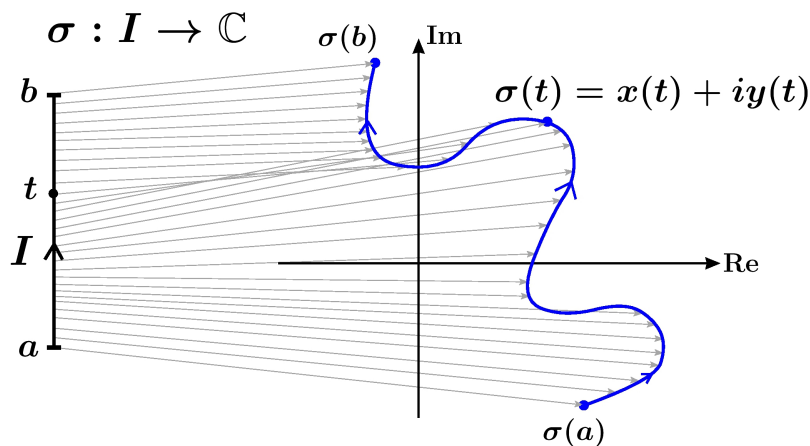


Fig. 2.20: A path $\sigma: I \rightarrow \mathbb{C}$ as a continuous mapping from a real interval I to the complex plane $\mathbb{C}$

- ❗ Note that the requirement of **continuity** is an essential part of the definition of a path!
- It is sometimes customary to express the path function $\sigma(t)$ with real part $x(t)$ and imaginary part $y(t)$

$$\sigma(t) = x(t) + iy(t)$$

The functions $x(t)$ and $y(t)$ are real functions on the interval I .

- ✦ In most cases, it is useful to think of σ as the trajectory of a physical particle in the complex plane. At each time $t \in I$, $\sigma(t) = x(t) + iy(t)$ is the point at which the particle is located at time t . Thus, the function σ is more than a description of a set of points in the complex plane. It is a description of the motion of a particle from which one may derive the position, velocity, and acceleration of the particle at each time t .

- ✦ **Example 5:** The canonical example of a path is the unit circle $|z| = 1$, whose parametric representation is

$$\sigma(t) = e^{it} = \cos t + i \sin t$$

and the time parameter t ranges over $[0, 2\pi]$. In terms of motion, this is the trajectory of a particle moving along the unit circle $|z| = 1$ at constant speed in the counterclockwise direction. See Figure 2.21.

- ✦ **Example 6:** Another typical example is the **spiral** $\sigma(t) = te^{it}$. In this case one may choose a larger time interval, such as $I = [0, 12\pi]$, during which the spiral makes 6 revolutions. See Figure 2.22.

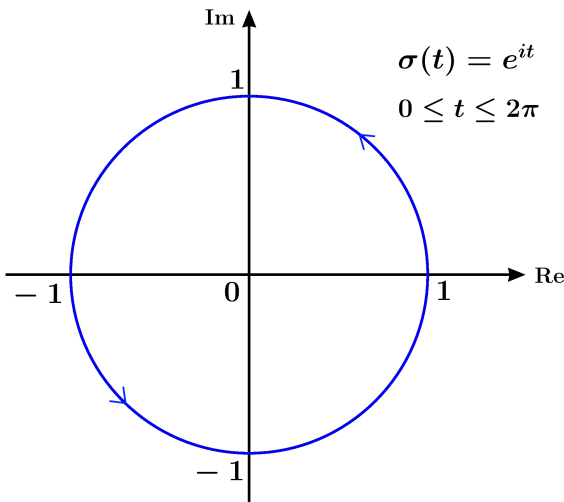


Fig. 2.21: The path $\sigma(t) = e^{it}$, $0 \leq t \leq 2\pi$ - the unit circle

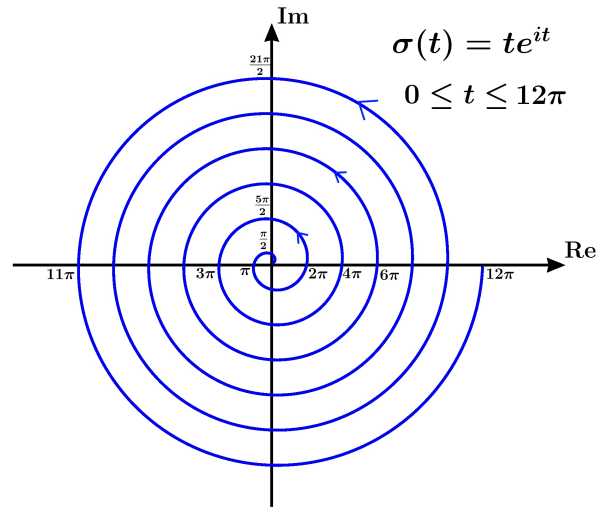


Fig. 2.22: The path $\sigma(t) = te^{it}$, $0 \leq t \leq 12\pi$ - a spiral

- ✦ A path $\sigma: [a, b] \rightarrow \mathbb{C}$ is called a **closed path** if $\sigma(a) = \sigma(b)$. Otherwise, it is called an **open path**. The path σ is called a **simple path** if it does not intersect itself at an interior time $a < t < b$. Figure 2.23 shows four typical kinds of paths.
- ✦ A simple closed path is also called a **Jordan path**, after the mathematician [Camille Jordan](#), who proved the [fundamental theorem concerning it](#) (Theorem 2.13 in Section 2.3). Equivalently, $\sigma: [a, b] \rightarrow \mathbb{C}$ is a Jordan path if $\sigma(a) = \sigma(b)$ and σ is one-to-one on $[a, b)$.

Elementary Properties of Sets

- ✦ We could continue listing more and more kinds of sets in the complex plane, but, as noted, this

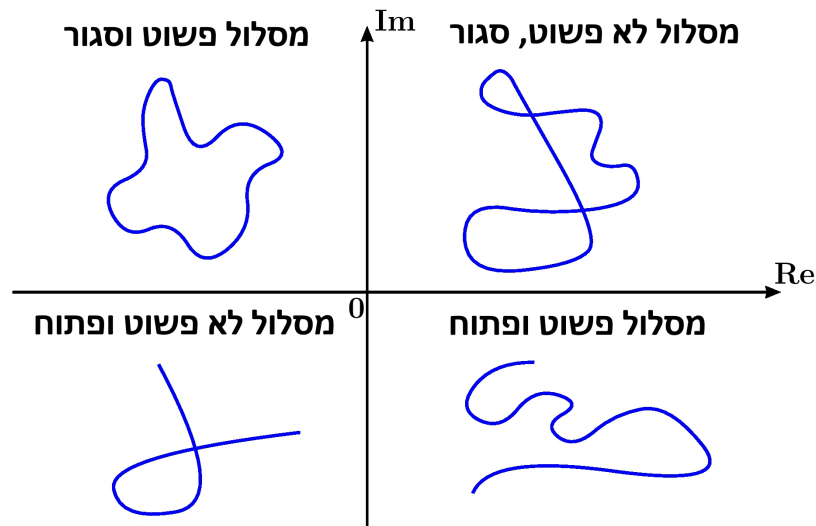


Fig. 2.23: Four typical kinds of paths

is only a partial list of elementary sets from which more complicated sets can be constructed by operations such as union, intersection, and difference. The purpose of this list is only to convey the simple forms of expression by which sets in the complex plane may be defined. We will encounter further examples later in this text.

- We now present several elementary properties by which sets may be classified. The more important classification by topological structure will be presented in the next section.

i Definition 2.1: A set $S \subseteq \mathbb{C}$ is called a **finite set** if it contains finitely many points. Otherwise, S is called an **infinite set**.

i Definition 2.2: Let S be a set of points in the complex plane. The set S is called **bounded** if there exists a disk D such that $S \subseteq D$.

- This may also be stated in the form familiar from the real line: The set S is **bounded** if there exists a positive real number $R > 0$ such that, for every $z \in S$, $|z| \leq R$.

i Definition 2.3: The **diameter (diameter)** of a bounded set S is defined as the supremum of the distances between any two points of the set

$$\text{diam}(S) = \sup \{ |z_1 - z_2| : z_1, z_2 \in S \}$$

- The existence of the diameter when S is bounded follows because S is contained in some disk $D(0, R)$, and therefore, if $z_1, z_2 \in S$, then necessarily $|z_1 - z_2| \leq 2R$. Thus the set $\{|z_1 - z_2| : z_1, z_2 \in S\}$ is bounded above by $2R$ and consequently has a supremum.

i Definition 2.4: A set $S \subseteq \mathbb{C}$ is called **symmetric with respect to the real axis** if

$$\forall z \in \mathbb{C} : z \in S \iff \bar{z} \in S$$

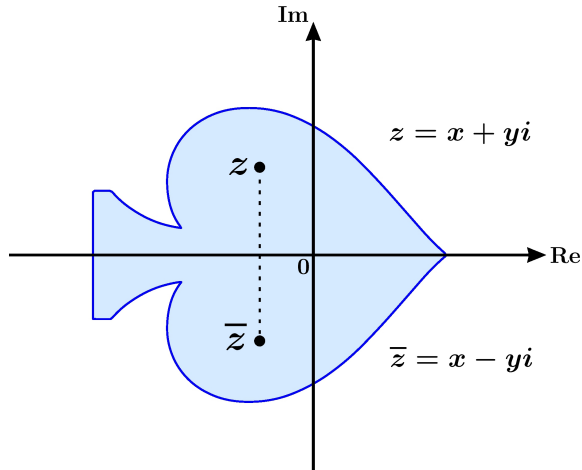
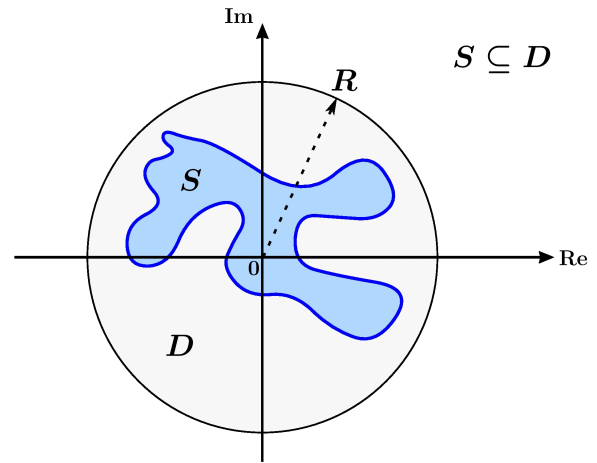


Fig. 2.24: Symmetry with respect to the real axis

Fig. 2.25: A bounded set S

i **Definition 2.5:** A set $S \subseteq \mathbb{C}$ is called **symmetric with respect to the imaginary axis** if

$$\forall z \in \mathbb{C} : z \in S \iff \overline{-z} \in S$$

For clarity, if $z = x + yi$, then $\overline{-z} = -x + yi$, which is the reflection of z across the imaginary axis.

i **Definition 2.6:** A set $S \subseteq \mathbb{C}$ is called **symmetric with respect to the origin** if

$$\forall z \in \mathbb{C} : z \in S \iff -z \in S$$

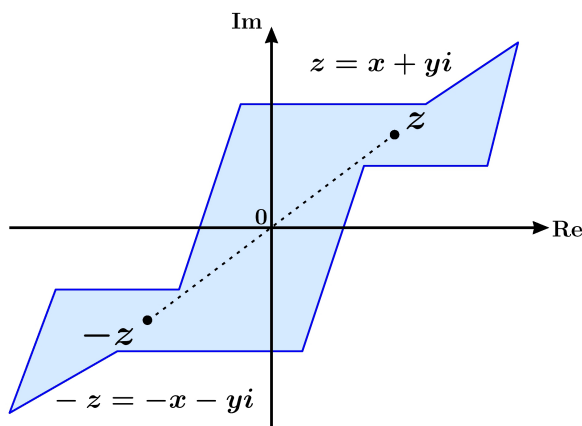


Fig. 2.26: Symmetry with respect to the origin

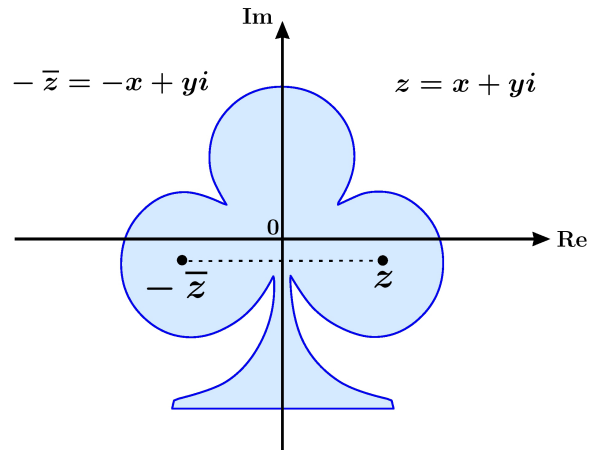


Fig. 2.27: Symmetry with respect to the imaginary axis

2.2 Topology in the Complex Plane

- The mathematical field called [topology](#) studies the properties and characteristics of sets that are preserved under **continuous deformations** such as contraction, expansion, bending, rotation, and so forth. For example, topologically, a square and a disk are equivalent because a square can be continuously deformed into a disk. A disk is not topologically equivalent to an annulus (see Figure [2.28](#)).

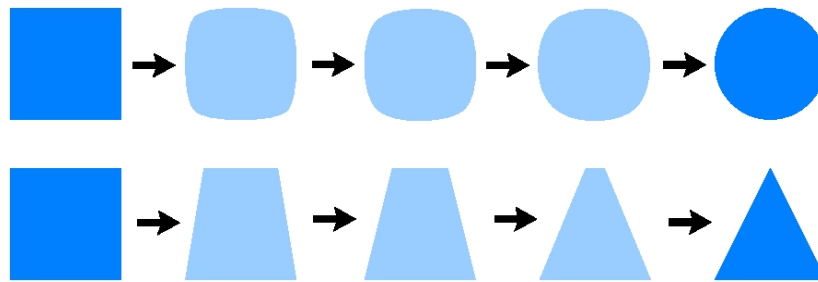


Fig. 2.28: A square and a disk are topologically equivalent because the square can be continuously deformed into the disk. A square and a triangle are also topologically equivalent.

- Another example of topological equivalence between two shapes that look very different from one another is shown in Figure [2.29](#). Topologically, the lock on the left side of the figure is simply a disk with two holes! Thus the lock symbol is topologically equivalent to a simple button.



Fig. 2.29: The lock symbol is topologically equivalent to the button symbol - topologically, each is a disk with two holes

- Figure [2.30](#) shows how a coffee mug is topologically equivalent to a doughnut (in three-dimensional space). Viewing the YouTube video linked to this figure is recommended for more interesting examples!
- The most fundamental topological distinction is between **open sets** and **closed sets**. These two fundamental concepts can be used to construct and analyze most theorems in topology, and therefore a thorough understanding of them is needed for the rest of the course. Before that, however, we need the concepts of the open disk and the closed disk.

i Definition 2.7: Let z_0 be a complex number and $r > 0$ a positive real number. The set of points

$$\mathbb{D}(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}$$

is called an **open disk** in the complex plane (or an **open ball**) about the point z_0 . The point z_0 is called the **center of the disk**, and the real number r is called the **radius of the disk**.



Fig. 2.30: A topologist is a person who cannot distinguish between a doughnut and a coffee mug

The set

$$\overline{D}(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| \leq r \}$$

is called a **closed disk** in the complex plane (or a **closed ball**) about the point z_0 with radius r .

- In most books, the boundary of an open set is drawn with a dashed line and the boundary of a closed set with a solid line. Figure 2.32 shows examples of an open disk and a closed disk. Note the dashed line surrounding the open disk $D(z_0, r)$.

i **Definition 2.8:** Let S be a set of points in the complex plane $\mathbb{C}$, and let $z_0 \in \mathbb{C}$.

- The point z_0 is called an **interior point** of S if there exists an open disk about z_0 that is entirely contained in S . That is, there exists a real number $r > 0$ such that $D(z_0, r) \subseteq S$.
- The point z_0 is called an **exterior point** of S if there exists an open disk about z_0 that does not intersect S . That is, there exists a real number $r > 0$ such that $D(z_0, r) \cap S = \emptyset$.
- The point z_0 is called a **boundary point** of the set S if it is neither an interior point nor an exterior point of the set S .

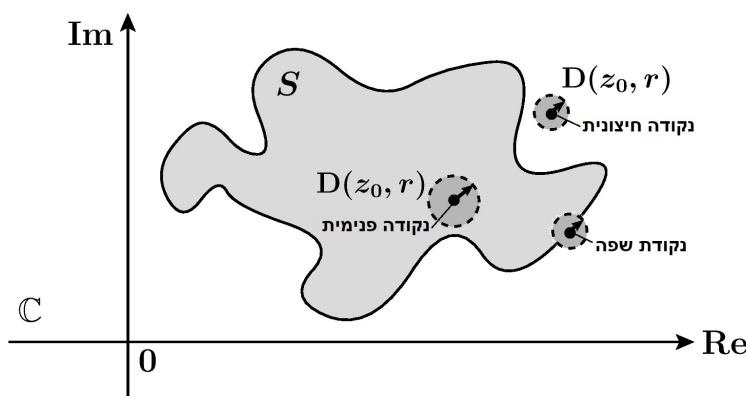


Fig. 2.31: An interior point, an exterior point, and a boundary point

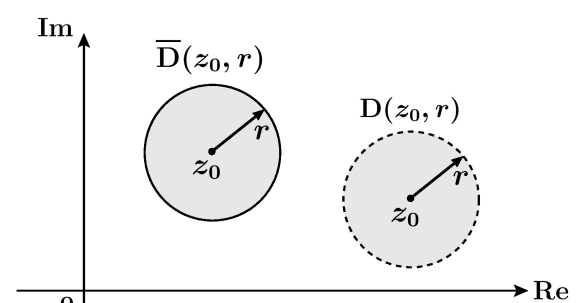


Fig. 2.32: An open disk $D(z_0, r)$ and a closed disk $\overline{D}(z_0, r)$

i Proposition 2.1: A point z_0 is a boundary point of a set S if and only if, for every $r > 0$, the open disk $D(z_0, r)$ intersects both S and its complement $\mathbb{C} - S$. That is,

$$\forall r > 0 : \quad D(z_0, r) \cap S \neq \emptyset \quad \text{and} \quad D(z_0, r) \cap (\mathbb{C} - S) \neq \emptyset$$

➤ A simple pictorial example of an interior point, an exterior point, and a boundary point is shown in Figure 2.31. We will encounter more complicated examples later.

◆ **Example 7:** Let $S = D(z_0, r)$ be an open disk of radius r centered at the point z_0 . It is easy to see in Figure 2.33 that every point inside the disk is an interior point. The boundary of the disk is the circle $|z - z_0| = r$, and every point outside the closed disk is an exterior point.

$$\partial D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

Of course, the only difference between the open disk $D(z_0, r)$ and the closed disk $\overline{D}(z_0, r)$ is that the closed disk includes its boundary, whereas the open disk does not. Every point outside the closed disk $\overline{D}(z_0, r)$ is an exterior point of S .

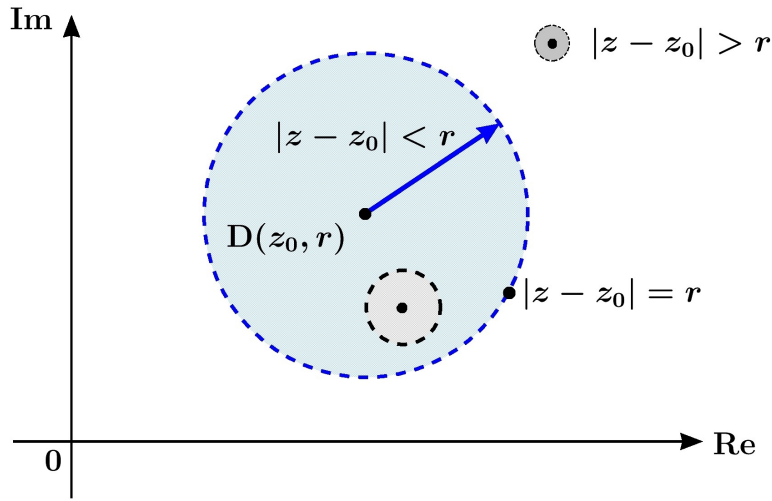


Fig. 2.33: Every point inside the disk $D(z_0, r)$ is an interior point, every point on the circle is a boundary point, and every point outside the circle is an exterior point

Note that every point z on the circle $|z - z_0| = r$ is not an exterior point of S , even though it does not belong to S !

i Definition 2.9: Let S be a set of points in the complex plane $\mathbb{C}$.

- The set of all interior points of S is called the **interior (Interior) of S** and is denoted by $\text{Int}(S)$.
- The set of all exterior points of S is called the **exterior (Exterior) of S** and is denoted by $\text{Ext}(S)$.
- The set of all boundary points of S is called the **boundary (Boundary) of S** and is

denoted by ∂S .

- d. The union of the set S with its boundary points ∂S is called the **closure (Closure)** of S and is denoted by $\bar{S}$. Of course, we may write: $\bar{S} = S \cup \partial S$.

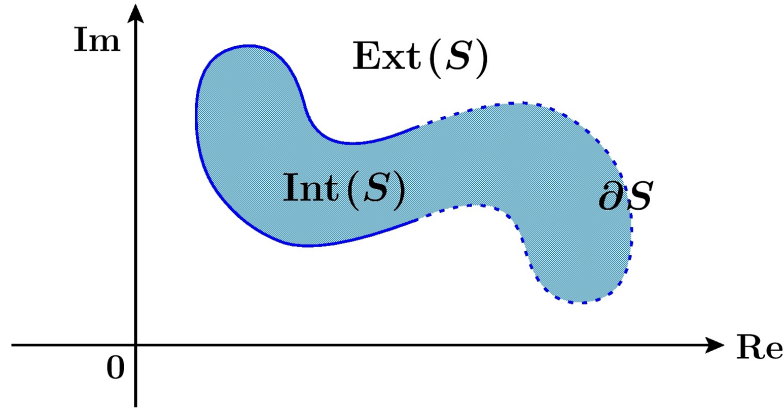


Fig. 2.34: Interior, exterior, and boundary of a set of points S : $\text{Int}(S)$, $\text{Ext}(S)$, ∂S

In Figure 2.34, note that some points of the boundary ∂S belong to the set S (solid line), whereas some boundary points do not belong to S (dashed line). In either case, the points on the dashed portion of the boundary are not exterior points! They are not contained in the exterior of S ! They belong to the boundary ∂S .

❖ **Example 8:** Let S be the set obtained by intersecting the following two disks:

$$S = \{z \in \mathbb{C} : |z - (4 + 4i)| \leq 3 \text{ and } |z - (7 + 4i)| < 3\}$$

A sketch of S appears in Figure 2.35. Note that the boundary of S consists of two circular arcs

$$\begin{aligned} A_1 &= \{z \in \mathbb{C} : |z - (4 + 4i)| < 3 \text{ and } |z - (7 + 4i)| = 3\} \\ A_2 &= \{z \in \mathbb{C} : |z - (4 + 4i)| = 3 \text{ and } |z - (7 + 4i)| \leq 3\} \\ \partial S &= A_1 \cup A_2 \end{aligned}$$

The characterization of the arc A_1 should be read as follows: The arc A_1 consists of all points on the right-hand circle ($|z - (7 + 4i)| = 3$) that lie inside the left-hand circle ($|z - (4 + 4i)| < 3$). Note that the arc A_1 (dashed line) does not belong to the set S !

Similarly, it is not difficult to find the interior, exterior, and closure of S

$$\begin{aligned} \text{Int}(S) &= \{z \in \mathbb{C} : |z - (4 + 4i)| < 3 \text{ and } |z - (7 + 4i)| < 3\} \\ \text{Ext}(S) &= \{z \in \mathbb{C} : |z - (4 + 4i)| > 3 \text{ or } |z - (7 + 4i)| > 3\} \\ \bar{S} &= \{z \in \mathbb{C} : |z - (4 + 4i)| \leq 3 \text{ and } |z - (7 + 4i)| \leq 3\} \end{aligned}$$

Figure 2.37 shows the interior of S . The difference between S and $\text{Int}(S)$ is the arc A_2 .

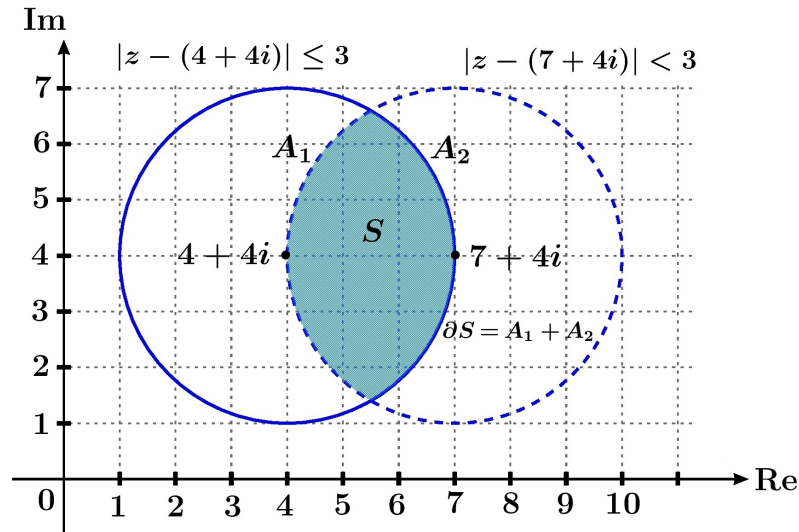


Fig. 2.35: Interior, exterior, and boundary of S from Example 8: $\text{Int}(S)$, $\text{Ext}(S)$, ∂S

Figure 2.39 shows the boundary of S . Figure 2.36 shows the exterior of S . Figure 2.38 shows the closure of S . Note that $\text{Ext}(S)$ does not include the two arcs A_1 and A_2 .

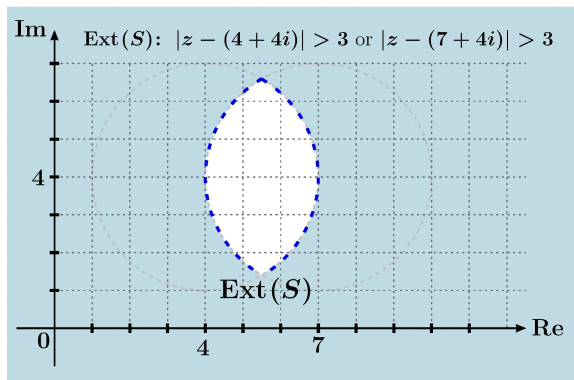


Fig. 2.36: The exterior of the set S

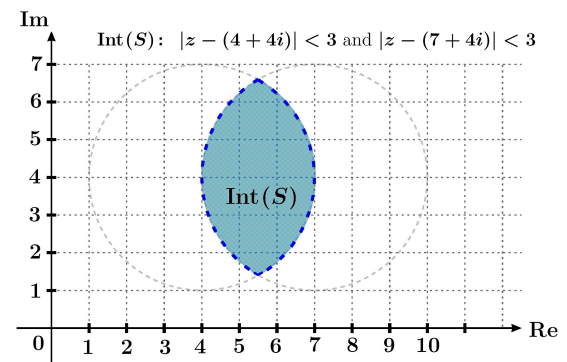


Fig. 2.37: The interior of the set S

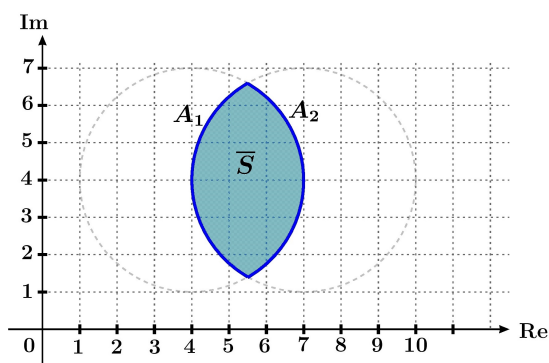


Fig. 2.38: The closure of the set S

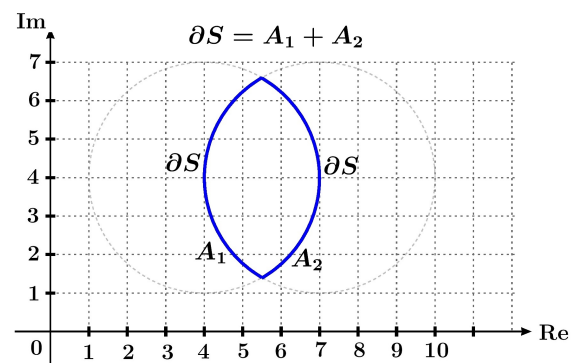


Fig. 2.39: The boundary of the set S

i Proposition 2.2: Let S be a set of points in the complex plane $\mathbb{C}$. Then the interior, boundary, and exterior of S always form a partition of $\mathbb{C}$ into three disjoint sets.

$$(2.1) \quad \text{Int}(S) \cup \partial S \cup \text{Ext}(S) = \mathbb{C}$$

$$(2.2) \quad \text{Int}(S) \cap \partial S = \emptyset$$

$$(2.3) \quad \text{Ext}(S) \cap \partial S = \emptyset$$

$$(2.4) \quad \text{Int}(S) \cap \text{Ext}(S) = \emptyset$$

✓ Proof: It is clear from the definition that $\text{Int}(S)$, $\text{Ext}(S)$, and ∂S are disjoint sets. By definition, every point that is neither interior nor exterior is automatically a boundary point. Therefore, the definition implies that the union of the interior, exterior, and boundary is necessarily $\mathbb{C}$.

i Corollary 2.3: Let S be a set of points in the complex plane $\mathbb{C}$. Then

$$(2.5) \quad \overline{S} = \text{Int}(S) \cup \partial S$$

$$(2.6) \quad \partial S = \overline{S} - \text{Int}(S)$$

$$(2.7) \quad \text{Ext}(S) = \mathbb{C} - \overline{S}$$

That is:

- ◆ The closure of S is the disjoint union of its interior and boundary.
- ◆ The boundary of S is the difference between the closure and the interior of S .
- ◆ The exterior of S is the complement of the closure of S .

➤ Simple examples of all seven identities can be seen in Figures [2.39-2.36](#).

i Definition 2.10: Let S be a set of points in the complex plane $\mathbb{C}$. The set S is called an **open set** if every point belonging to it is also an interior point of it. The set S is called a **closed set** if its complement $\mathbb{C} - S$ is an open set.

◆ Example 9:

- a. It is clear from the definition that the empty set $S = \emptyset$ and the set $S = \mathbb{C}$ are both open and closed at the same time (the student is asked to determine why).
- b. Every open disk ($|z - z_0| < r$) is an open set, and every closed disk ($|z - z_0| \leq r$) is a closed set.
- c. The half-planes $\text{Im}(z) > 0$, $\text{Im}(z) < 0$, $\text{Re}(z) > 0$, and $\text{Re}(z) < 0$ are open sets.
- d. The half-planes $\text{Im}(z) \geq 0$, $\text{Im}(z) \leq 0$, $\text{Re}(z) \geq 0$, and $\text{Re}(z) \leq 0$ are closed sets.
- e. All four quadrants of the plane including the axes are closed sets.
- f. All four quadrants of the plane (excluding the axes) are open sets.



g. Every circular sector of the form

$$\{z \in \mathbb{C} : 0 < |z| < r, \quad \theta_1 < \text{Arg}(z) < \theta_2\}$$

is an open set.

i Proposition 2.4: A set S is open if and only if it contains none of its boundary points.

✓ **Proof:** We must prove both directions.

First direction: suppose that S is open. If z is a boundary point of S , then for every $r > 0$, $D(z, r) \cap (\mathbb{C} - S) \neq \emptyset$. That is, there is no $r > 0$ such that $D(z, r) \subseteq S$. Therefore, z is not an interior point of S , and hence $z \notin S$.

Second direction: suppose that the set S contains none of its boundary points. Let $z \in S$. Then z is not an exterior point, because by Proposition 2.2, $\text{Int}(S) \cap \text{Ext}(S) = \emptyset$. Thus, by Definition 2.8, the only remaining possibility is that z is an interior point. We have therefore proved that every point $z \in S$ is also an interior point of S , and hence S is an open set. ■

⊛ In Example 8, the set S is neither open nor closed. Thus, not every set in the complex plane is necessarily open or closed! In a certain sense, most sets are neither open nor closed. A set of this kind is sometimes called a **half-open** set (or **half-closed**). This terminology is not particularly successful, however, because the word “half” is inappropriate in most cases.

i Proposition 2.5: Let S be a set of points in the complex plane $\mathbb{C}$. Then

- (a) The interior of S is an open set.
- (b) The exterior of S is an open set.
- (c) The closure of S is a closed set.

✓ **Proof:**

(a) By Proposition 2.2, $\text{Int}(S) \cap \partial S = \emptyset$. That is, the interior $\text{Int}(S)$ contains no boundary point, and therefore, by Proposition 2.4, it is an open set.

(b) It follows immediately from Proposition 2.1 that $\partial \text{Ext}(S) = \partial S$. Since, by Proposition 2.2, the boundary of S is disjoint from the exterior of S , the exterior $\text{Ext}(S)$ contains none of its boundary points, and therefore, by Proposition 2.4, $\text{Ext}(S)$ is an open set.

(c) By Corollary 2.3, the complement of $\overline{S}$ is $\text{Ext}(S)$, which is an open set by the preceding part. Therefore, by definition, $\overline{S}$ is a closed set.

⊛ **A brief review of set theory:** Let I be an index set. For every $\alpha \in I$, let S_α be a set of complex numbers. The notation $\{S_\alpha\}_{\alpha \in I}$ denotes the **family of sets** S_α , for $\alpha \in I$. The index set I may be finite or infinite, ordered or unordered. The **union** of the family of sets is defined by

$$\bigcup_{\alpha \in I} S_\alpha = \{z \in \mathbb{C} : \exists \alpha \in I [z \in S_\alpha]\}$$



The **intersection** of the family of sets is defined by

$$\bigcap_{\alpha \in I} S_{\alpha} = \{ z \in \mathbb{C} : \forall \alpha \in I [z \in S_{\alpha}] \}$$

The **difference** of two sets S_1 and S_2 is defined by

$$S_1 - S_2 = \{ z \in \mathbb{C} : z \in S_1 \text{ and } z \notin S_2 \}$$

Union, intersection, and difference are the principal set operations by which we construct new sets from existing ones. Any elementary text on set theory contains proofs of the following two formulas, which are sometimes called **De Morgan's laws**:

$$\begin{aligned} S - \bigcup_{\alpha \in I} S_{\alpha} &= \bigcap_{\alpha \in I} (S - S_{\alpha}) \\ S - \bigcap_{\alpha \in I} S_{\alpha} &= \bigcup_{\alpha \in I} (S - S_{\alpha}) \end{aligned} \tag{2.8}$$

i Theorem 2.6: Let $\{S_{\alpha}\}_{\alpha \in I}$ be a family of sets in the complex plane $\mathbb{C}$.

(a) If all the sets S_{α} are open, then their union $\bigcup_{\alpha \in I} S_{\alpha}$ is also open.

(b) If all the sets S_{α} are closed, then their intersection $\bigcap_{\alpha \in I} S_{\alpha}$ is also closed.

✓ Proof:

(a) Set $S = \bigcup_{\alpha \in I} S_{\alpha}$ and suppose that all the sets S_{α} are open. We prove that S is open. Let $z \in S$ be arbitrary. Then there exists an index $\alpha \in I$ such that $z \in S_{\alpha}$. Since S_{α} is open, there exists $r > 0$ such that $D(z, r) \subseteq S_{\alpha}$. Since $S_{\alpha} \subseteq S$, it follows that $D(z, r) \subseteq S$. We have proved that every point $z \in S$ is an interior point of S , so S is open.

(b) Set $S = \bigcap_{\alpha \in I} S_{\alpha}$ and suppose that all the sets S_{α} are closed. We prove that $\mathbb{C} - S$ is open, and hence that S is closed. By De Morgan's laws (2.8),

$$\mathbb{C} - S = \mathbb{C} - \bigcap_{\alpha \in I} S_{\alpha} = \bigcup_{\alpha \in I} (\mathbb{C} - S_{\alpha})$$

Since S_{α} is closed, $\mathbb{C} - S_{\alpha}$ is open, and therefore, by the preceding part, the union on the right-hand side is open. ■

- In summary, we have shown that a union of open sets is open, and that an intersection of closed sets is closed. What about an intersection of open sets and a union of closed sets?
- It turns out that an intersection of open sets is not always open unless the family of open sets is finite.



❖ **Example 10:** For every real number $r > 0$, define the set $S_r = D(0, r)$. That is, S_r is the open disk centered at the origin $z = 0$ with radius r . It is not difficult to prove that

$$\bigcap_{r>0} S_r = \{0\}$$

Clearly, the singleton $\{0\}$ is not open, although S_r is open for every $r > 0$. We have therefore obtained an example of a family of open sets whose intersection is not open.

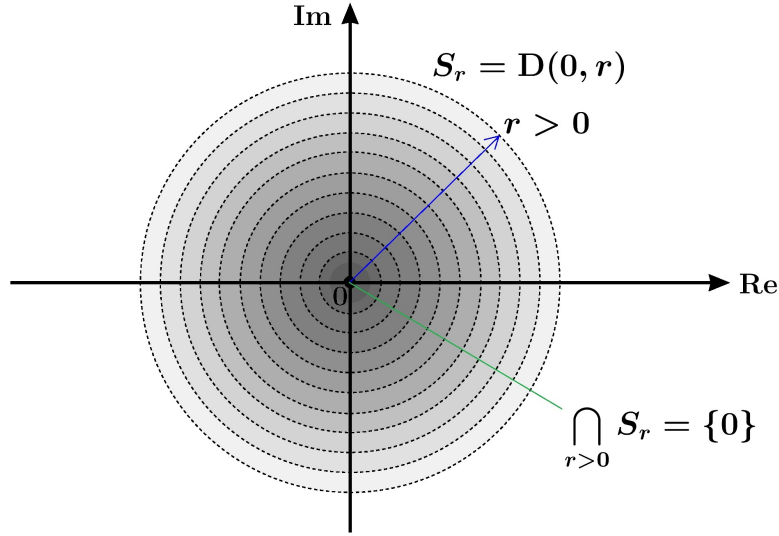


Fig. 2.40: The intersection of all the open disks $S_r = D(0, r)$, $r > 0$, is the singleton $\{0\}$

➤ Similarly, a union of closed sets is not always closed unless the family of closed sets is finite. It is easy to verify that the union of the closed disks $\overline{D}(0, r)$, $0 \leq r < 1$, is the open unit disk

$$D(0, 1) = \bigcup_{0 \leq r < 1} \overline{D}(0, r)$$

We have obtained an example of a family of closed sets whose union is not closed! In the next theorem, we prove that if the family of sets is finite, then both questions have affirmative answers.

i Theorem 2.7: Let $\{S_k\}_{k=1}^n$ be a finite family of sets in the complex plane $\mathbb{C}$.

(a) If all the sets S_k are open, then their intersection $\bigcap_{k=1}^n S_k$ is also open.

(b) If all the sets S_k are closed, then their union $\bigcup_{k=1}^n S_k$ is also closed.

✓ **Proof:**

(a) Set $S = \{S_k\}_{k=1}^n$. Suppose that all the sets S_k are open. Let $z \in S$. For every k , there exists a real number $r_k > 0$ such that $D(z, r_k) \subseteq S_k$. Let $r = \min \{r_k : 1 \leq k \leq n\}$. Then $D(z, r) \subseteq D(z, r_k) \subseteq S_k$. Thus, the disk $D(z, r)$ is contained in the intersection S , so z is an interior point of S . We have proved that every point z of S is an interior point, and hence S is open.

(b) As in the proof of the preceding theorem, this part follows from the preceding part and De

Morgan's law

$$\mathbb{C} - \bigcup_{k=1}^n S_k = \bigcap_{k=1}^n (\mathbb{C} - S_k)$$

■

2.3 Connectedness

- The property of connectedness allows us to classify sets in the complex plane according to different levels of complexity and thereby to study and investigate them in order of complexity. Connected sets are easier to handle and understand, and serve as building blocks for more complicated sets. An important practice in understanding and handling complicated sets is to decompose them into simpler connected components.
- In the present course, we will encounter three kinds of connectedness:
 - (a) Topological connectedness, or simply connectedness (connected)
 - (b) Path connectedness (pathwise connected)
 - (c) Simple connectedness (Simply connected)

i Definition 2.11: Let $S \subseteq \mathbb{C}$ be a set of points in the complex plane. A pair of sets (A, B) is called a **separation of S** if the following conditions hold:

- (a) $A \cap B = \emptyset$
- (b) $S \subseteq A \cup B$
- (c) $A \cap S \neq \emptyset$
- (d) $B \cap S \neq \emptyset$

❖ **Example 11:** If A is the open left half-plane and B is the closed right half-plane, then (A, B) is a separation of the unit disk $|z| < 1$.

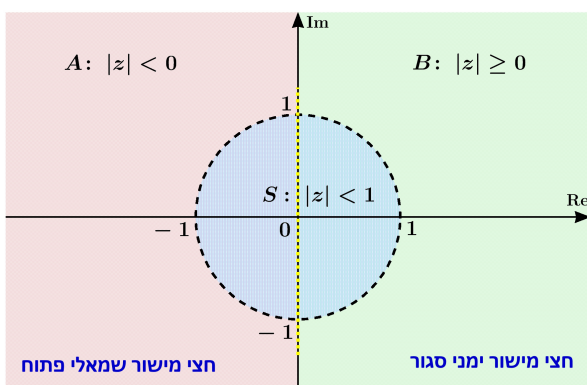


Fig. 2.41: A separation of the open unit disk by half-planes

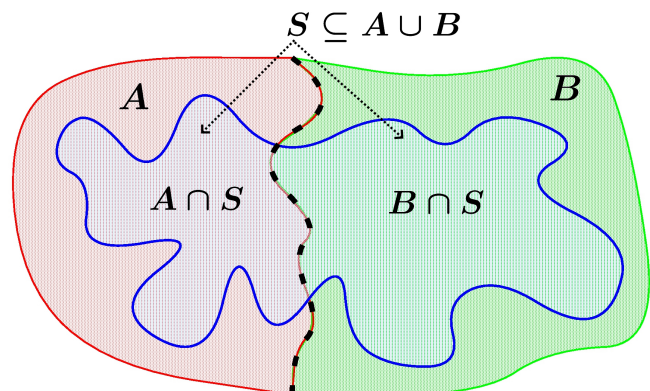


Fig. 2.42: A separation (A, B) of a set S

- A separation (A, B) of a set S is called an **open separation** if the sets A and B are open.
- Note that the separation of the unit disk in Figure 2.41 is not an open separation! In fact, we shall soon discover that the unit disk has no open separation!

i Definition 2.12: Let $S \subseteq \mathbb{C}$ be a set of points in the complex plane. The set S is called **topologically connected** if it has no open separation.

- In other words, the set S is topologically connected if, for every separation (A, B) of S , at least one of the two sets A and B is not open.
- It is customary to abbreviate the term **topologically connected** to **connected**.

i Proposition 2.8: Every path in the complex plane is a connected set¹.

- ✓ **Proof:** We prove the proposition for a finite path and leave the proof for the other kinds of paths to the exercises. Let $\sigma: [a, b] \rightarrow \mathbb{C}$ be a finite path, and let $S = \sigma([a, b])$ be the image of σ . Suppose, for contradiction, that S is not connected. That is, there exists an open separation (A, B) of S . Without loss of generality, suppose that $\sigma(a) \in A$, and define

$$t_0 = \inf \{ t \in [a, b] : \sigma(t) \in B \}$$

Clearly $t_0 > a$, because A contains a neighborhood of $\sigma(a)$ and therefore contains some initial portion of the path. If $\sigma(t_0) \in B$, then there exists $r > 0$ such that $D = \mathbb{D}(\sigma(t_0), r) \subseteq B$. The point $\sigma(t_0)$ is the center of the disk D , and hence D contains a point $\sigma(t)$ for some $t < t_0$, contradicting the definition of t_0 . Similarly, we prove that if $\sigma(t_0) \in A$, then there exists $\delta > 0$ such that, for every t in the interval $[t_0, t_0 + \delta]$, $\sigma(t) \in A$, contradicting the definition of t_0 . ■

i Corollary 2.9: Every segment $[z_0, z_1]$ in the complex plane is a connected set².

- ✓ **Proof:** A segment $[z_0, z_1]$ is a special case of a path $\sigma: [0, 1] \rightarrow \mathbb{C}$ whose formula is

$$\sigma(t) = z_0 + t(z_1 - z_0)$$

The proposition is, of course, also true for an open segment, a half-open segment, a ray, and a line. ■

i Corollary 2.10: Every closed disk in the complex plane is a connected set.

- ✓ **Proof:** The theorem follows easily from Theorem 2.11, which we shall prove shortly, so we present only the idea of the proof. Suppose that (A, B) is an open separation of the closed disk $\overline{\mathbb{D}}(z_0, r)$. Choose a point $z_1 \in A \cap \overline{\mathbb{D}}(z_0, r)$, and a point $z_2 \in B \cap \overline{\mathbb{D}}(z_0, r)$. Since z_1 and z_2 are points in the disk, the segment $[z_1, z_2]$ is also contained in the disk. Clearly, (A, B) is

¹ More precisely, one must distinguish between a path as a function $\sigma: [a, b] \rightarrow \mathbb{C}$ and the set of points belonging to it, which is its image $\sigma([a, b])$. The proposition, of course, refers to this set.

² The proposition is, of course, also true for an open segment, a half-open segment, a ray, and a line.



also an open separation of the segment $[z_1, z_2]$, contradicting the preceding corollary. ■

- A similar proof for an ellipse, square, triangle, and so forth may become more complicated. The matter becomes still more complicated for concave sets or sets containing “holes” (see Figures 2.43 and 2.44). We therefore need an additional tool to help us prove that familiar simple sets are indeed connected. This additional tool is the concept of polygonal connectedness (and subsequently the concept of path connectedness).

i Definition 2.13: A set S is called **polygonally connected** if, for every pair of points $z_1, z_2 \in S$, there exists a polyline within S joining z_1 to z_2 .

- Figures 2.43 and 2.44 show two examples of polygonally connected sets. Note that in Figure 2.44, the set does not include the white holes inside it, and some effort is therefore required to join any two of its points by a polyline.

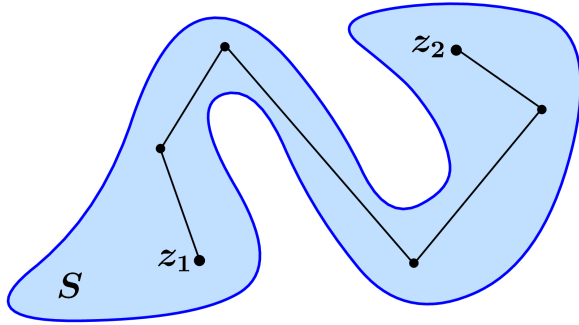


Fig. 2.43: A polygonally connected set

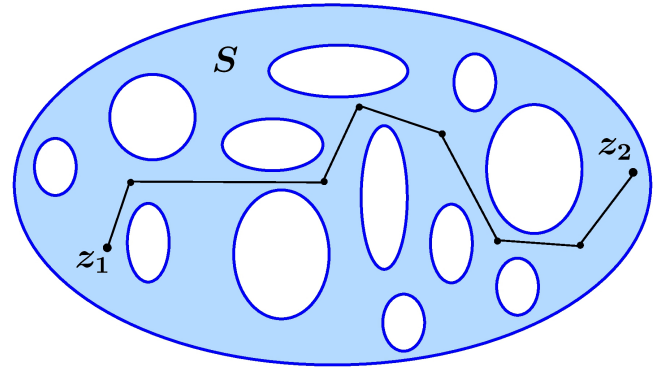


Fig. 2.44: A polygonally connected set

i Theorem 2.11: An open set $S \subseteq \mathbb{C}$ is connected if and only if it is polygonally connected.

- ✓ **Proof:** We must prove both directions.

Direction 1: Suppose that S is connected. We prove that S is polygonally connected. If S is empty, then the assertion is vacuously true. Therefore, suppose that $S \neq \emptyset$. Choose a point $z_0 \in S$ and define

$$S_0 = \{ z \in S : z \text{ dna } z_0 \text{ gninioj } S \text{ ni enilylop a si ereht} \}$$

Thus, S_0 is the set of all points $z \in S$ that can be reached from z_0 by a polyline contained in S . We prove that S_0 is open: if $z_1 \in S_0$, then $z_1 \in S$, and hence there exists an open disk $\mathbb{D}(z_1, r) \subseteq S$. If $z \in \mathbb{D}(z_1, r)$, then we can join it to the point z_1 by a straight line and continue from there to z_0 . Thus, $\mathbb{D}(z_1, r) \subseteq S_0$, and consequently S_0 is open. Next, we prove that $S_0 = S$. It will then follow that any two points $z_1, z_2 \in S$ can be joined by a polyline passing through z_0 , completing the proof. Suppose, for contradiction, that $S_0 \neq S$. We prove that $S - S_0$ is open, obtaining an open separation $(S_0, S - S_0)$ of S , contrary to the assumption that S is connected. Let $z_2 \in S - S_0$. Thus, z_2 cannot be joined to z_0 by

a polyline. There exists an open disk $\mathbb{D}(z_2, r) \subseteq S$. No point $z \in \mathbb{D}(z_2, r)$ can be joined to z_0 , because otherwise one could travel from z_2 to z along a straight line and then continue to z_0 . Hence $\mathbb{D}(z_2, r) \cap S_0 = \emptyset$, and therefore $\mathbb{D}(z_2, r) \subseteq S - S_0$. Thus, $S - S_0$ is open, and we have obtained an open separation $(S_0, S - S_0)$ of S , contrary to the hypothesis.

Direction 2: Suppose that S is polygonally connected. We prove that S is connected. Suppose, for contradiction, that S is not connected. Then there exists an open separation (A, B) of S . Choose a point $z_1 \in A$ and a point $z_2 \in B$. By hypothesis, there exists a polyline $[w_1, w_2, \dots, w_n]$ contained in S such that $w_1 = z_1$ and $w_n = z_2$. Consider each of the segments $[w_k, w_{k+1}]$, $k = 1, 2, \dots, n - 1$. If any one of them intersects both sets A and B , then (A, B) is also an open separation of the segment $[w_k, w_{k+1}]$, contradicting Corollary 2.9. Therefore, the only remaining possibility is that, for every k , either $[w_k, w_{k+1}] \subseteq A$ or $[w_k, w_{k+1}] \subseteq B$. Clearly, not all segments can lie on the same side of the separation, so there exists k such that $[w_k, w_{k+1}] \subseteq A$ and $[w_{k+1}, w_{k+2}] \subseteq B$ (or vice versa). Thus, $w_{k+1} \in A$ and $w_{k+1} \in B$, contradicting the assumption that $A \cap B = \emptyset$. ■

i Definition 2.14: An open, topologically connected set is called a **domain (Domain)**.

i Definition 2.15: A domain D containing a point z is called a **neighborhood of the point z** . A domain D containing a set S is called a **neighborhood of the set S** .

- Our notion of a neighborhood is slightly more general than usual³, but it is more practical.
- It must be emphasized that the equivalence between topological connectedness and polygonal connectedness in Theorem 2.11 holds only for open sets! For general sets (which are not necessarily open), the concept of polygonal connectedness ceases to work, as the following example shows. To cover general sets, we need the more general concept of **path connectedness**.
- ◆ **Example 12:** Define a set S consisting of the two disks $|z - 2| \leq 1$ and $|z + 2| \leq 1$, and the upper half of the circle $|z - i| = 2$ (see Figure 2.45). It is clear that this set should be connected, but it is not polygonally connected, because every path joining a point in the left disk to a point in the right disk must pass through the upper arc and therefore cannot be a polyline.

i Definition 2.16: A set S is called **path connected** if, for every pair of points $z_1, z_2 \in S$, there exists a path $\sigma: [a, b] \rightarrow S$ such that $z_1 = \sigma(a)$ and $z_2 = \sigma(b)$.

- It is not difficult to prove that the set S from Example 12 is path connected (although it is not polygonally connected). Moreover, the set S is also topologically connected. Are path connectedness and topological connectedness equivalent for general sets? The answer is no. Path connectedness implies topological connectedness, but not conversely, as the next example shows.

³ A neighborhood is usually defined as an open disk about the point

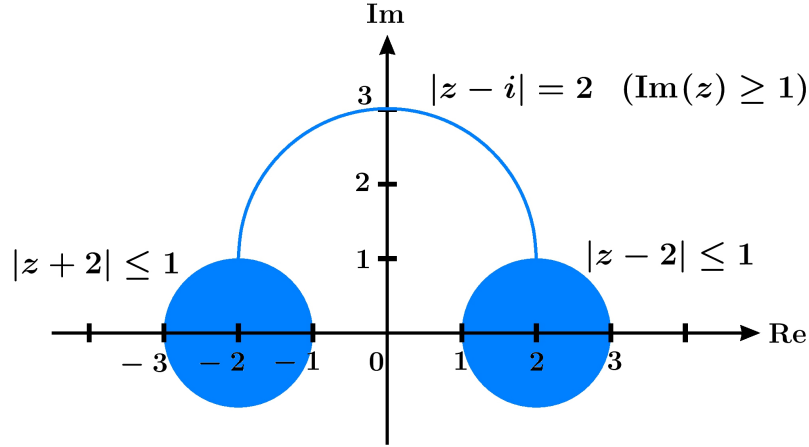


Fig. 2.45: The set S is path connected but not polygonally connected

◆ **Example 13:** Define the set S as the union of the graph of the function $y = \sin \frac{1}{x}$ over the positive x -axis and the segment $x = 0$, $-1 \leq y \leq 1$ (the red segment on the y -axis in Figure 2.46).

$$S = \left\{ x + yi \in \mathbb{C} : y = \sin \frac{1}{x}, 0 < x < \infty \right\} \cup \{ yi : -1 \leq y \leq 1 \}$$

It is clear from the definition that the red segment $I = \{ yi : -1 \leq y \leq 1 \}$ is disconnected from the graph of the function $\sin \frac{1}{x}$, so the set S is not path connected, but S is topologically connected. Indeed, if (A, B) is an open separation of S , then Corollary 2.9 implies that the entire segment I must be contained in either A or B . Without loss of generality, suppose that $I \subseteq A$. The point $z = 0$ belongs to A , so there exists $r > 0$ such that $D(0, r) \subseteq A$. Clearly, the disk $D(0, r)$ intersects the graph of the function, and hence the set A also intersects the graph. We have thus obtained that (A, B) is also an open separation of the path $\sigma(t) = t + i \sin \frac{1}{t}$, contradicting Proposition 2.8.

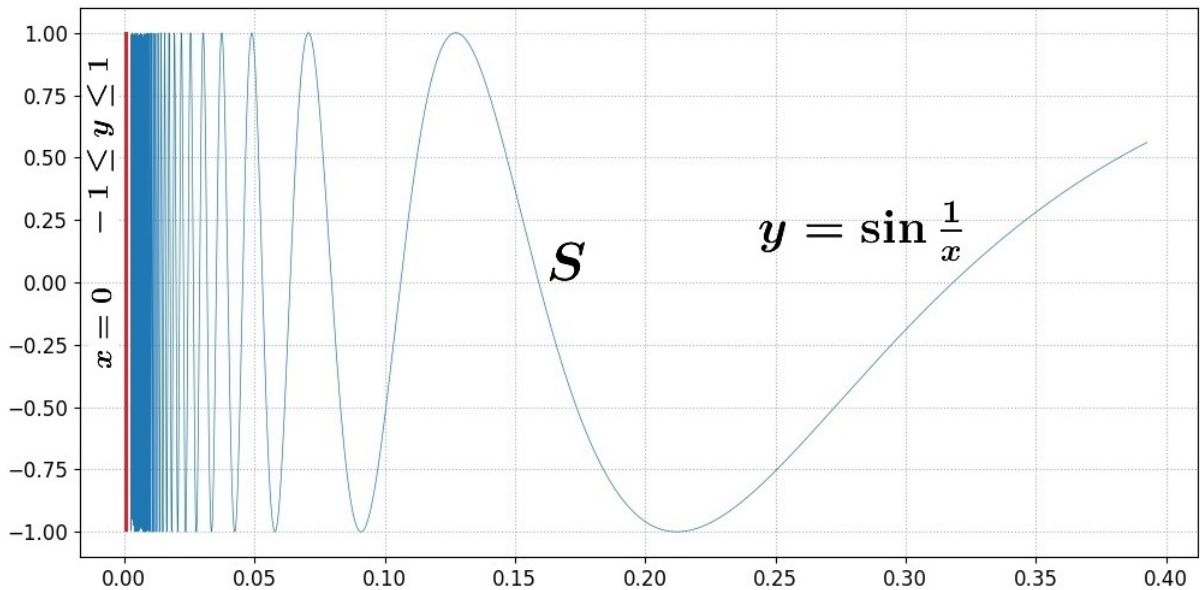


Fig. 2.46: The set S , consisting of the graph of the function and the segment on the y -axis, is topologically connected but not path connected

i Theorem 2.12: Every path-connected set is also topologically connected.

✓ **Proof:** This is an immediate consequence of Proposition 2.8. Let S be a path-connected set. Suppose, for contradiction, that S is not topologically connected. That is, there exists an open separation (A, B) of S . Choose a point $z_1 \in A$ and a point $z_2 \in B$. By hypothesis, there exists a path σ joining z_1 to z_2 and contained in S . Clearly, (A, B) would be an open separation of the path, contradicting Proposition 2.8. ■

➤ We conclude the section with the third concept, simple connectedness (simply connected). Simple connectedness is a much stronger property than the preceding two kinds of connectedness. It describes a domain without holes.

➤ Before giving the definition, however, we need the concepts of the interior and exterior of a simple closed path. Although these two concepts seem very natural and intuitive, the mathematical proof guaranteeing their existence is very complicated and lies beyond the scope of the course. We content ourselves with stating the Jordan theorem for simple closed paths.

i Theorem 2.13: The Jordan Curve Theorem for the Complex Plane (Camille Jordan 1838-1922)

Every simple closed path σ divides the complex plane into two disjoint domains called the **interior of σ** and the **exterior of σ** . The interior of σ is a bounded set topologically equivalent to an open disk. The exterior of σ is an unbounded set topologically equivalent to a punctured plane.



Fig. 2.47: Camille Jordan
1838-1922

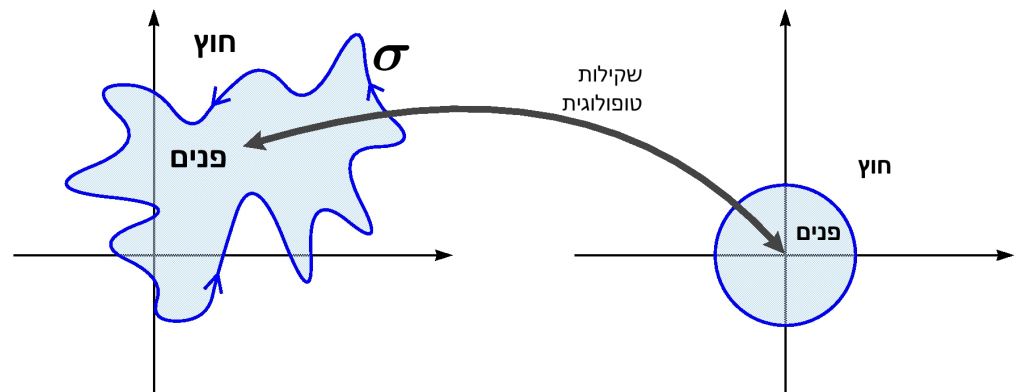


Fig. 2.48: Every simple closed path divides the plane into an interior and an exterior

➤ Recall that the term “domain” appearing in the theorem denotes an open and (topologically) connected set!

➤ Several proofs of this theorem may be found in the following discussion. All lie beyond the scope of the course; interested students are invited to consult them:

 [Nice proofs to the Jordan curve theorem](#)

i Definition 2.17: A set S in the complex plane is called **simply connected (simply connected)** if it is path connected and, for every simple closed path (Jordan path) contained in S , the interior of σ is also contained in S .

- In simple terms, a set S is simply connected if it is path connected and contains no holes.
- It follows from the definition that every simply connected set is path connected, but the converse is false. Figure 2.44 (page 60) shows a set that is path connected but not simply connected. The set in Figure 2.43 (page 60), by contrast, is simply connected.
- Inspection of the above examples reveals that the boundary of a set that is not simply connected is disconnected (the holes are separated from the outer boundary). It turns out that this is an equivalent characterization of simple connectedness, and some books use it as the definition of this kind of connectedness. We formulate it as a theorem.

i Theorem 2.14: A path-connected set S is simply connected if and only if its boundary ∂S is connected.

- ✓ **Proof:** We give only an informal proof: if S is not simply connected, then it has at least one hole, and hence ∂S also contains the boundary of the hole, separated from the outer boundary. If the set S has no holes, then its boundary is a Jordan curve, which is of course a simple closed curve and is therefore connected by Proposition 2.8.

i Definition 2.18: Let S be a given set. A nonempty subset $C \subseteq S$ is called a **connected component of S** if C is connected and there is no larger connected subset $C \subset D \subseteq S$.

- In brief, a **connected component of S** is a maximal connected subset of S .
- ◆ **Example 14:** Let S be the set of all points in the unit disk $|z| < 1$ that satisfy

$$\operatorname{Im}(z) \neq \pm \operatorname{Re}(z)$$

That is, z does not lie on either of the lines $y = x$ or $y = -x$. It is easy to see that S consists of four distinct connected components, as shown in Figure 2.49. For example, we describe the component S_1

$$S_1 = \left\{ z \in \mathbb{C} : |z| < 1 \text{ and } -\frac{\pi}{2} < \operatorname{Arg}(z) < \frac{\pi}{2} \right\}$$

i Definition 2.19: A family of sets $\{S_\alpha\}_{\alpha \in I}$ is called a **decomposition of a set $S \subseteq \mathbb{C}$ into connected components** if

- (a) For every index $\alpha \in I$, S_α is a connected component of S .
- (b) For every pair of distinct indices $\alpha, \beta \in I$, $S_\alpha \cap S_\beta = \emptyset$.

- In the next theorem, we prove that there is a unique way to decompose a set into connected



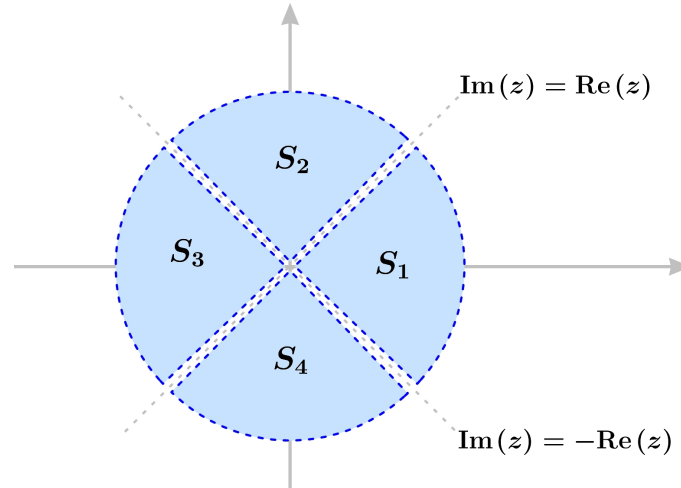


Fig. 2.49: The set S is disconnected but has four connected components

components.

i Theorem 2.15: Let $S \subseteq \mathbb{C}$. Then S has a unique decomposition into connected components.

✓ Proof: For every point $z \in S$, define the set

$$S_z = \{ w \in S : \sigma \subseteq S \text{ path from } z \text{ to } w \text{ in } S \}$$

Clearly, S_z is connected: if $w_1, w_2 \in S_z$, then there is a path σ_1 in S joining w_1 to z , and there is a path σ_2 in S joining w_2 to z . Concatenating σ_1 with the reverse of σ_2 gives a path joining w_1 and w_2 . Clearly, the family of sets $\{S_z\}_{z \in S}$ contains many duplicates. If there is a path in S joining z_1 to z_2 , then $S_{z_1} = S_{z_2}$. On the other hand, if there is no path joining z_1 to z_2 , then $S_{z_1} \cap S_{z_2} = \emptyset$ (why?). After removing all these duplicates, we are left with a disjoint family of connected sets $\{S_z\}_{z \in I}$. Clearly, $S = \bigcup_{z \in I} S_z$. ■

✦ Exercise 2.1: Prove that if S is closed, then all connected components of S must be closed sets.

2.4 Sequences of Complex Numbers

➤ The final kind of sets to be discussed is sequences of complex numbers. As in the real case, the subject of sequences is essential for understanding and handling the concepts of functions, continuity, differentiability, integrability, and so forth. The complex case is completely analogous. We begin with the definition of the limit of a sequence of complex numbers.

i Definition 2.20: A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ converges to the limit z if the sequence of distances $\{|z - z_n|\}_{n=1}^{\infty}$ converges to zero.

➤ The sequence of distances is a real sequence, and hence its limit is familiar to us.

- All the principal theorems concerning sequences in the real number field are also valid for complex sequences, and all the proofs are nearly identical, so we shall not formulate these theorems separately. The main differences are that complex sequences have a two-dimensional aspect and possess a richer and more complicated visual representation than real sequences.
- The fundamental property underlying all other properties of complex sequences is the fact that a complex sequence is simply a combination of two real sequences: $z_n = a_n + ib_n$. This makes it possible to decompose every assertion about a complex sequence into two equivalent assertions about real sequences.

i Proposition 2.16: A complex sequence $z_n = a_n + ib_n$ converges to the limit $z = a + bi$ if and only if $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} b_n = b$.

✓ **Proof:**

Direction 1: Suppose that $\lim_{n \rightarrow \infty} z_n = z$. By identity 9 in Theorem 1.2,

$$|a - a_n| = |\operatorname{Re}(z - z_n)| \leq |z - z_n|$$

Therefore, $\lim_{n \rightarrow \infty} |a - a_n| = 0$, and hence $\lim_{n \rightarrow \infty} a_n = a$. In the same way, we prove that $\lim_{n \rightarrow \infty} b_n = b$.

Direction 2: Suppose that $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} b_n = b$. Then

$$|z - z_n| = |(a - a_n) + (b - b_n)i| \leq |a - a_n| + |b - b_n| \xrightarrow{n \rightarrow \infty} 0$$

We have used the triangle inequality. ■

◆ **Example 15:**

- a. The sequence $\left\{ \frac{2n}{n+1} + ie^{-\frac{1}{n}} \right\}_{n=1}^{\infty}$ converges to the limit $2 + i$ because

$$\lim_{n \rightarrow \infty} \frac{2n}{n+1} = 2, \quad \lim_{n \rightarrow \infty} e^{-\frac{1}{n}} = 1$$

- b. We compute the limit of the sequence $\left\{ \frac{n+i}{n-i} \right\}_{n=1}^{\infty}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n+i}{n-i} &= \lim_{n \rightarrow \infty} \frac{(n+i)^2}{(n-i)(n+i)} = \lim_{n \rightarrow \infty} \frac{n^2 - 1 + 2ni}{n^2 + 1} \\ &= \lim_{n \rightarrow \infty} \frac{n^2 - 1}{n^2 + 1} + \lim_{n \rightarrow \infty} \frac{2n}{n^2 + 1} i = 1 \end{aligned}$$

- Most definitions and limit theorems from real differential calculus can be naturally generalized to the complex number field. Because of limitations of time and scope, we shall not do so in



this course. We illustrate the idea by proving the Bolzano-Weierstrass theorem for the complex number field.

i Theorem 2.17: If the sequence $\{z_n\}_{n=1}^{\infty}$ converges to the limit z , then

(a) the sequence $\{z_n\}_{n=1}^{\infty}$ is bounded.

(b) $\lim_{n \rightarrow \infty} \overline{z_n} = \overline{z}$

(c) $\lim_{n \rightarrow \infty} |z_n| = |z|$

✓ Proof: (a) The sequence of distances $\{|z - z_n|\}_{n=1}^{\infty}$ converges and is therefore bounded by some real number M .

$$|z_n| = |z_n - z + z| \leq |z_n - z| + |z| \leq M + |z|$$

(b) Suppose that $z_n = x_n + iy_n$ and $z = x + iy$. By hypothesis, $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, and therefore

$$\lim_{n \rightarrow \infty} \overline{z_n} = \lim_{n \rightarrow \infty} (x_n - iy_n) = \lim_{n \rightarrow \infty} x_n - i \lim_{n \rightarrow \infty} y_n = x - iy = \overline{z}$$

(c) By the triangle inequality (Theorem 1.3), $||z_n| - |z|| \leq |z_n - z| \rightarrow 0$. ■

i Theorem 2.18: (Bolzano-Weierstrass)

Every bounded sequence of complex numbers has a convergent subsequence.

✓ Proof: Let $z_n = a_n + ib_n$ be a bounded sequence of complex numbers. That is, there exists a positive real number $R > 0$ such that, for every n , $|z_n| \leq R$. Since $|a_n| \leq |z_n| \leq R$ and $|b_n| \leq |z_n| \leq R$, the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are bounded. By the Bolzano-Weierstrass theorem for real sequences, there exists a convergent subsequence $\{c_n\}_{n=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$. Let $u_n = c_n + id_n$ be the subsequence of $\{z_n\}_{n=1}^{\infty}$ corresponding to $\{c_n\}_{n=1}^{\infty}$. Applying the Bolzano-Weierstrass theorem once more, now to the real sequence d_n , yields a convergent subsequence of $\{z_n\}_{n=1}^{\infty}$. ■

i Theorem 2.19: A set $S \subseteq \mathbb{C}$ is closed if and only if, for every sequence of points $\{z_n\}_{n=1}^{\infty}$ in S converging to a limit $z = \lim_{n \rightarrow \infty} z_n$, we also have $z \in S$.

✓ Proof: Direction 1: If S is closed and $z_n \in S$ converges to a point $z \notin S$, then, by the definition of a limit, for every $\varepsilon > 0$ the neighborhood $D(z, \varepsilon)$ intersects S , and hence z is a boundary point of S and must therefore belong to S .

Direction 2: Suppose that, for every convergent sequence $z_n \in S$, we also have $\lim_{n \rightarrow \infty} z_n \in S$. We prove that the set $\mathbb{C} - S$ is open. Let $z \in \mathbb{C} - S$. We must prove that z is an interior point; that is, there exists $r > 0$ such that $D(z, r) \subseteq \mathbb{C} - S$. Suppose, for contradiction, that no such r exists. Then, for every n , the neighborhood $D(z, \frac{1}{n})$ intersects S . Thus, for



every n , there exists $z_n \in \mathbb{D}(z, \frac{1}{n}) \cap S$, and hence $z_n \rightarrow z$. By the hypothesis, $z \in S$, a contradiction. $\blacksquare$

i Definition 2.21: A set S is called a **compact set** if S is closed and bounded.

Exercise 2.2: Let S be a compact set, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of open sets such that $S \subseteq \bigcup_{n=1}^{\infty} A_n$. Prove that there exists m such that $S \subseteq \bigcup_{n=1}^m A_n$.

Hint: This is one of the important properties of a compact set: **Every open cover has a finite subcover**. Suppose, for contradiction, that this is false, and construct a sequence of points $\{z_n\}_{n=1}^{\infty}$ in the set S such that, for every n , $z_n \notin \bigcup_{k=1}^n A_k$.

i Proposition 2.20: (Cantor's lemma) Let $\{K_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty compact sets. Then

(a) $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

(b) If, in addition, $\lim_{n \rightarrow \infty} \text{diam}(K_n) = 0$, then there exists a point z_0 such that $\bigcap_{n=1}^{\infty} K_n = \{z_0\}$.

Proof: A decreasing sequence means that $K_{n+1} \subseteq K_n$. We present the idea of the proof and leave the details for the student to complete as an exercise. For every n , choose a point $z_n \in K_n$. By compactness, the sequence $\{z_n\}_{n=1}^{\infty}$ has a convergent subsequence. Thus, without loss of generality, we may suppose that $z_n \rightarrow z_0$. It is not difficult to prove that, for every n , $z_0 \in K_n$. Therefore, $z_0 \in \bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Part b: Recall that the expression $\text{diam}(K_n)$ denotes the diameter of K_n (see Definition 2.3). If the intersection contained two distinct points z_1 and z_2 , then the distance between them would be $d = |z_1 - z_2| > 0$, and hence, by the definition of diameter, for every n , $\text{diam}(K_n) \geq d$, contradicting the assumption that $\text{diam}(K_n) \rightarrow 0$. $\blacksquare$

i Definition 2.22: A point z_0 is called an **accumulation point** of a set S if, for every $\varepsilon > 0$,

$$\dot{\mathbb{D}}(z_0, \varepsilon) \cap S \neq \emptyset$$

That is, every punctured neighborhood of z_0 intersects S .

Remark: See Definition 3.2 of a punctured neighborhood (page 91).

Example 16: Every point on the circle $|z| = R$ is an accumulation point of the open disk $|z| < R$. Every point on the circle $|z| = R$ is also an accumulation point of the punctured plane $|z| > R$.

Exercise 2.3: A point z_0 is an accumulation point of a set S if and only if there exists a sequence of distinct points $z_n \in S$ such that $\lim_{n \rightarrow \infty} z_n = z_0$, and the sequence of distances $|z_n - z_0|$ is a strictly decreasing sequence of positive numbers.

Hint: Construct z_n inductively. Begin with $\varepsilon_1 = 1$. There exists $z_1 \in S$ such that $0 < |z_1 - z_0| < 1$.



Step $n + 1$: Take $\varepsilon_{n+1} = \text{Min} \left\{ \frac{|z_n - z_0|}{2}, \frac{1}{2^{n+1}} \right\}$.

i Proposition 2.21: A set S is closed if and only if it contains all its accumulation points.

i Definition 2.23: A set D is called **dense in a set S** if, for every point $z \in S$, either $z \in D$ or z is an accumulation point of D .

Example 17: The set of “rational” complex numbers, $D = \{a + bi : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$.

i Proposition 2.22: A set D is dense in a set S if and only if every point $z \in S$ is the limit of a sequence of points $z_n \in D$.

Remark: Unlike the case of an accumulation point, the sequence z_n may be constant, $z_n = z$, when $z \in D$. For example, the singleton $S = \{i\}$ is dense in itself.

i Definition 2.24: (Infinite Limit) A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ **converges to infinity (∞)** if the sequence of its absolute values $\{|z_n|\}_{n=1}^{\infty}$ tends to infinity (as a real sequence).

Remark: Later in the course, we will discover that in the complex number system there is no distinction between $+\infty$ and $-\infty$. They are actually the very same point! Therefore, from this point onward, we will not encounter any further mention of the expression $-\infty$.

Exercises for chapter 2

Hint: At the end of the list there is [a link to a Google notebook](#) containing final and complete solutions to some of the exercises. Additional solutions will be added over time (I hope with help from the students). We would be very pleased to receive additional student solutions to exercises that have not yet been solved!



1. Sketch the following sets in the complex plane:

- | | | |
|----------------------------------|--------------------------|----------------------------|
| (a) $ z - 1 + i = 1$ | (b) $ z + i \leq 5$ | (c) $ z - 3i \geq 3$ |
| (d) $ z - 3i + z + 3i = 8$ | (e) $ z - 1 = z + i $ | (f) $ z - 1 \geq z + i $ |
| (g) $\text{Re}(\bar{z} - i) = 2$ | (h) $ 2\bar{z} + i = 4$ | (i) $z^2 + \bar{z}^2 = 2$ |

2. Give as simple a verbal description as possible of each of the following point sets:

- | | | |
|--|---------------------|--|
| (a) $\left \frac{z-1}{z+1} \right = 2$ | (b) $ z^2 - 1 = 1$ | (c) $\arg\left(\frac{z-1}{z+i}\right) = \frac{\pi}{3}$ |
|--|---------------------|--|



3. Find the locus of all points z in the plane corresponding to the parametric description

$$z - z_0 = r \left(\frac{1 + it}{1 - it} \right)$$

where $r > 0$ is a given fixed real number and $-\infty < t < \infty$ is a real parameter.



Hint: Multiply by the conjugate of the denominator to obtain $z - z_0 = r \left[\frac{(1-t^2)+2it}{1+t^2} \right]$, and then, using the substitution $t = \tan \frac{\theta}{2}$, pass to the representation $z - z_0 = re^{2i\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Use the identities $\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$, $\cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$. The locus is therefore a circle.



4. Let z_1 , z_2 , and z_3 be three distinct points in the complex plane that do not lie on the same line. Find as simple an algebraic characterization as possible of the open region bounded by the triangle formed by these points.

Hint: The skull symbol indicates that the problem may be too difficult and may therefore be postponed to a more advanced stage of the course without impeding progress.



5. Let z_1 , z_2 , z_3 , and z_4 be four distinct points in the complex plane that form the vertices of a rectangle. Find as simple an algebraic characterization as possible of the open rectangular region bounded by the rectangle (see Figure 2.50).

Hint: First solve Exercise 27 in Chapter ?? . Translate the rectangle so that the point z_1 coincides with the origin, and then rotate the rectangle about the origin so that the point z_2 lands on the real axis.

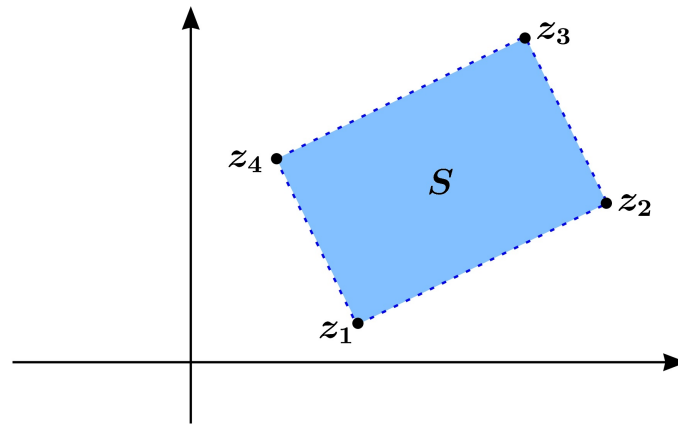


Fig. 2.50: A general rectangular region in the complex plane

6. Let S be a subset of $\mathbb{C}$. Prove that
- ∂S is a closed set.
 - $\partial(\partial S) = \partial S$
 - $\text{Int}(\partial S) = \emptyset$
7. Let S be a set of complex numbers, and let C be a closed set such that $S \subseteq C$. Prove that $\overline{S} \subseteq C$. (That is, $\overline{S}$ is the smallest closed set containing S).



8. Prove that every finite set of points in the complex plane is closed.



9. Prove: if a set S is both open and closed, then necessarily $S = \mathbb{C}$ or $S = \emptyset$.



10. Determine which of the following sets are open, closed, bounded, compact, or connected. For each set, find its interior, closure, and boundary. Whenever possible, find all its connected components.



- (a) $\{z \in \mathbb{C} : |z - 5| > 1\}$ (b) $\{z \in \mathbb{C} : |z - 5| > 1 \text{ and } |z + 5| > 1\}$
 (c) $\{z \in \mathbb{C} : z^2 \in \mathbb{R}\}$ (d) $\{z \in \mathbb{C} : |z + 5| > 1 \text{ and } |z + 5| < 3\}$
 (e) $\{z \in \mathbb{C} : |\operatorname{Re}(z)| \leq 1 \text{ and } |z| > 2\}$ (f) $\{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Re}(z) \neq |z|\}$



11. Construct an example of each of the following:

- a. A simply connected domain D whose closure $\overline{D}$ is not simply connected.
 b. A domain D that is not simply connected, but whose closure $\overline{D}$ is simply connected.
 c. A closed, simply connected set whose interior is disconnected.
 d. A bounded, simply connected domain whose boundary is not a Jordan path.
 e. An open disconnected set D whose closure $\overline{D}$ is simply connected.

Hint: a. A slit annulus, b. A punctured disk, c. Examine Figure 2.45, d. A slit disk, e. A cut disk.

12. Prove that every path $\sigma: I \rightarrow \mathbb{C}$ is a connected set, where I may be any of the following intervals: (a, b) , $[a, b)$, $(a, b]$, $(-\infty, b)$, (a, ∞) , $(-\infty, \infty)$,



13. Prove that $\lim_{n \rightarrow \infty} z^n = 0$ if and only if $|z| < 1$.



14. Prove or disprove by a counterexample: If $\lim_{n \rightarrow \infty} z_n = z$, then $\lim_{n \rightarrow \infty} \operatorname{Arg}(z_n) = \operatorname{Arg}(z)$.

15. Determine whether each of the following sequences converges or diverges. Find the limit if the sequence converges.

(a) $\frac{(1+i)^n}{n}$ (b) $\frac{i^n}{n}$ (c) $n \left(\frac{1+i}{2} \right)^n$ (d) ni^n

16. Prove that $\lim_{n \rightarrow \infty} (\sqrt{n+2i} - \sqrt{n+i}) = 0$.

17. Prove that $\lim_{n \rightarrow \infty} \sqrt{n} (\sqrt{n+2i} - \sqrt{n+i}) = \frac{i}{2}$.

18. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers satisfying

$$\lim_{n \rightarrow \infty} |z_n| = |z_0|$$

$$\lim_{n \rightarrow \infty} \operatorname{Arg}(z_n) = \operatorname{Arg}(z_0)$$

Prove that $\lim_{n \rightarrow \infty} z_n = z_0$.



19. Prove or disprove by a counterexample:

- a. If S is connected, then its complement $\mathbb{C} - S$ is also connected.
 b. If D_1 and D_2 are two non-disjoint domains, then $D_1 \cup D_2$ is also a domain.
 c. If D_1 and D_2 are two non-disjoint domains, then $D_1 \cap D_2$ is also a domain.
 d. If D_1 is a domain, D_2 is open and simply connected, and $D_2 \subset D_1$, then the difference



$D_1 - D_2$ is connected.

- e. If S is disconnected, then its closure $\overline{S}$ is also disconnected.
- f. If A is dense in B and B is dense in C , then A is dense in C .
- g. There exist two disjoint nonempty sets A and B such that A is dense in B and B is dense in A .
- h. If $\{S_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonempty closed sets, then $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$.
- i. If D_1 and D_2 are non-disjoint simply connected domains, then $D_1 \cup D_2$ is also a simply connected domain.

Notes: b. The term “non-disjoint” means that $D_1 \cap D_2 \neq \emptyset$. c. Try joining two banana-shaped domains. h. A sequence of sets $\{S_n\}_{n=1}^{\infty}$ is called decreasing if, for every n , $S_n \supseteq S_{n+1}$. i. Choose an annulus and divide it into two parts ...

20. Let $a, c \in \mathbb{R}$, with $c > 0$. Describe the set of points satisfying the equation

$$|z - a| - |z + a| = 2c$$

for every possible value of a and c .



21. Find a parametrization $x(t) + iy(t)$, $a \leq t \leq b$, of the ellipse $|z - 1| + |z + 1| = 4$.



22. The distance $d(z, S)$ from a point z to a nonempty set S is defined as the infimum of all real numbers $|z - s|$, over all $s \in S$:

$$d(z, S) = \inf_{s \in S} |z - s|$$

Let S be a closed subset of $\mathbb{C}$, and let $z \notin S$. Prove that

- (a) $d(z, S) > 0$
- (b) There exists a point $s \in S$ such that $d(z, S) = |z - s|$.



23. The distance $d(S_1, S_2)$ between two nonempty sets $S_1, S_2 \subseteq \mathbb{C}$ is defined by

$$d(S_1, S_2) = \inf_{\substack{z \in S_1 \\ w \in S_2}} |z - w|$$

If S_1 and S_2 are two disjoint compact sets, prove that

- (a) $d(S_1, S_2) > 0$
- (b) There exist points $z_1 \in S_1$ and $z_2 \in S_2$ such that $d(S_1, S_2) = |z_1 - z_2|$.

24. Continuing Exercise 23, give an example of two closed, nonempty, disjoint sets S_1 and S_2 such that $d(S_1, S_2) = 0$.

Hint: Consider the hyperbola $y = 1/x$.



25. Continuing Exercise 23, does the assertion remain true if only one set is compact and the other



is closed?

Hint: If S_1 is compact and S_2 is closed, surround S_1 by a disk and expand the disk until it intersects S_2 .



- 26.** Prove that a closed set is connected if and only if it cannot be decomposed into two disjoint nonempty closed sets.

Hint: Let S be a closed set, and suppose that $S = S_1 \cup S_2$ is a decomposition of S into two disjoint closed sets S_1 and S_2 . We may assume that S is bounded (why?). By Exercise 23, $r = d(S_1, S_2) > 0$. Define $A_1 = \bigcup_{z \in S_1} D(z, r/3)$, $A_2 = \bigcup_{z \in S_2} D(z, r/3)$. Prove that (A_1, A_2) is an open separation of S .
Second direction: Suppose that (B_1, B_2) is an open separation of S . Show that $(S - B_1, S - B_2)$ is a closed separation of S .



- 27.** Prove: if S is closed and simply connected, then the boundary of S is connected.



- 28.** Let S be an open set, and let σ be a bounded path contained entirely in S .

- Prove that there exists a positive real number $\varepsilon > 0$ such that, for every point z on the path σ , the open disk $D(z, \varepsilon)$ is contained entirely in S .
- Prove that there exist n open disks $D(z_1, \varepsilon), D(z_2, \varepsilon), \dots, D(z_n, \varepsilon)$, contained in S that cover the path σ (that is, $\sigma \subseteq \bigcup_{k=1}^n D(z_k, \varepsilon)$), and whose centers lie on the path σ . See Figure 2.51.

Hint: This is an important auxiliary assertion that will help us in Chapter 5 when proving the Cauchy-Goursat theorem 5.8, which is one of the important theorems in the course. This assertion recalls the finite-cover property of a compact set (see Exercise 2.2). The path σ is a compact set (why?), and the family of all open disks $D(z, r)$ contained in S is a cover of the path σ . Therefore, by the result of Exercise 2.2 (page 68), there exists a finite family of such disks covering the path. The only difficulty is that these disks are not guaranteed to have the same radius or to have their centers on the path. Another interesting approach is to consider the distance r between the boundary ∂S of S and the path σ . By Exercise 6, ∂S is closed. By Exercises 23 and 25, the distance r is well defined and positive, $r > 0$. It is easy to prove that if z is a point on the path σ , then the disk $D(z, r)$ is contained in S . It is easy to choose suitable points along the path at intervals smaller than r in order to cover it (but this also follows automatically from compactness). See Figure 2.51.

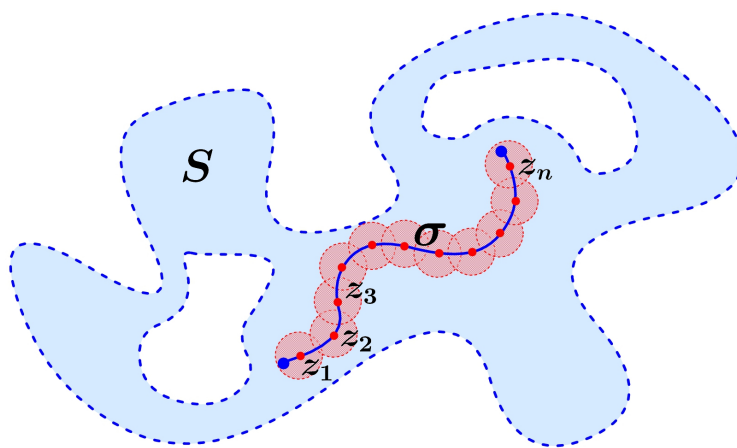


Fig. 2.51: Illustration for Exercise 28



- 29.** Use the result of Exercise 28 to give another proof of the first direction of Theorem 2.11. That is: if S is open and topologically connected, then S is polygonally connected.



30. Two paths $\sigma_1: [a_1, b_1] \rightarrow \mathbb{C}$ and $\sigma_2: [a_2, b_2] \rightarrow \mathbb{C}$ are called **equivalent** if

$$\sigma_1([a_1, b_1]) = \sigma_2([a_2, b_2])$$

That is, they describe the same curve with two different parametrizations. Prove: For every path $\sigma: [a, b] \rightarrow \mathbb{C}$, there exists an equivalent path $\gamma: [0, 1] \rightarrow \mathbb{C}$.

Hint: Construct a continuous function $f: [0, 1] \rightarrow [a, b]$ and define the path $\gamma(t) = \sigma(f(t))$.



31. Let D be a given domain, and let S be a subset of the boundary of D ($S \subseteq \partial D$). Prove that the set $D \cup S$ is connected.



32. Let D be a given domain, and let $z_1, z_2 \in \partial D$ be two distinct points on the boundary of D . Prove that there exists a simple path σ joining z_1 to z_2 all of whose interior points are interior points of D .

Hint: In other words, the path σ is contained in D except for its endpoints z_1 and z_2 , which lie on the boundary of D . Apply Exercise 31 to the set $D \cup \{z_1, z_2\}$.



33. Sketch the plane set W described by

$$W = \left\{ (x, \sin \frac{1}{x}) : 0 < x < 1 \right\} \cup \{0\} \times [-2, 1] \cup [0, 1] \times \{-2\} \cup \{1\} \times [-2, \sin(1)]$$

(a) Prove that W is closed and bounded (hence compact), and simply connected.

(b) Prove that W is in fact a simple, open, bounded curve of infinite length!

(c) Try to give a parametric description $\sigma(t) = x(t) + y(t)i$, $\sigma: [a, b] \rightarrow \mathbb{C}$, of the curve W .

(d) Does there exist a path from the point $(0.5, -1)$ to the point $(0.5, 2)$ that does not intersect W ?

Hint: The curve W is sometimes called the Warsaw “Circle” (Warsaw Circle), although it resembles a rectangle more closely, and although it is not a closed curve! (Why?). We already encountered its first part in Example 13 (page 62). Although it is bounded and simple (it does not intersect itself), its length is infinite! It is advisable to decompose it into four paths, define each one separately, and then concatenate them ($\sigma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$). After drawing the sketch, the point $(0.5, -1)$ will appear approximately at the center of the “circle”, whereas the point $(0.5, 2)$ appears outside the “circle”. Suppose that a path γ exists between these two points, and use Exercise 23 concerning the meaning of the distance between σ and γ . For further study, follow this link: mathoverflow.net



34. Rectilinear connectedness: A polyline is called a **rectilinear line** if all its sides are parallel to either the real or the imaginary axis. Rectilinear lines are widely used in the design of electronic chips. A set S is called **rectilinearly connected** if, for every pair of distinct points $z_1, z_2 \in S$, there exists a rectilinear line joining z_1 and z_2 (see Figures 2.52 and 2.53). Prove that an open set S is topologically connected if and only if it is rectilinearly connected.

Hint: One may give a proof similar to that of Theorem 2.11, or use the result of Exercise 28 to give a shorter proof.



35. Prove that the annulus $4 < |z| < 5$ is polygonally connected. What is the smallest number of



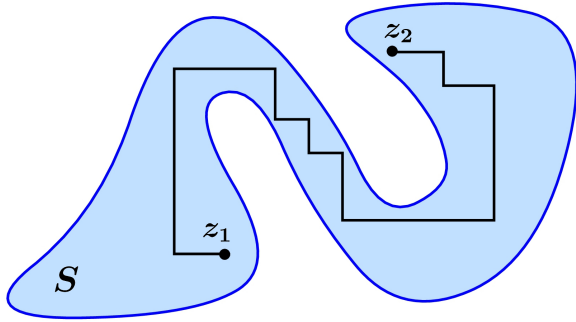


Fig. 2.52: A rectilinearly connected set

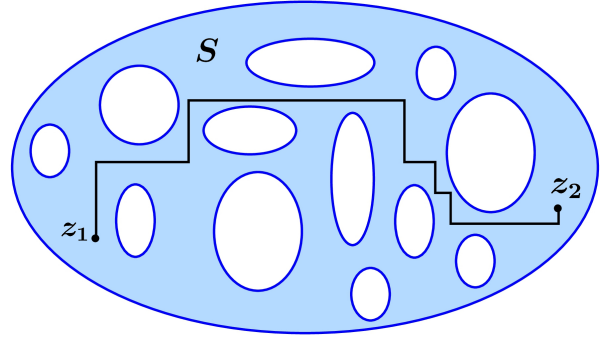


Fig. 2.53: A rectilinearly connected set

segments needed to join any two points in the annulus? What is the smallest number of segments needed to join any two points in the annulus by a rectilinear path? (Exercise 34).

Remark: This is a nontrivial mathematical challenge, but it is not directly related to the course material. It may therefore be skipped, although trying it is recommended.



36. To appreciate the complexity of proving the Jordan theorem, try to prove it for a triangle. That is, let z_1 , z_2 , and z_3 be three points in the complex plane that do not lie on the same line. Prove that the triangle they form, $[z_1, z_2, z_3, z_1]$, divides the complex plane into two open connected sets. The first set is bounded (the interior of the triangle), and the second is unbounded (the exterior of the triangle). What about a quadrilateral and a pentagon?

Hint: Try to find a simple algebraic characterization of a point z lying inside the triangle $[z_1, z_2, z_3, z_1]$, as requested in Exercise 4. Show that the set of all points satisfying this characterization is open and connected. Similarly, find a simple algebraic characterization of a point z lying outside the triangle $[z_1, z_2, z_3, z_1]$. Show that this set is also open and connected, but is unbounded.



37. Read the following Wikipedia page on the **Weierstrass function** and use it to construct a bounded path

$$\sigma: [0, 1] \rightarrow \mathbb{C}$$

of infinite length.

WIKIPEDIA The Weierstrass Function

Hint: Understanding the definition of the Weierstrass function requires a good acquaintance with series, discussed in Chapter 6, but solving the exercise does not require understanding the definition of the function. All we need is the existence of such a function; determine how to use it to define a closed path of infinite length.



38. A point z is called r -interior to a set U if $D(z, r) \subseteq U$. Prove that, for every $r > 0$, the set of all r -interior points of U is closed.

One possible idea: If $z_n \rightarrow z_0$, then $D(z_n, r) \rightarrow D(z_0, r)$.



39. Let U be an open set. Prove that there exists an increasing sequence of compact sets $\{K_n\}_{n=1}^{\infty}$, $K_n \subseteq K_{n+1}$, such that $U = \bigcup_{n=1}^{\infty} K_n$.

Hint: Exercise 38 may help.



40. Prove or disprove by a counterexample: If D is a simply connected domain and $S \subseteq D$ is a



closed, simply connected set, then the difference $D - S$ is a domain (that is, an open connected set).

Hint: Review Exercise [33](#).

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Chapter 3

Complex Functions

3.1 Basic Concepts

- Having surveyed the subject of sets, in the present chapter we discuss functions of complex numbers and the many concepts associated with them. A complex function $f: S \rightarrow \mathbb{C}$ is a single-valued correspondence from a set of complex numbers $S \subseteq \mathbb{C}$ to $\mathbb{C}$. The term “single-valued” means that every $z \in S$ is assigned exactly one value $f(z) \in \mathbb{C}$.
- The set S is called the **domain of the function**; in many cases it is indeed a domain, in the sense that it is open and connected.
- The set of all values of the function is called the **image of f** and is denoted by

$$\text{image}(f) = \{ f(z) : z \in S \}$$

Similarly, for every subset $A \subseteq S$, we define the **image of A under f** by

$$f(A) = \{ f(z) : z \in A \}$$

Thus, we may also write $\text{image}(f) = f(S)$.

- As in Calculus 1, we will define and study the concepts of limit, continuity, and differentiability for complex functions, and learn how they are related to real functions. We will also need to extend the exponential, logarithmic, and trigonometric functions to complex numbers, and determine whether these extensions preserve their properties.
- The function $f(z)$ may also be written in **Cartesian form**

$$f(z) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions of the two real variables x and y :

$$\begin{cases} u(x, y) = \operatorname{Re}(f(z)) \\ v(x, y) = \operatorname{Im}(f(z)) \end{cases}$$

For example, the function $f(z) = z^2$ may also be written in the following Cartesian form:

$$f(x + iy) = (x + iy)^2 = (x^2 - y^2) + 2xyi$$

Therefore, in this case,

$$\begin{cases} u(x, y) = x^2 - y^2 \\ v(x, y) = 2xy \end{cases}$$

Similarly, the conjugation function $f(z) = \bar{z}$ may be written as follows:

$$f(x + iy) = x - iy$$

and hence, in this case,

$$\begin{cases} u(x, y) = x \\ v(x, y) = -y \end{cases}$$

- In simple cases, the domain S and the range of the function can be drawn, but the graph of the function cannot be drawn because it is a surface in the four-dimensional space $\mathbb{R}^4$. Nevertheless, in simple cases, graphs may be drawn by using color for the fourth dimension (examples follow).
- ◆ **Example 1:** One way to visualize a function is to map differently colored strips from the domain to the range. The image of each strip is colored in the same color as the strip, thereby allowing us to visualize the deformation produced by the function. We illustrate this with the function $f(z) = z^2$. Figure 3.1 shows how the function acts on vertical strips, and Figure 3.2 shows how it acts on horizontal strips. A third way to illustrate the action of a complex function is to apply it to a grid of squares of different colors, as shown in Figure 3.3.

◆ **Exercise 3.1:** Show that the function $f(z) = z^2$ maps vertical lines $(a + ti)$ to parabolas. What happens to horizontal lines?

A Multivalued Function and Its Branches

- Naturally, in addition to ordinary (single-valued) functions, we must also deal with multivalued functions, such as the argument function $\arg$ and logarithmic functions. If $f: S \rightarrow \mathbb{C}$ is a multivalued function, for convenience we assume that the expression $f(z)$ denotes the set of all possible values of f at the point z .
- The usual practice for a multivalued function is to select one value (from infinitely many possible

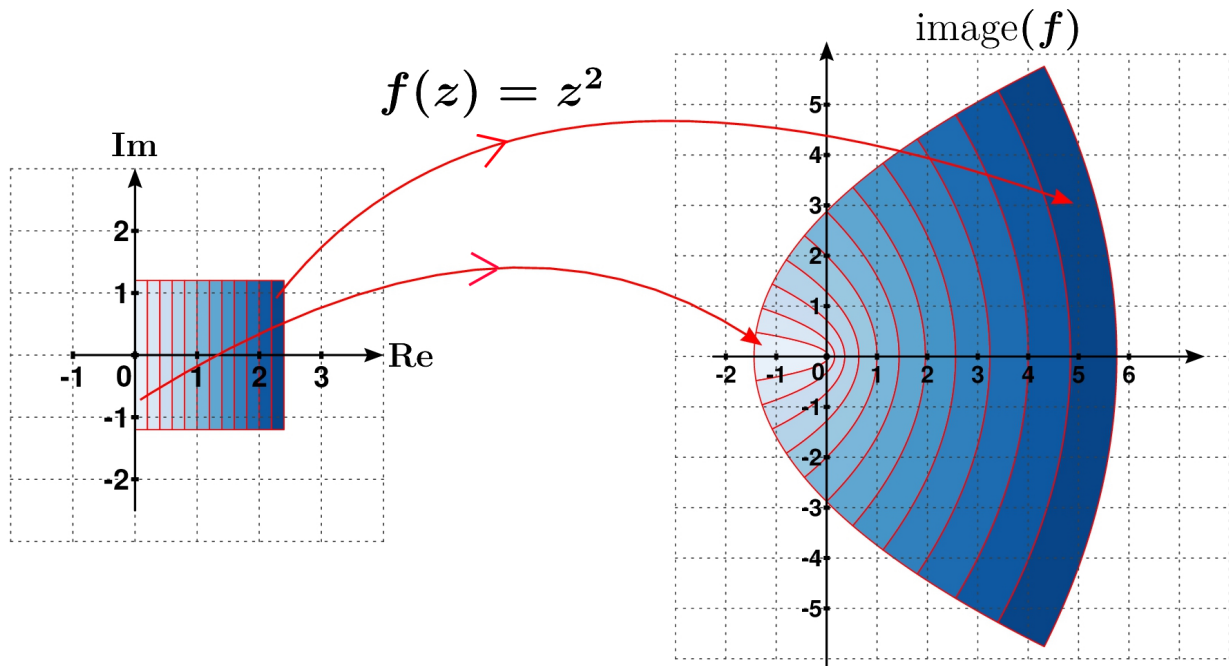


Fig. 3.1: The function $f(z) = z^2$ maps vertical strips to arcs in the range. Each shade of blue in the domain corresponds to the same shade in the range.

values) at every point of its domain.

i Definition 3.1: Let $f: S \rightarrow \mathbb{C}$ be a multivalued function. A single-valued function is called a **branch of f on a domain $D \subseteq S$** if, for every $z \in D$, $g(z) \in f(z)$.

- Recall that a “domain” is an open connected set. It is clear from the definition that every multivalued function may have infinitely many different branches. In most cases encountered later, the challenge will be to find branches with “good” properties (such as continuity and differentiability), even at the cost of giving up some points of the original domain.

3.2 The Elementary Functions

- We have already encountered the elementary functions in calculus courses. They are the most common and useful functions encountered in most areas of science, and we therefore devote the present section to them.
- As with functions of a real variable, here too the elementary functions can be constructed according to the framework of basic functions and elementary operations. The basic functions are the building blocks, and the elementary operations are used to construct increasingly complicated functions.
- **The basic functions:**
 - (a) The constant functions $f(z) = c$, for a constant $c \in \mathbb{C}$
 - (b) The identity function $f(z) = z$

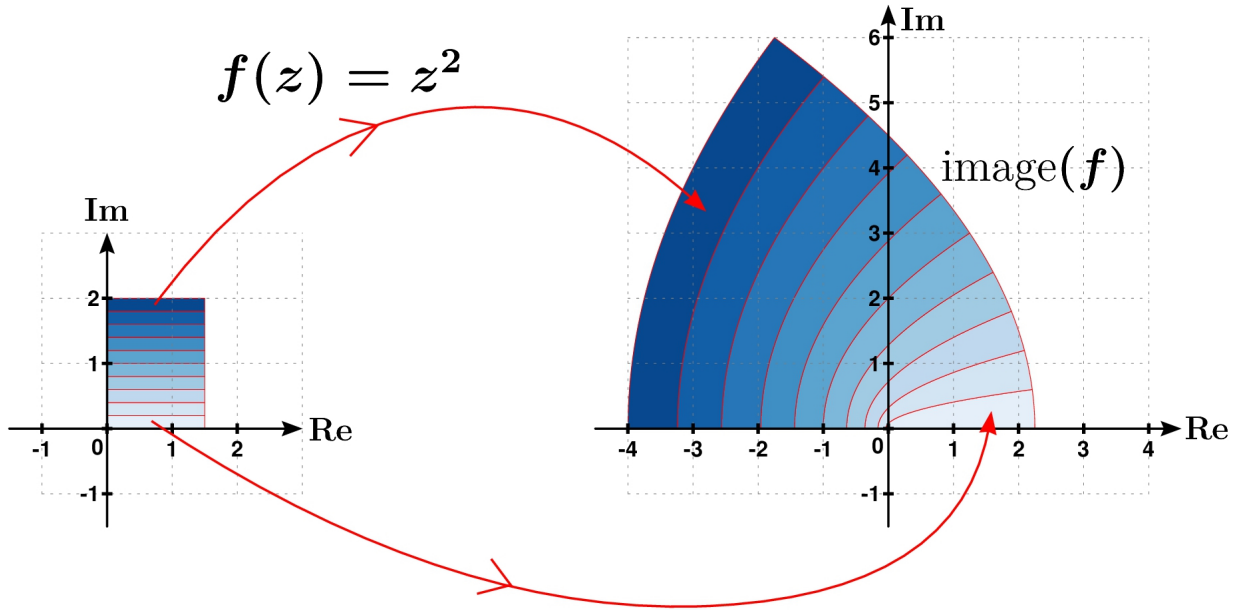


Fig. 3.2: The function $f(z) = z^2$ maps horizontal strips to arcs in the range. Each shade of blue in the domain corresponds to the same shade in the range.

- (c) The exponential functions $f(z) = c^z$
 - (d) The power functions $f(z) = z^c$
 - (e) The trigonometric functions: $\sin(z)$, $\cos(z)$, $\tan(z)$
 - (f) The logarithmic functions $f(z) = \log_c z$
 - (g) The inverse trigonometric functions: $\arcsin(z)$, $\arccos(z)$, $\arctan(z)$
- **The elementary operations:** addition, subtraction, multiplication, division, and composition.
 - **The elementary functions** are all functions that can be constructed from the basic functions by means of the elementary operations.
 - For example, the functions $f(z) = cz^n$ are obtained from the basic functions $f(z) = c$ and $f(z) = z$ by using multiplication $n + 1$ times:

$$cz^n = c \cdot \underbrace{z \cdot z \cdot \dots \cdot z}_{n \text{ times}}$$

- By adding functions of the form az^k , we can define the **polynomials**

$$f(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$$

- The **rational functions** are obtained by dividing two polynomials $P(z)$ and $Q(z)$:

$$f(z) = \frac{P(z)}{Q(z)} = \frac{a_0 + a_1z + a_2z^2 + \dots + a_nz^n}{b_0 + b_1z + b_2z^2 + \dots + b_mz^m}$$

- Composition of two functions is defined by $g(f(z))$. That is, by applying the function g to the

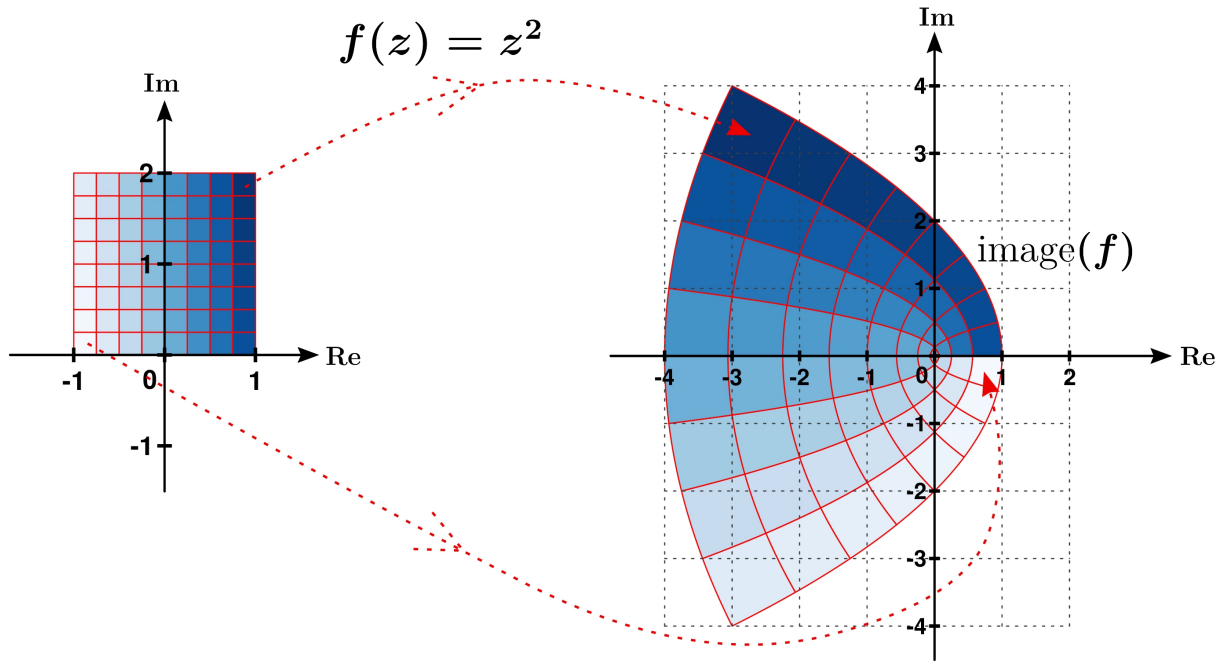


Fig. 3.3: The function $f(z) = z^2$ maps a square grid to a distorted grid in the range. Each shade of blue in the domain corresponds to the same shade in the range.

output of the function f .

- For example, the function e^{z^2+5} is obtained by composing the polynomial $f(z) = z^2 + 5$ with the exponential function $g(z) = e^z$:

$$g(f(z)) = g(z^2 + 5) = e^{z^2+5}$$

Composition is not commutative!

$$f(g(z)) = f(e^z) = e^{2z} + 5$$

In most cases, $g(f(z)) \neq f(g(z))$.

- We now survey the basic functions in detail. Because of time constraints, we leave some assertions for the student as homework.

The Function e^z

- At this stage, we present the definition of the exponential function e^z for a complex number $z = x + iy$:

$$e^z = e^x e^{iy} = e^x (\cos y + i \sin y)$$

The expression e^x is well defined for a real number x , and we defined the expression e^{iy} in the first chapter (Euler's formula (1.8)), so we now have a complete definition of e^z for every $z \in \mathbb{C}$.

➤ The Cartesian form of the function e^z is

$$e^z = e^{x+yi} = e^x \cos y + ie^x \sin y \quad (3.1)$$

➤ Of course, if the imaginary part vanishes ($y = 0$), then the function e^z coincides with the real function e^x , and hence e^z is an **extension** of the real function e^x to the complex number field!

➤ The function e^z has **period** $2\pi i$. That is, for every $z \in \mathbb{C}$,

$$e^{z+2\pi i} = e^z \quad (3.2)$$

The crucial consequence of this fact is that e^z is not one-to-one and is therefore not invertible! Consequently, the logarithm function is multivalued. Recall that the real function e^x is invertible on all of $\mathbb{R}$, and its inverse $\ln x$ is single-valued.

⬠ **Exercise 3.2:** Figure 3.4 shows a color map of e^z . The student is asked to provide a precise justification for the fact that vertical lines in the domain map to circles about the origin in the range, whereas horizontal lines in the domain map to oblique lines through the origin. For example, the horizontal line $y = \frac{\pi}{4}$ maps to the oblique line $y = \frac{\pi}{4}x$.

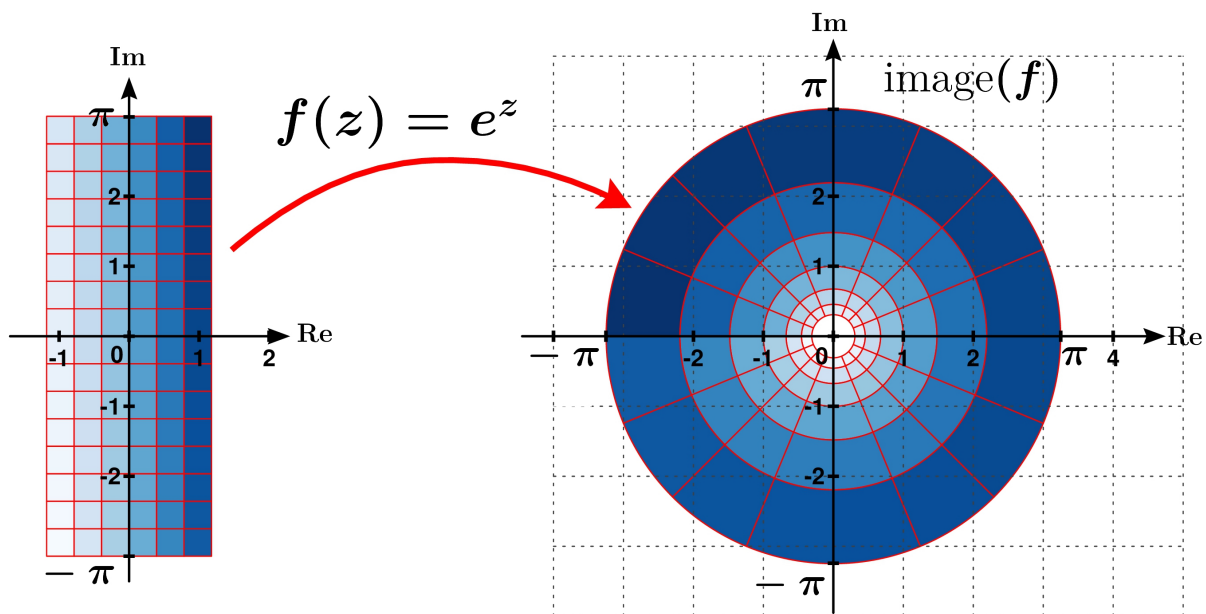


Fig. 3.4: A color map of the function $f(z) = e^z$: vertical lines map to circles, and horizontal lines map to oblique lines

➤ Elementary identities that are easy to prove:

$$\begin{aligned}
 e^{z_1+z_2} &= e^{z_1} e^{z_2} \\
 e^{z_1-z_2} &= \frac{e^{z_1}}{e^{z_2}} \\
 |e^z| &= e^{\operatorname{Re}(z)} \\
 \operatorname{Arg}(e^z) &= \operatorname{Im}(z)
 \end{aligned} \tag{3.3}$$

The Logarithm Function $\log z$

➤ For clarity, we use the symbol **ln** to denote the real logarithm function only, and the symbol **log** for the complex natural logarithm.

➤ In the real case, the natural logarithm function $y = \ln x$ is defined to be the solution of the equation

$$e^y = x \tag{3.4}$$

where x is given and y is the unknown. The real function e^y is positive and continuous, and strictly increasing on all of $\mathbb{R}$, and satisfies $\lim_{y \rightarrow -\infty} e^y = 0$, $\lim_{y \rightarrow \infty} e^y = \infty$. It is therefore a one-to-one function from $\mathbb{R}$ to $\mathbb{R}^+$. Consequently, for every real $x > 0$, equation (3.4) has a unique solution y . This solution is denoted by $\ln x$, and this is the definition of the natural logarithm function.

➤ In the complex number field, the equation $e^w = z$ has infinitely many solutions w for each given $z \neq 0$. Indeed, by formula (3.2), for every integer n , $e^{w+2n\pi i} = e^w$. We try to characterize all the solutions by a single formula. For this purpose, we use the polar representation $z = |z|e^{i \arg(z)}$:

$$e^w = e^{x+iy} = e^x e^{iy} = z = |z|e^{i \arg(z)}$$

Since $|e^{iy}| = |e^{i \arg(z)}| = 1$, it follows that $|z| = e^x$, and hence $x = \ln |z|$. Also, $e^{iy} = e^{i \arg(z)}$, and therefore $y = \arg(z)$. We have obtained

$$w = x + iy = \ln |z| + i \arg(z)$$

Thus, the definition of the natural logarithm over the complex number field is

$$\log z = \ln |z| + i \arg(z) \tag{3.5}$$

This is, of course, a multivalued function because the argument function $\arg$ is multivalued.

- The function $\log z$ is not defined for $z = 0$ because the (real) function $\ln |z|$ is defined only for $|z| > 0$ (and the argument function is likewise undefined for $z = 0$).
- If z is a positive real number ($z = x + 0i$), then $\log z$ coincides with the real function $\ln x$. Thus, the complex function $\log z$ is an extension of the real function $\ln x$.
- If z is a negative real number $x < 0$, then $\log x = \ln |x| + (2n + 1)\pi i$. Therefore, over the complex number field, the logarithm function is also defined for negative numbers!

◆ **Example 2:** $\log e = 1 + 2n\pi i$, $n \in \mathbb{Z}$

❗ **Proposition 3.1:** For every $z \neq 0$,

$$e^{\log z} = z \quad (3.6)$$

- Equality (3.6) holds, of course, for all infinitely many possible values of $\log z$.

❗ **Proposition 3.2:** For every $z \in \mathbb{C}$,

$$\log e^z = z + 2n\pi i, \quad n \in \mathbb{Z} \quad (3.7)$$

- The **principal branch of the logarithm function** $\text{Log}(z)$ is obtained by replacing the argument function $\arg$ with the principal branch of the argument function Arg (which is single-valued!).

$$\text{Log}(z) = \ln |z| + i\text{Arg}(z) \quad (3.8)$$

Although this is an ordinary single-valued function, like $\text{Arg}(z)$ it is not continuous on the negative x -axis. Later in the chapter, we prove that it is differentiable and continuous on the slit plane.

- A basic identity: for every $z \neq 0$,

$$e^{\text{Log}(z)} = z \quad (3.9)$$

- ⚠ **Warning:** The logarithmic identities familiar from the real number field do not always carry over to the complex field! For example, because the range of the function $\text{Arg}(z)$ is restricted to the interval $(-\pi, \pi]$, the familiar identity $\ln ab = \ln a + \ln b$ (a and b positive real numbers) is no longer valid for complex numbers! For example, take $z_1 = z_2 = e^{\frac{3\pi i}{2}}$. Then

$$\text{Log}(z_1 z_2) = \text{Log}\left(e^{\frac{3\pi i}{2}} e^{\frac{3\pi i}{2}}\right) = \text{Log}\left(e^{3\pi i}\right) = \text{Log}\left(e^{\pi i}\right) = \pi i$$



but

$$\operatorname{Log}(z_1) + \operatorname{Log}(z_2) = \operatorname{Log}\left(e^{\frac{3\pi i}{2}}\right) + \operatorname{Log}\left(e^{\frac{3\pi i}{2}}\right) = -\frac{\pi i}{2} - \frac{\pi i}{2} = -\pi i$$

Further information about the logarithm function will be given later in the course. We continue our survey of the basic functions.

✦ **Exercise 3.3:** Figure 3.5 shows a color map of the function $\operatorname{Log}(z)$. Try to determine the kinds of curves to which the function maps vertical and horizontal lines. Try to prove your conjecture.

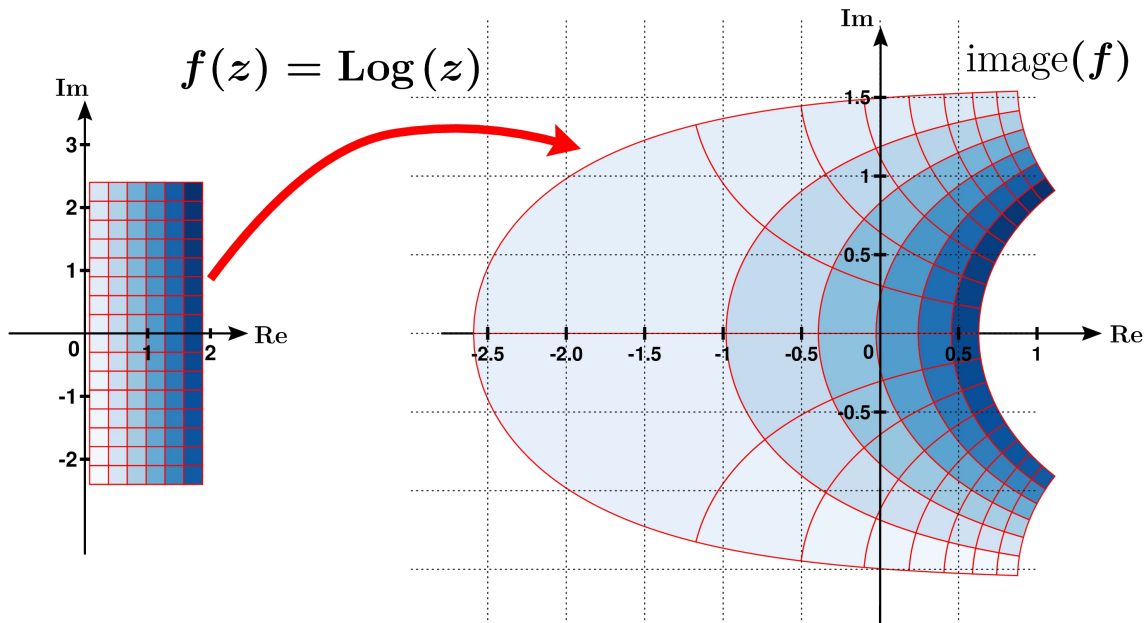


Fig. 3.5: A color map of the function $f(z) = \operatorname{Log}(z)$ for a rectangular grid in the right half-plane

✦ **Example 3:** We compute $\operatorname{Log}(-1)$ and $\operatorname{Log}(i)$.

$$\operatorname{Log}(-1) = \ln|-1| + i\operatorname{Arg}(-1) = \pi i$$

$$\operatorname{Log}(i) = \ln|i| + i\operatorname{Arg}(i) = \frac{\pi i}{2}$$

The Power Functions z^c

✦ For $c = 0$, $z^c = 1$ for every $z \neq 0$.

✦ The power function z^c , for a fixed complex number $c \in \mathbb{C} - \{0\}$, is defined by

$$z^c = e^{c \log z} \tag{3.10}$$

Since $\log z$ is multivalued, z^c is multivalued as well. The function is not defined at the point $z = 0$.

◆ **Example 4:**

$$i^{-5i} = e^{-5i \log i} = e^{-5i(\frac{\pi i}{2} + 2n\pi i)} = e^{\frac{5\pi}{2} - 10n\pi}$$

Thus, i raised to the power $-5i$ has infinitely many possible values, all of them real!

◆ **Example 5:**

$$9^{\frac{1}{2}} = e^{\frac{1}{2}(\ln 9 + 2n\pi i)} = e^{\frac{1}{2} \ln 9} \cdot e^{n\pi i} = 3e^{n\pi i}$$

It is important not to confuse the expression $\sqrt{z}$, which is a single-valued function obtained by choosing a principal branch, with the expression $z^{\frac{1}{2}}$, which denotes a multivalued function!

- It follows that the power function has infinitely many branches. The **principal branch of the power function** is the branch obtained by substituting the principal branch of the logarithm in formula (3.10):

$$z^c = e^{c \text{Log}(z)} \quad (3.11)$$

In most contexts, we use the expression z^c to denote the principal branch of the power function (unless another branch is chosen explicitly). In most practical cases, the domain of z^c excludes the slit $(-\infty, 0]$ in order to obtain continuity and differentiability on the slit plane.

- ◆ **Example:** We compute the principal branch of $z^{\frac{2}{3}}$.

$$\begin{aligned} z^{\frac{2}{3}} &= e^{\frac{2}{3} \text{Log}(z)} = e^{\frac{2}{3}[\ln|z| + i \text{Arg}(z)]} = e^{\frac{2}{3} \ln|z|} \cdot e^{\frac{2i}{3} \text{Arg}(z)} \\ &= \sqrt[3]{|z|^2} \left[\cos \frac{2 \text{Arg}(z)}{3} + i \sin \frac{2 \text{Arg}(z)}{3} \right] \end{aligned}$$

◆ **Exercise 3.4:** What is the meaning of the assertion $z^a z^b = z^{a+b}$? And is it true? Is the assertion true for the principal branch?

🔗 **Hint:** Infinitely many branches are involved on both sides of the equality. The assertion is not true in general! It is true for the principal branch.

◆ **Exercise 3.5:** What is the meaning of the assertion $z^{ab} = (z^a)^b$? And is it true? Is the assertion true for the principal branch?

The Exponential Functions c^z

- As with the power functions, we may define the exponential functions by

$$c^z = e^{z \log c}$$

Once again we obtain a multivalued function, although in this case the multivaluedness is regarded



as less significant than usual,

$$c^z = e^{z \log c} = e^{z(\ln |c| + i(\operatorname{Arg}(c) + 2n\pi))} = e^{z(\operatorname{Log}(c) + 2n\pi i)} \quad (n \in \mathbb{Z})$$

For reasons that will become clear later, it is customary to regard the last formula as a description of an infinite family of single-valued functions rather than a multivalued function with many branches.

- Thus, in most cases encountered, the expression c^z will refer to the single-valued function corresponding to $n = 0$. That is,

$$c^z = e^{z \operatorname{Log}(c)} \quad (3.12)$$

- For the principal branch of c^z , the following identities are easy to prove:

$$\begin{aligned} c^{z_1} c^{z_2} &= c^{z_1 + z_2} \\ \frac{c^{z_1}}{c^{z_2}} &= c^{z_1 - z_2} \\ c^z c^{-z} &= 1 \end{aligned} \quad (3.13)$$

The Trigonometric Functions

- The complex trigonometric functions are defined in terms of the exponential function e^z :

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad (3.14)$$

Here too, it is easy to verify that if the imaginary part vanishes ($z = x + 0i$), then we obtain the familiar real functions $\sin x$ and $\cos x$. That is, the functions $\sin z$ and $\cos z$ are **extensions** of the real functions $\sin x$ and $\cos x$ to the complex numbers.

- Both functions have period 2π (not $2\pi i$!). It is easy to prove that, for every integer n ,

$$\begin{aligned} \sin(z + 2n\pi) &= \sin z \\ \cos(z + 2n\pi) &= \cos z \end{aligned} \quad (3.15)$$

➤ The tangent and cotangent functions are defined, as expected, by

$$\begin{aligned}\tan z &= \frac{\sin z}{\cos z} = \frac{i(e^{-iz} - e^{iz})}{e^{iz} + e^{-iz}} \\ \cot z &= \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}\end{aligned}\tag{3.16}$$

➤ The functions e^z , $\sin z$, and $\cos z$ are defined for every complex number! The functions $\tan z$ and $\cot z$ have more restricted domains. We leave it as an exercise for the student to find these domains.

❖ **Exercise 3.6:** Try to prove the assertion shown in Figure 3.6 - the function $\sin z$ maps vertical lines to hyperbolas and horizontal lines to ellipses.

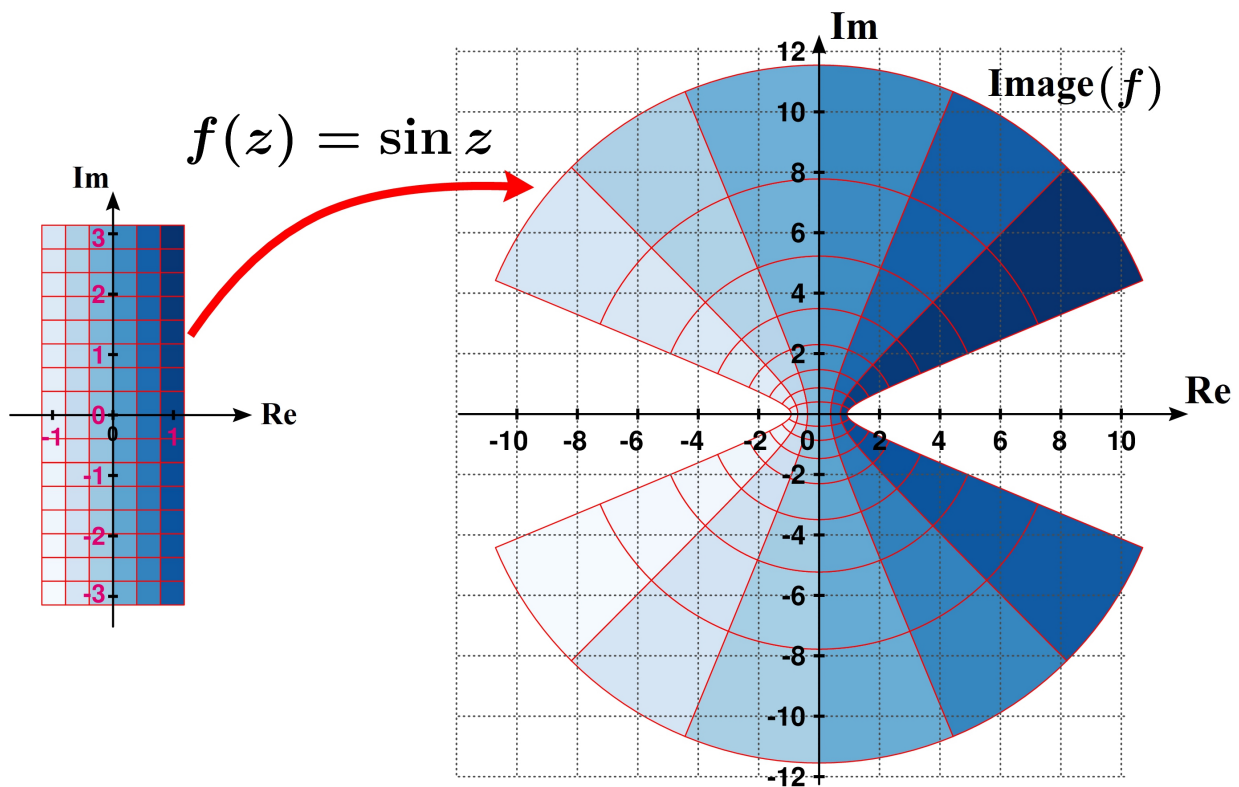


Fig. 3.6: A color map of the function $f(z) = \sin z$: vertical lines map to hyperbolas, and horizontal lines map to ellipses

Basic Identities

➤ From the large variety of trigonometric identities, we present a small number of important ones. Because of time constraints, we leave their proofs to the student as an exercise. Note that identities for real numbers do not carry over automatically to complex numbers, and hence the

identities generalized to complex numbers require proof.

$$\begin{aligned}\sin^2 z + \cos^2 z &= 1 \\ \sin(x + yi) &= \sin x \cdot \cosh y + i \cos x \sinh y \\ \cos(x + iy) &= \cos x \cdot \cosh y - i \sin x \sinh y\end{aligned}\tag{3.17}$$

Note the difference between the last two identities and their counterparts in real trigonometry, which do not involve the hyperbolic functions!

📌 **Reminder:** The real hyperbolic functions were defined as follows (x real):

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

➡ The **definition** of the hyperbolic functions for complex numbers is identical:

$$\begin{aligned}\sinh(z) &= \frac{e^z - e^{-z}}{2} = \frac{e^{2z} - 1}{2e^z} \\ \cosh(z) &= \frac{e^z + e^{-z}}{2} = \frac{e^{2z} + 1}{2e^z} \\ \tanh(z) &= \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1} \\ \coth(z) &= \frac{\cosh(z)}{\sinh(z)} = \frac{e^z + e^{-z}}{e^z - e^{-z}} = \frac{e^{2z} + 1}{e^{2z} - 1}\end{aligned}\tag{3.18}$$

We emphasize that these are definitions, not identities! (Note the background color.)

➡ Additional trigonometric identities:

$$\begin{aligned}\sin\left(\frac{\pi}{2} - z\right) &= \cos z & \sin(z + \pi) &= -\sin z \\ \cos\left(\frac{\pi}{2} - z\right) &= \sin z & \cos(z + \pi) &= -\cos z\end{aligned}\tag{3.19}$$

Inverse Trigonometric and Hyperbolic Functions

➡ Since the trigonometric functions were defined in terms of the exponential function e^z , we can calculate their inverse functions using elementary algebra. For example, to obtain **arcsin** z , solve



for w as a function of z in the following equation:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

It is easy to see that

$$e^{iw} - \frac{1}{e^{iw}} = 2iz$$

Multiplying both sides by the expression e^{iw} gives the quadratic equation

$$(e^{iw})^2 - 2ize^{iw} - 1 = 0$$

and therefore

$$e^{iw} = \frac{2iz \pm \sqrt{4 - 4z^2}}{2} = iz + (1 - z^2)^{\frac{1}{2}}$$

Of course, both branches of $(1 - z^2)^{\frac{1}{2}}$ must be taken into account. Therefore,

$$iw = \log [iz + (1 - z^2)^{\frac{1}{2}}]$$

Thus, we have obtained the following formula:

$$\arcsin z = -i \log [iz + (1 - z^2)^{\frac{1}{2}}] \quad (3.20)$$

Similarly, we obtain the formulas

$$\begin{aligned} \arccos z &= -i \log [iz + (z^2 - 1)^{\frac{1}{2}}] \\ \arctan z &= \frac{i}{2} [\log(1 - iz) - \log(1 + iz)] \\ \operatorname{arccot} z &= \frac{i}{2} [\log(1 - \frac{i}{z}) - \log(1 + \frac{i}{z})] \end{aligned} \quad (3.21)$$

It is advisable to try at least one of the identities as a homework exercise!

- ❗ Note that all the inverse trigonometric functions are multivalued. To obtain a branch of **arcsin**, for example, one must choose a branch of **log** and a branch of $(1 - z^2)^{\frac{1}{2}}$.
- **The inverse hyperbolic functions:** Similarly, the inverse functions of the hyperbolic functions can be found: **arcsinh**, **arccosh**, **arctanh**, **arccoth**. They will not be used in the present course. Formulas and a detailed explanation of the inverse hyperbolic functions may be found at the following link:



The complex inverse trigonometric and hyperbolic functions

3.3 Limits and Continuity of a Complex Function

- We now define the concepts of limit and continuity for a complex function. For this purpose, we need the concept of a **punctured neighborhood** of a point.

i Definition 3.2: (Punctured neighborhood) Let z_0 be a given point, and let $r > 0$ be a positive real number. A **punctured neighborhood** of z_0 is the open disk about z_0 with the point z_0 removed:

$$\begin{aligned}\dot{D}(z_0, r) &= \{z \in \mathbb{C} : 0 < |z - z_0| < r\} \\ &= D(z_0, r) - \{z_0\}\end{aligned}$$

- The concept of a limit for complex functions is a straightforward generalization of the familiar concept of a limit from differential calculus.

i Definition 3.3: Let $f: S \rightarrow \mathbb{C}$ be a function defined in a punctured neighborhood of a point z_0 (that is, there exists $r > 0$ such that $\dot{D}(z_0, r) \subseteq S$). We say that f has the limit w at the point z_0 , and write $\lim_{z \rightarrow z_0} f(z) = w$, if, for every $\varepsilon > 0$, there exists $r > 0$ such that $f(\dot{D}(z_0, r)) \subseteq D(w, \varepsilon)$.

- This may also be written symbolically as

$$\forall \varepsilon > 0, \exists r > 0, \forall z \in \mathbb{C}: 0 < |z - z_0| < r \implies |f(z) - w| < \varepsilon$$

- The definition does not require the function to be defined at the point z_0 in order to have a limit there. Most interesting examples encountered later concern the existence of a limit at a point where the function is not defined.
- The basic idea of the limit of a function at a point z_0 is that, as we approach z_0 (while remaining distinct from it!), the value of f approaches the limit w . The punctured neighborhood $\dot{D}(z_0, r)$ is the set of all points z lying at a positive distance less than r from the point z_0 . The set $f(\dot{D}(z_0, r))$ is its image. Another way to define the limit is by setting

$$\rho(r) = \sup \{ |f(z) - w| : 0 < |z - z_0| < r \}$$

Then the assertion $\lim_{z \rightarrow z_0} f(z) = w$ is equivalent to the assertion $\lim_{r \rightarrow 0} \rho(r) = 0$.



i Proposition 3.3: $\lim_{z \rightarrow z_0} f(z) = w$ if and only if $\lim_{z \rightarrow z_0} |f(z) - w| = 0$.

✓ Proof: Define the function $g(z) = f(z) - w$ and use the definition of a limit. ■

➤ We shall not give an example of proving the existence of a limit directly from the definition. The following theorem reduces the limit of a complex function to two limits of functions of two variables, which are familiar from Calculus 2.

i Theorem 3.4: Let $f(x + iy) = u(x, y) + iv(x, y)$ be a function defined in a punctured neighborhood of $z_0 = x_0 + iy_0$. Then $\lim_{z \rightarrow z_0} f(z) = a + bi$ if and only if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = a \qquad \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = b$$

✓ Proof: The proof is very similar to the proof of Proposition 2.16. We therefore omit the details.

Direction 1: Suppose that the limit exists: $\lim_{z \rightarrow z_0} f(z) = a + bi$. We prove the other two limits.

Let $\varepsilon > 0$. There exists $r > 0$ such that, for every $z \in \dot{D}(z_0, r)$, $|f(z) - f(z_0)| < \varepsilon$. Choose $\delta = r/2$, and suppose that $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$. Then $|x - x_0| < \delta$ and $|y - y_0| < \delta$, and therefore

$$\begin{aligned} |z - z_0| &= \sqrt{(x - x_0)^2 + (y - y_0)^2} < \sqrt{r^2/4 + r^2/4} = \frac{\sqrt{2}}{2}r < r \\ \implies |f(z) - (a + bi)| &< \varepsilon \\ \implies \sqrt{[u(x, y) - a]^2 + [v(x, y) - b]^2} &< \varepsilon \\ \implies |u(x, y) - a| < \varepsilon \text{ and } |v(x, y) - b| &< \varepsilon \end{aligned}$$

We have obtained

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |u(x, y) - a| < \varepsilon \text{ and } |v(x, y) - b| < \varepsilon$$

and hence, by the definition of a limit from Calculus 2, $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = a$,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = b.$$

Direction 2: Suppose that the two real limits exist. Let $\varepsilon > 0$. By the definition of a limit, we may suppose that there exists $\delta > 0$ such that, for every $x, y \in \mathbb{R}$,

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \implies |u(x, y) - a| < \varepsilon/2, \quad |v(x, y) - b| < \varepsilon/2$$

The triangle inequality gives

$$\begin{aligned} |f(z) - (a + bi)| &= \sqrt{(u(x, y) - a)^2 + (v(x, y) - b)^2} \\ &\leq |u(x, y) - a| + |v(x, y) - b| < \varepsilon \end{aligned}$$

Therefore, $\lim_{z \rightarrow z_0} f(z) = a + bi$. ■

➤ The last theorem can be used to prove the following theorem (or one may simply use the definition of a limit). We leave the details to the reader.

i Theorem 3.5: Let $f(z)$ and $g(z)$ be functions defined in a punctured neighborhood of z_0 , and suppose that the limits exist: $\lim_{z \rightarrow z_0} f(z) = u$, $\lim_{z \rightarrow z_0} g(z) = v$. Then the following limits also exist:

$$\begin{aligned}\lim_{z \rightarrow z_0} [f(z) + g(z)] &= \lim_{z \rightarrow z_0} f(z) + \lim_{z \rightarrow z_0} g(z) = u + v \\ \lim_{z \rightarrow z_0} [f(z) - g(z)] &= \lim_{z \rightarrow z_0} f(z) - \lim_{z \rightarrow z_0} g(z) = u - v \\ \lim_{z \rightarrow z_0} f(z) \cdot g(z) &= \lim_{z \rightarrow z_0} f(z) \cdot \lim_{z \rightarrow z_0} g(z) = uv \\ \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} &= \frac{\lim_{z \rightarrow z_0} f(z)}{\lim_{z \rightarrow z_0} g(z)} = \frac{u}{v} \quad (v \neq 0)\end{aligned}$$

i Theorem 3.6: (Limit of a composite function) Let $f: S \rightarrow \mathbb{C}$ be a function defined in a punctured neighborhood of $z_0 = x_0 + iy_0$ with limit $\lim_{z \rightarrow z_0} f(z) = a + bi$, and let $g: T \rightarrow \mathbb{C}$ be a function defined in a punctured neighborhood of $w_0 = a + bi$ with limit $\lim_{z \rightarrow w_0} g(z) = c + di$. Then the composite function $g(f(z))$ has a limit at z_0 , and $\lim_{z \rightarrow z_0} g(f(z)) = c + di$.

✓ **Proof:** Write f and g in Cartesian form:

$$\begin{aligned}f(x + iy) &= u_1(x, y) + v_1(x, y)i \\ g(x + iy) &= u_2(x, y) + v_2(x, y)i\end{aligned}$$

Then, by Theorem 3.4,

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u_1(x, y) = a, \quad \lim_{(x,y) \rightarrow (x_0,y_0)} v_1(x, y) = b$$

Express the composite function in Cartesian form:

$$\begin{aligned}g(f(x + iy)) &= g(u_1(x, y) + v_1(x, y)i) \\ &= u_2(u_1(x, y), v_1(x, y)) + v_2(u_1(x, y), v_1(x, y))i\end{aligned}$$

For convenience, define the functions

$$\begin{aligned}u(x, y) &= u_2(u_1(x, y), v_1(x, y)) \\ v(x, y) &= v_2(u_1(x, y), v_1(x, y))\end{aligned}$$

and obtain a clean Cartesian form for the composite function:

$$g(f(x + iy)) = u(x, y) + v(x, y)i$$

By the appropriate limit theorems from Calculus 2, we know that

$$\begin{aligned} \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) &= \lim_{(x,y) \rightarrow (x_0,y_0)} u_2(u_1(x, y), v_1(x, y)) = c \\ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) &= \lim_{(x,y) \rightarrow (x_0,y_0)} v_2(u_1(x, y), v_1(x, y)) = d \end{aligned}$$

Therefore, by Theorem 3.4, $\lim_{z \rightarrow z_0} g(f(z)) = c + di$. ■

➤ A concise formula expressing the theorem (under the necessary conditions, of course) is

$$\lim_{z \rightarrow z_0} g(f(z)) = g(\lim_{z \rightarrow z_0} f(z)) \quad (3.22)$$

➤ The next theorem will be useful for proving the nonexistence of a limit (or discontinuity).

i Theorem 3.7: Let $f: S \rightarrow \mathbb{C}$ be a function defined in a punctured neighborhood of z_0 . If the limit $\lim_{z \rightarrow z_0} f(z)$ exists, then, for every path $\sigma: I \rightarrow \mathbb{C}$ passing through $z_0 = \sigma(t_0)$, we have $\lim_{t \rightarrow t_0} f(\sigma(t)) = w$.

✓ **Proof:** Let $\varepsilon > 0$. By hypothesis, there exists $r > 0$ such that, for every z in $\dot{D}(z_0, r)$, $|f(z) - f(w)| < \varepsilon$. By the continuity of the path σ , there exists $\delta > 0$ such that, for every t in the neighborhood $(t_0 - \delta, t_0 + \delta)$, $|\sigma(t) - \sigma(t_0)| < r$. Thus, for every t in the neighborhood $(t_0 - \delta, t_0 + \delta)$, $|f(\sigma(t)) - f(\sigma(t_0))| < \varepsilon$. ■

◆ **Example 6:** Consider the function $f(z) = \frac{z}{|z|}$. It is defined in a punctured neighborhood of $z = 0$ (but not at the point $z = 0$ itself!). We prove that the limit $\lim_{z \rightarrow 0} f(z)$ does not exist. For this purpose, it suffices to consider the path $\sigma(t) = t$ (the x -axis):

$$f(\sigma(t)) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

The limit $\lim_{t \rightarrow 0} f(\sigma(t))$ does not exist, and therefore the limit $\lim_{z \rightarrow 0} f(z)$ does not exist.

➤ Because of the complete analogy with the real case, we leave the following theorem for the student to prove as an exercise.

i Theorem 3.8: (Heine) Let $f: S \rightarrow \mathbb{C}$ be a function defined in a punctured neighborhood of z_0 . The limit $\lim_{z \rightarrow z_0} f(z) = w$ exists if and only if, for every sequence $\{z_n\}_{n=1}^{\infty}$ converging to



the point z_0 , $\lim_{n \rightarrow \infty} f(z_n) = w$.

Continuity

i Definition 3.4: Let $f: S \rightarrow \mathbb{C}$ be a function defined in a neighborhood of the point $z_0 \in S$. We say that f is **continuous at the point** z_0 if the limit exists: $\lim_{z \rightarrow z_0} f(z) = f(z_0)$. We say that f is **continuous on a set** D if f is continuous at every point $z \in D$.

Remarks:

- Note that in the first part of the definition, the neighborhood in question is an ordinary neighborhood (not punctured!). That is, f must also be defined at the point z_0 .
- Continuity at a point has two components: the existence of a limit at z_0 , and, in addition, that this limit equals $f(z_0)$ (the value of f at the point z_0).
- The set S need not be open or closed, but for a function f to be continuous at a point $z_0 \in S$, the point z_0 must be an interior point of S .
- The assertion that a function f is continuous on a set of points $D \subseteq S$ is meaningful only if every point $z \in D$ is an interior point of S . The set D need not be open or closed, but every point $z \in D$ must have a neighborhood $\mathbb{D}(z, r) \subseteq S$ on which f is defined. This assertion is equivalent to the existence of an open set $E \subseteq S$ such that $D \subseteq E$.

➤ The following theorem is an immediate consequence of Theorem 3.4.

i Theorem 3.9: Let $f(x + iy) = u(x, y) + iv(x, y)$ be a function defined on a set S . Then f is continuous on S if and only if the real functions $u(x, y)$ and $v(x, y)$ are continuous on S .

The following theorem is a consequence of Theorems 3.5 and 3.6.

i Theorem 3.10: Let $f(z)$ and $g(z)$ be two functions continuous on a set S . Then

- The sum and difference $f(z) \pm g(z)$ are continuous functions on S .
- The product $f(z)g(z)$ is a continuous function on S .
- The quotient $f(z)/g(z)$ is a continuous function on S (at every point where $g(z) \neq 0$).
- The composition $g(f(z))$ of two continuous functions is continuous.

◆ **Example 7:** The function $f(z) = \bar{z}$ is continuous throughout the complex plane $\mathbb{C}$, because its Cartesian form is $f(x + iy) = x - yi$. The functions $u(x, y) = x$ and $v(x, y) = -y$ are known to be continuous real functions on $\mathbb{R}^2$.

◆ **Example 8:** The function $f(z) = |z|$ is continuous throughout the complex plane $\mathbb{C}$, because its Cartesian form is $f(x + iy) = \sqrt{x^2 + y^2} + 0i$. From Calculus 2, we know that $u(x, y) = \sqrt{x^2 + y^2}$ is continuous throughout the real plane $\mathbb{R}^2$.



Corollary 3.11: If the limit $\lim_{z \rightarrow z_0} f(z) = w$ exists, then $\lim_{z \rightarrow z_0} |f(z)| = |w|$.

Proof: Without loss of generality, suppose that $f(z_0) = w$. Then f is continuous at the point z_0 . We saw earlier that $g(z) = |z|$ is continuous, and clearly $|f(z)| = g(f(z))$. It follows from part d of Theorem 3.10 that $|f(z)|$ is also continuous. ■

Example 9: The functions $\text{Re}(z)$ and $\text{Im}(z)$ are continuous throughout the complex plane. This is seen immediately by writing them explicitly: $\text{Re}(x + iy) = x$, $\text{Im}(x + iy) = y$.

Example 10: The function $f(z) = e^z$ is continuous throughout the real plane, because by definition the Cartesian form of the function e^z is

$$e^z = e^{x+yi} = e^x \cos y + ie^x \sin y$$

The functions $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$ are continuous throughout the real plane $\mathbb{R}^2$, and hence, by Theorem 3.9, e^z is continuous on $\mathbb{C}$.

Example 11: The principal branch of the logarithm function $\text{Log}(z)$ is continuous on the slit plane, but is not continuous along the slit $(-\infty, 0)$ (although it is defined for every negative real number $-\infty < x < 0$!).

$$\text{Log}(z) = \ln |z| + i \text{Arg}(z) = \ln \sqrt{x^2 + y^2} + i \text{atan2}(y, x)$$

The function $u(x, y) = \ln \sqrt{x^2 + y^2}$ is continuous on the punctured real plane $\mathbb{R}^2 - \{(0, 0)\}$, and in the first chapter we saw that the function atan2 is continuous on the slit plane. $\mathbb{C} - (-\infty, 0]$. Therefore, by Theorem 3.9, the principal branch of the logarithm function $\text{Log}(z)$ is continuous on the slit plane (Figure 3.7).

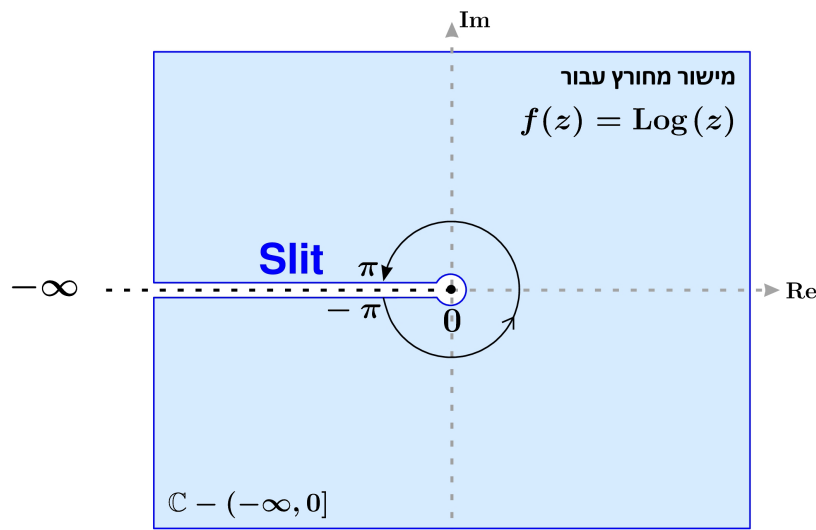


Fig. 3.7: The slit plane: $\mathbb{C} - (-\infty, 0]$. A complex plane with a slit along the negative x -axis (Complex Slit Plane)

- All elementary functions whose definitions are based on a branch of the logarithm function $\log z$ or the argument function exhibit the same “slit phenomenon”, except that the slit may move to another ray, sometimes become a finite segment, and sometimes even a curved path.
- ❖ **Example 12:** The elementary function $\mathbf{Log}(3 - z)$ (obtained by composing the functions $f(z) = 3 - z$ and $g(z) = \mathbf{Log}(z)$) is defined on the punctured plane $\mathbb{C} - \{3\}$ but is not continuous along the slit $(3, +\infty)$.

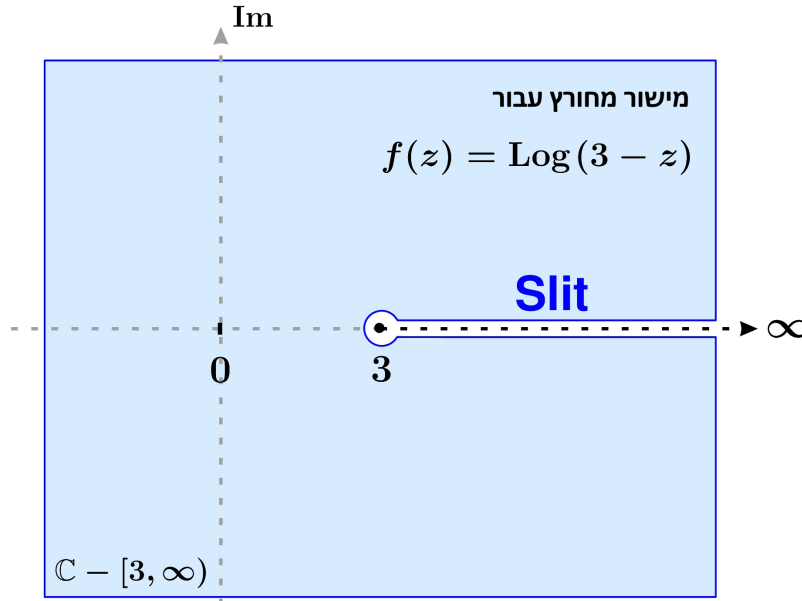


Fig. 3.8: The slit plane $\mathbb{C} - [3, -\infty)$ for the function $\mathbf{Log}(3 - z)$

- ❖ **Example 13:** In formula (3.20), we found that

$$\arcsin z = -i \log \left[iz + (1 - z^2)^{\frac{1}{2}} \right]$$

This is, of course, a multivalued function with many branches. The multiplicity of branches comes from both the logarithm function $\log$ and the power function $(1 - z^2)^{\frac{1}{2}}$. The principal branch of $\arcsin$ is defined by

$$\mathbf{Arcsin} z = -i \mathbf{Log} \left(iz + \sqrt{|1 - z^2|} e^{\frac{i}{2} \mathbf{Arg}(1 - z^2)} \right) \quad (3.23)$$

where $\sqrt{|1 - z^2|} e^{\frac{i}{2} \mathbf{Arg}(1 - z^2)}$ is the principal branch of $(1 - z^2)^{\frac{1}{2}}$, and, of course, $\mathbf{Log}$ is the principal branch of $\log$. The function $\mathbf{Arg}(1 - z^2)$ is not defined at the points $z = -1$ and $z = 1$. All the other expressions are defined for every $z \in \mathbb{C}$. Thus, the domain of $\mathbf{Arcsin}$ is $\mathbb{C} - \{-1, 1\}$.

The function $1 - z^2$ maps the rays $[1, \infty)$ and $(-\infty, -1]$ to the ray $(-\infty, 0]$, which is the slit of $\mathbf{Arg}$. Therefore, the expression $e^{\frac{i}{2} \mathbf{Arg}(1 - z^2)}$ is not continuous on the domain $S = (-\infty, -1) \cup (1, \infty)$. The continuous expression $\sqrt{|1 - z^2|}$ does not vanish on the domain S , and hence the product $\sqrt{|1 - z^2|} e^{\frac{i}{2} \mathbf{Arg}(1 - z^2)}$ is also not continuous on S . Thus, the domain

of continuity of **Arcsin** is

$$D = \mathbb{C} - ((-\infty, -1] \cup [1, \infty))$$

The domain D is shown in Figure 3.9, together with the two slits $(-\infty, -1]$ and $[1, \infty)$.

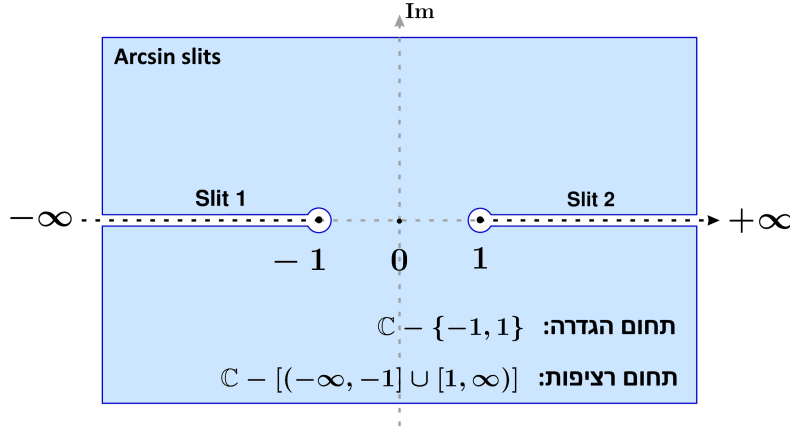


Fig. 3.9: The function **Arcsin** has two slits: $(-\infty, -1] \cup [1, \infty)$

We leave it for the student to prove that the expression $iz + \sqrt{|1 - z^2|} e^{\frac{i}{2}\text{Arg}(1-z^2)}$ appearing in formula (3.23) never lies on the ray $(-\infty, 0]$, and consequently there are no additional discontinuity points beyond those already found.

- In many cases, finding the slit of an elementary function may be a difficult task, but at this stage we wish to characterize points lying on a slit and distinguish them from ordinary points. The following definition gives a simple criterion for identifying a slit point.

i Definition 3.5: A point z_0 is called a **slit point** of a function $f(z)$ if there exists $\varepsilon > 0$ such that, for every $0 < r < \varepsilon$, the function $f(z)$ has a jump along the circular path $\sigma_{z_0, r}(t) = z_0 + re^{it}$.

- In other words, a point z_0 is a **slit point** if, for every $0 < r < \varepsilon$, the function

$$g(t) = f(z_0 + re^{it})$$

has a jump discontinuity on the interval $-\pi < t \leq \pi$. Figure 3.10 shows a small circular path about a slit point of the function **Arg**(z). The jump of **Arg** between the points z_1 and z_2 is $2\pi - 2\varepsilon$, and it approaches 2π as the radius of the circle decreases. Such a phenomenon does not occur at points that do not belong to the slit!

- In the same way, one can prove that all the elementary functions listed in the preceding section are continuous on their domains except along a one-dimensional slit. We leave it for the student to find the slit in each case in the proof of the following theorem.

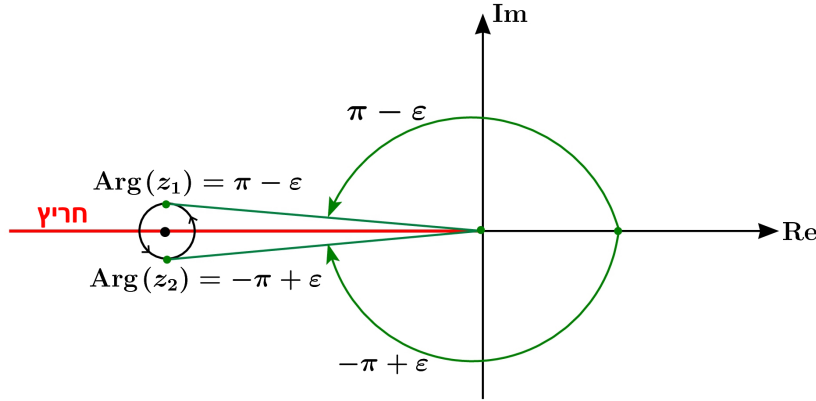


Fig. 3.10: The function Arg makes a jump of size close to 2π on small circles about a slit point

i Theorem 3.12: Every elementary function is continuous (except at slit points).

✓ Proof: The elementary functions were listed in detail in the preceding section. We go through the list once more and examine continuity.

- a. The constant functions $f(z) = c$, for a constant $c \in \mathbb{C}$:
are continuous throughout the complex plane.
- b. The identity function $f(z) = z$:
is continuous throughout the complex plane.
- c. The exponential functions $f(z) = e^z$ (principal branch):
The principal branch was defined by $e^{z \log c}$. In Example 10, we proved that e^z is continuous throughout the complex plane. The same proof applies to e^{cz} .
- d. The power functions $f(z) = z^c$:
The function was defined by $z^c = e^{c \log z}$. This is a composition of the two continuous functions e^z and $\log z$ (on the slit plane). Therefore, by Theorem 3.10, it is continuous throughout the slit plane.
- e. The trigonometric functions:
The functions $\sin z$ and $\cos z$ were defined in terms of the exponential functions e^{iz} and e^{-iz} , and hence, by Theorem 3.10, they are continuous throughout the complex plane. The functions $\tan z$ and $\cot z$ were defined by $\frac{\sin z}{\cos z}$ and $\frac{\cos z}{\sin z}$, so, by part c of Theorem 3.10, the function $\tan z$ is continuous throughout its domain: the points z at which $\cos z \neq 0$. The function $\cot z$ is likewise continuous throughout its domain: all points z at which $\sin z \neq 0$.
- f. The logarithmic functions $f(z) = \log_c z$:
In Example 11, we proved that the principal branch of the logarithm function Log is continuous on the slit plane. A similar explanation applies to $\log z$: The natural logarithm function $\log z$ was defined by $\ln |z| + \arg(z)$. The real function $\ln |z|$ is continuous throughout its domain $\mathbb{C} - \{0\}$. Every continuous branch of $\arg(z)$ is discontinuous across its corresponding slit, and hence every branch of $\log z$ is continuous on a slit plane. The function $\log_c z$ is simply $\frac{1}{\log c} \log z$.

g. The inverse trigonometric functions:

In Example 13, we treated the function Arcsin . We leave it as an exercise for the student to find the domain and the region of continuity of Arccos , Arctan , and Arccot . ■

❖ **Example 14:** The principal branch of the argument function $\text{Arg}(z)$ is not continuous at the point $z = -1$. Use the path $\sigma(t) = \cos t + i \sin t$ for $t \in (-\pi, \pi]$. Here the left-hand limit at $t = \pi$ must be used: $\lim_{t \rightarrow \pi^-} \text{Arg}(\sigma(t)) = \pi$, whereas $\lim_{t \rightarrow -\pi^+} \text{Arg}(\sigma(t)) = -\pi$.

➤ The elementary functions are formed from the basic functions by addition, subtraction, multiplication, division, and composition of functions. We may therefore summarize the discussion by saying that every elementary function is continuous on the common domain of analyticity of the basic functions from which it was constructed.

A Continuous Function on a Compact Set

➤ Recall from Definition 2.21 that a compact set is closed and bounded. Such sets have a special property expressed by the following theorem.

❗ **Theorem 3.13:** A continuous function maps a compact set to a compact set.

✔ **Proof:** Let $f: S \rightarrow \mathbb{C}$ be a continuous function, and let $D \subseteq S$ be a compact set. We use Theorem 2.19 to prove that $f(D)$ is closed. Let $\{w_n\}_{n=1}^\infty$ be a sequence of points in $f(D)$ converging to a limit w . We prove that $w \in f(D)$.

There exists a sequence of points $\{z_n\}_{n=1}^\infty$ in D such that $w_n = f(z_n)$. By the Bolzano-Weierstrass theorem 2.18, there exists a convergent subsequence $\{z_{n_k}\}_{k=1}^\infty$ of z_n . Since D is closed, $z = \lim_{k \rightarrow \infty} z_{n_k} \in D$. The continuity of f implies that $\lim_{k \rightarrow \infty} f(z_{n_k}) = f(z) = w$. Therefore, $w \in f(D)$.

It remains to prove that $f(D)$ is bounded. Suppose, for contradiction, that $f(D)$ is unbounded. Then there exists a sequence of points $\{w_n\}_{n=1}^\infty$ in $f(D)$ such that $\lim_{n \rightarrow \infty} w_n = \infty$. There exists a sequence of points $\{z_n\}_{n=1}^\infty$ in D such that, for every n , $w_n = f(z_n)$. By the Bolzano-Weierstrass theorem 2.18, there exists a convergent subsequence $\{z_{n_k}\}_{k=1}^\infty$ of z_n . Suppose that $z = \lim_{k \rightarrow \infty} z_{n_k}$. As before, we obtain

$$\lim_{k \rightarrow \infty} w_{n_k} = \lim_{k \rightarrow \infty} f(z_{n_k}) = f(z)$$

Thus, the sequence $\{w_n\}_{n=1}^\infty$ has a subsequence converging to a finite limit, contrary to the assumption $w_n \rightarrow \infty$. ■

❗ **Corollary 3.14: (Weierstrass)** If f is continuous on a compact set D , then $|f(z)|$ attains a minimum and a maximum in D .

❖ **Exercise 3.7:** Let $S \subseteq \mathbb{C}$ be a given domain. Prove that a function $f: S \rightarrow \mathbb{C}$ is continuous if

and only if, for every open set B , $f^{-1}(B)$ is open.

❖ **Exercise 3.8:** Let $S \subseteq \mathbb{C}$ be a given domain. Prove that a function $f: S \rightarrow \mathbb{C}$ is continuous if and only if, for every closed set B , $f^{-1}(B)$ is closed.

❗ **Proposition 3.15:** The continuous image of a connected set is connected.

✔ **Proof:** Let f be continuous and let S be connected. If $f(S)$ is disconnected, then there exists an open separation (A, B) of $f(S)$. By Exercise 3.7, $(f^{-1}(A), f^{-1}(B))$ is an open separation of S , contradicting the connectedness of S . ■

❗ **Proposition 3.16:** The continuous image of a path-connected set is path connected.

✔ **Proof:** Let f be continuous and let S be path connected. Let $w_1, w_2 \in f(S)$ be two distinct points in the image of S . There exist $z_1, z_2 \in S$ such that $w_1 = f(z_1)$ and $w_2 = f(z_2)$. Let σ be a path joining z_1 to z_2 . Then $f(\sigma(t))$ is a path joining w_1 to w_2 . ■

Limit at Infinity

- In addition to the ordinary limit, there are situations in which we need to understand the behavior of a function $f(z)$ when the argument z tends to infinity (that is, when $|z| \rightarrow \infty$).
- The **limit at infinity** of a function $f(z)$ is defined as the value $w \in \mathbb{C}$ approached by $f(z)$ as $|z|$ tends to infinity. Here is the formal definition.

❗ **Definition 3.6:** Let $f(z)$ be a function defined on a punctured plane $|z| > R$. We say that $f(z)$ has the **limit at infinity** $w \in \mathbb{C}$, and write $\lim_{z \rightarrow \infty} f(z) = w$ if, for every $\varepsilon > 0$, there exists a real number M such that, for every $z \in \mathbb{C}$, if $|z| > M$, then $|f(z) - w| < \varepsilon$.

Remark: A necessary condition for the existence of a limit at infinity is that the function be defined on some punctured plane. In some books, and occasionally here as well, we write $\lim_{|z| \rightarrow \infty} f(z) = w$. Of course, this means the same thing. See the additional remark following Definition 3.7.

- The following assertion follows immediately from the definition of the limit of a complex function.

❗ **Proposition 3.17:** $\lim_{|z| \rightarrow \infty} f(z) = 0$ if and only if $\lim_{|z| \rightarrow \infty} |f(z)| = 0$

❖ **Example 15:** The limit at infinity of the function $f(z) = \frac{3z-5}{4-iz}$ is $-3i$. This can be seen from the following simple calculation:

$$\lim_{|z| \rightarrow \infty} \frac{3z-5}{4-iz} = \lim_{|z| \rightarrow \infty} \frac{3-5/z}{4/z-i} = \frac{3-0}{0-i} = 3i$$

Clearly, $\lim_{|z| \rightarrow \infty} 5/z = \lim_{|z| \rightarrow \infty} |5/z| = \lim_{|z| \rightarrow \infty} 5/|z| = 0$. The same applies to $\lim_{|z| \rightarrow \infty} 4/z = 0$.



i Definition 3.7: Let $f(z)$ be a function defined on a punctured plane $|z| > R$. We say that the function $f(z)$ **tends to infinity as z tends to infinity**, and write $\lim_{|z| \rightarrow \infty} f(z) = \infty$ if, for every real number N , there exists a real number $M > 0$ such that, for every $z \in \mathbb{C}$, if $|z| > M$, then $|f(z)| > N$.

➤ In some books, this is written as follows: $\lim_{|z| \rightarrow \infty} |f(z)| = \infty$, to prevent confusion, because the function may tend to $-\infty$ along some paths. But, as we shall see in the next chapter when we encounter the **extended complex plane**, the point $+\infty$ and the point $-\infty$ are exactly the same point!

i Proposition 3.18: If $P(z)$ is a nonconstant polynomial, then $\lim_{|z| \rightarrow \infty} P(z) = \infty$.

✓ **Proof:** Suppose that $P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$, $a_n \neq 0$. Then

$$\begin{aligned} |P(z)| &= |a_0 + a_1z + a_2z^2 + \cdots + a_nz^n| = \left| z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + a_n \right) \right| \\ &= |z|^n \cdot \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + a_n \right| \end{aligned}$$

Each fraction $\frac{a_k}{z^{n-k}}$ tends to zero as $|z| \rightarrow \infty$. We are left with $|a_n| > 0$ and with $|z|^n$, which of course tends to infinity as $|z| \rightarrow \infty$. Therefore, $\lim_{|z| \rightarrow \infty} |P(z)| = \infty$. ■

i Proposition 3.19: Let $P(z)$ be a polynomial of degree m and $Q(z)$ a polynomial of degree n :

$$\begin{aligned} P(z) &= a_0 + a_1z + a_2z^2 + \cdots + a_mz^m \\ Q(z) &= b_0 + b_1z + b_2z^2 + \cdots + b_nz^n \end{aligned}$$

Then

(a) If $m < n$, then $\lim_{|z| \rightarrow \infty} \frac{P(z)}{Q(z)} = 0$.

(b) If $m = n$, then $\lim_{|z| \rightarrow \infty} \frac{P(z)}{Q(z)} = \frac{a_n}{b_n}$.

(c) If $m > n$, then $\lim_{|z| \rightarrow \infty} \frac{P(z)}{Q(z)} = \infty$.

✓ **Proof:**

$$\begin{aligned} \left| \frac{P(z)}{Q(z)} \right| &= \left| \frac{a_0 + a_1z + a_2z^2 + \cdots + a_mz^m}{b_0 + b_1z + b_2z^2 + \cdots + b_nz^n} \right| = \frac{|a_0 + a_1z + a_2z^2 + \cdots + a_mz^m|}{|b_0 + b_1z + b_2z^2 + \cdots + b_nz^n|} \\ &= \frac{|z|^m \cdot \left| \frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \frac{a_2}{z^{m-2}} + \cdots + a_m \right|}{|z|^n \cdot \left| \frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \frac{b_2}{z^{n-2}} + \cdots + b_n \right|} \end{aligned}$$



(a) If $m < n$, then $n - m > 0$, and therefore

$$\left| \frac{P(z)}{Q(z)} \right| = \frac{\left| \frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \frac{a_2}{z^{m-2}} + \cdots + a_m \right|}{|z|^{n-m} \cdot \left| \frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \frac{b_2}{z^{n-2}} + \cdots + b_n \right|} \xrightarrow{|z| \rightarrow \infty} \frac{|a_m|}{\infty \cdot |b_n|} = 0$$

Each of the fractions $\frac{a_k}{z^{m-k}}$, $\frac{b_k}{z^{n-k}}$, tends to zero as $|z| \rightarrow \infty$. Thus, the numerator tends to a_m . But in the denominator, the expression $|z|^{n-m} \rightarrow \infty$ causes the quotient to tend to zero.

(b) If $m = n$, then

$$\left| \frac{P(z)}{Q(z)} \right| = \frac{|z|^n \cdot \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + a_n \right|}{|z|^n \cdot \left| \frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \frac{b_2}{z^{n-2}} + \cdots + b_n \right|} \xrightarrow{|z| \rightarrow \infty} \frac{|a_m|}{|b_n|}$$

(c) If $m > n$, then $m - n > 0$, and therefore

$$\left| \frac{P(z)}{Q(z)} \right| = \frac{|z|^{m-n} \cdot \left| \frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \frac{a_2}{z^{m-2}} + \cdots + a_m \right|}{\left| \frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \frac{b_2}{z^{n-2}} + \cdots + b_n \right|} \xrightarrow{|z| \rightarrow \infty} \frac{\infty \cdot |a_m|}{|b_n|} = \infty$$

Infinite Limit at a Finite Point

➤ In addition, we must define an infinite limit at an ordinary point $z_0 \in \mathbb{C}$. For example, $\lim_{z \rightarrow 3} \frac{1}{z-3} = \infty$.

i Definition 3.8: Let $f(z)$ be a function defined in a punctured neighborhood of a point $z_0 \in \mathbb{C}$. We say that $f(z)$ **tends to infinity** at the point z_0 , and write $\lim_{z \rightarrow z_0} f(z) = \infty$, if, for every real number M , there exists a real number $\delta > 0$ such that, for every $z \in \dot{D}(z_0, \delta)$, $|f(z)| > M$.

➤ In logical notation, this is

$$\lim_{z \rightarrow z_0} f(z) = \infty \iff \forall M, \exists \delta > 0, \forall z: 0 < |z - z_0| < \delta \implies |f(z)| > M$$

Note again that this is actually a limit of the real absolute value $|f(z)|$, not of the complex value $f(z)$, which generally diverges in every direction toward infinity (including $-\infty$) near such a point. In the final chapter of this text, we define the concept of a **singular point**, which treats and classifies points of this kind more systematically.

◆ **Example 16:** $\lim_{z \rightarrow 0} \frac{1}{z} = \infty$.

◆ **Exercise 3.9:** Prove that $\lim_{z \rightarrow z_0} f(z) = \infty$ if and only if, for every sequence $z_n \rightarrow z$, $|f(z_n)| \rightarrow \infty$.

❖ **Example 17:** The limit $\lim_{z \rightarrow 0} e^{\frac{1}{z}}$ does not exist because for the sequence $z_n = \frac{1}{n}$ the limit is ∞ , whereas for the sequence $z_n = -\frac{1}{n}$ the limit is 0 .

3.4 Differentiability and Analyticity

❖ The derivative of a complex function is defined exactly like the derivative of a real function. For complex numbers, however, there is a stronger and more important property than differentiability, called **analyticity** (or holomorphicity). In the present section, we study the elementary properties of these two concepts.

❗ **Definition 3.9:** Let $f(z)$ be a complex function defined in a neighborhood of the point z_0 . We say that f is **differentiable at the point z_0** if the limit exists:

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

❖ Sometimes this is also written as

$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

where $h \in \mathbb{C}$ is sufficiently small to lie in the neighborhood of z on which the function f is defined.

❖ **Example 18:** We compute the derivative of $f(z) = z^3$:

$$\begin{aligned} f'(z) &= \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{(z+h)^3 - z^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{(z^3 + 3z^2h + 3zh^2 + h^3) - z^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3z^2h + 3zh^2 + h^3}{h} = \lim_{h \rightarrow 0} (3z^2 + 3zh + h^2) = 3z^2 \end{aligned}$$

As expected, we have obtained the same formula familiar from the real case, $(z^3)' = 3z^2$. This will be the case for most familiar functions.

❖ **Example 19:** The function $f(z) = \bar{z}$ is not differentiable at any point of the complex plane.

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \rightarrow 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}}{h}$$

The last limit does not exist, because if $h = x$ is real, the limit is 1 , whereas if $h = iy$ is



imaginary, the limit is -1 . Therefore, $f(z) = \bar{z}$ is nowhere differentiable!

❖ **Exercise 3.10:** Prove that the function $f(z) = |z|^2$ is differentiable only at the point $z = 0$!

❗ **Definition 3.10:** We say that $f(z)$ is differentiable on a set $S \subseteq \mathbb{C}$ if f is differentiable at every point $z \in S$.

❖ **Exercise 3.11:** Prove that $f(z) = z^n$ is differentiable on all of $\mathbb{C}$ and that $f'(z) = nz^{n-1}$ (n is a natural number).

❗ **Proposition 3.20: (Linear approximation formula)** Let $f: S \rightarrow \mathbb{C}$ be a function differentiable at the point z_0 . Then there exists a function $\rho(z)$ on the domain $S - \{z_0\}$ such that $\lim_{z \rightarrow z_0} \rho(z) = 0$, and for every $z \in S - \{z_0\}$,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0) \quad (3.24)$$

✔ **Proof:** Begin by defining the function $\rho(z)$:

$$\rho(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \quad (3.25)$$

It is clear from the definition that

$$\begin{aligned} \lim_{z \rightarrow z_0} \rho(z) &= \lim_{z \rightarrow z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right] = \lim_{z \rightarrow z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \right] - f'(z_0) \\ &= f'(z_0) - f'(z_0) = 0 \end{aligned}$$

Multiplying both sides of equation (3.25) by $(z - z_0)$ gives

$$\rho(z)(z - z_0) = f(z) - f(z_0) - f'(z_0)(z - z_0)$$

and transposing terms gives

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0) \quad \blacksquare$$

➤ Formula (3.24) is called the **linear approximation formula** because the expression $f(z_0) + f'(z_0)(z - z_0)$ is a linear function that becomes very close to $f(z)$ in a small neighborhood of z_0 . In addition to its computational use, it has important theoretical applications that we will encounter later.

❗ **Proposition 3.21:** If f is differentiable at the point z_0 , then f is continuous at z_0 .

✓ **Proof:** By Proposition 3.3, it suffices to prove that $\lim_{z \rightarrow z_0} |f(z) - f(z_0)| = 0$. By the preceding proposition, we may write

$$\begin{aligned} \lim_{z \rightarrow z_0} |f(z) - f(z_0)| &= \lim_{z \rightarrow z_0} |f'(z_0)(z - z_0) + \rho(z)(z - z_0)| \\ &= \lim_{z \rightarrow z_0} |f'(z_0) + \rho(z)| \cdot |z - z_0| = 0 \end{aligned}$$

By Proposition 3.20, $\lim_{z \rightarrow z_0} \rho(z) = 0$, $f'(z_0)$ is a constant, and hence $\lim_{z \rightarrow z_0} |f'(z_0) + \rho(z)| = |f'(z_0)|$. $\lim_{z \rightarrow z_0} |z - z_0| = 0$. Therefore, the limit of the product is zero. ■

i Definition 3.11: We say that $f(z)$ is **analytic (or holomorphic) at the point z_0** if there exists $r > 0$ such that f is differentiable on the neighborhood $D(z_0, r)$.

✧ Thus, for a function f to be analytic at a point, it must be differentiable at every point in a neighborhood of that point.

i Definition 3.12: We say that $f(z)$ is **analytic on a set $S \subseteq \mathbb{C}$** if f is analytic at every $z \in S$.

i Proposition 3.22: If the function $f(z)$ is analytic at the point z_0 , then $f(z)$ is analytic in a neighborhood of z_0 .

✓ **Proof:** By definition, f is differentiable in some neighborhood $D(z_0, r)$ of the point z_0 . If $z \in D(z_0, r)$, then clearly there exists $\varepsilon > 0$ such that $D(z, \varepsilon) \subseteq D(z_0, r)$. Thus, f is differentiable in a neighborhood of z , and hence f is analytic throughout the neighborhood of z_0 . ■

i Proposition 3.23: If $f(z)$ is differentiable on an open set S , then $f(z)$ is analytic on S .

i Definition 3.13: A function $f(z)$ is called an **entire function** if it is analytic throughout the complex plane $\mathbb{C}$.

✧ **Example 20:** Every polynomial $P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ is an entire function.

✧ **Example 21:** The logarithm function $\text{Log}(z)$ is not entire because it is not continuous on the slit $(-\infty, 0]$, and hence, by Proposition 3.21, it is not differentiable there. At present, we do not know the derivative of $\text{Log}(z)$.

✧ A multivalued function $f(z)$ usually “suffers” from the problem of a slit on which it is not even continuous (and hence, by Proposition 3.21, not analytic), but in many cases we wish to find a branch of f that is analytic on a given set S . Such a branch is called an **analytic branch of f on the set S** .

✧ The proof of the following theorem is identical to the proof of the corresponding theorem in

Calculus 1. One may consult any elementary text on differential calculus of real functions for the proof.

i Theorem 3.24: If f and g are functions differentiable at the point z , then their sum, difference, product, and quotient (if $g(z) \neq 0$) are differentiable, and

$$\begin{aligned} [f(z) + g(z)]' &= f'(z) + g'(z) \\ [f(z) - g(z)]' &= f'(z) - g'(z) \\ [f(z) \cdot g(z)]' &= f'(z)g(z) + f(z)g'(z) \\ [f(z)/g(z)]' &= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2} \end{aligned}$$

i Theorem 3.25: (The chain rule) Let $f: S \rightarrow \mathbb{C}$ and $g: T \rightarrow \mathbb{C}$ be two functions analytic on open sets S and T , and suppose that $f(S) \subseteq T$. Then the composite function $g(f(z))$ is analytic, and for every $z \in S$,

$$[g(f(z))]' = g'(f(z)) \cdot f'(z)$$

✓ Proof: Let $z_0 \in S$ be a given point. By definition,

$$(g \circ f)'(z_0) = \lim_{z \rightarrow z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0}$$

We must prove that the limit exists and equals $g'(f(z_0)) \cdot f'(z_0)$.

We first present the rough (but imprecise) idea of the proof.

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} &= \lim_{z \rightarrow z_0} \left[\frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \cdot \frac{f(z) - f(z_0)}{z - z_0} \right] \\ &= \lim_{z \rightarrow z_0} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \cdot \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= g'(f(z_0)) \cdot f'(z_0) \end{aligned}$$



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By Proposition 3.21, f is continuous at the point z_0 , and hence $\lim_{z \rightarrow z_0} (f(z) - f(z_0)) = 0$.

Therefore, by the definition of the derivative, $\lim_{z \rightarrow z_0} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} = g'(f(z_0))$.

The only difficulty with this idea is that the expression $f(z) - f(z_0)$ may sometimes vanish, especially when $f'(z_0) = 0$ (or, for example, when f is constant).

To overcome this obstacle, we use formula (3.24) from Proposition 3.20. Use the variable w for



the function g , and set $w_0 = f(z_0)$.

$$g(w) = g(w_0) + [g'(w_0) + \rho(w)](w - w_0)$$

where $\lim_{w \rightarrow w_0} \rho(w) = 0$. Moving $g(w_0)$ from the right-hand side to the left-hand side gives

$$g(w) - g(w_0) = [g'(w_0) + \rho(w)](w - w_0)$$

From the assumption $f(S) \subseteq T$, we conclude that, for every $z \in S$, there exists $w \in T$ such that $w = f(z)$. Therefore, for every $z \in S - \{z_0\}$,

$$g(f(z)) - g(f(z_0)) = [g'(f(z_0)) + \rho(f(z))][f(z) - f(z_0)]$$

Dividing both sides by $z - z_0$ gives

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = [g'(f(z_0)) + \rho(f(z))] \cdot \frac{f(z) - f(z_0)}{z - z_0}$$

It follows that

$$\begin{aligned} (g \circ f)'(z_0) &= \lim_{z \rightarrow z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} \\ &= \lim_{z \rightarrow z_0} [g'(f(z_0)) + \rho(f(z))] \cdot \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{z \rightarrow z_0} [g'(f(z_0)) + \rho(f(z))] \cdot \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= g'(f(z_0)) \cdot f'(z_0) \end{aligned}$$

and we have obtained the chain-rule formula. ■

i Theorem 3.26: (Derivative of an inverse function) Let $f: S \rightarrow T$ be analytic and let $g: T \rightarrow S$ be continuous, where S and T are open sets. Let $z_0 \in S$ and $w_0 = f(z_0) \in T$, with $f'(z_0) \neq 0$ and $g = f^{-1}$ in a neighborhood of w_0 . Then g is differentiable at w_0 , and

$$g'(w_0) = (f^{-1})'(w_0) = \frac{1}{f'(z_0)} \quad (3.26)$$

♥ Proof: First, we explain the meaning of the assertion “ $g = f^{-1}$ in a neighborhood of w_0 ”. It means that there exists $r > 0$ such that g is one-to-one on the neighborhood $T_0 = \mathbb{D}(w_0, r)$ (which is contained in T). The image of T_0 under g , $S_0 = g(T_0)$, is a subset of S , f is also one-to-one on S_0 , and the inverse function of f on S_0 is g . Under these conditions, it is clear

that, for every $w \in T_0$, $f(g(w)) = w$, and therefore

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{g(w) - g(w_0)}{f(g(w)) - f(g(w_0))} = \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}$$

Thus,

$$g'(w_0) = \lim_{w \rightarrow w_0} \frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{f'(z_0)}$$

In the calculation, we used the notation $z = g(w)$ and $z_0 = g(w_0)$. Unlike the difficulty encountered in the preceding proof, here $z \neq z_0$ because g is one-to-one¹. By the continuity of g , as w tends to w_0 , z tends to z_0 , and hence the limit in the denominator equals $f'(z_0)$. ■

➤ In the next chapter, after introducing the Cauchy-Riemann equations, we prove that if $f'(z_0) \neq 0$, then f is invertible in some neighborhood of z_0 . Formula (3.26) can therefore be derived without requiring the function g . We conclude the section with a restricted version of L'Hopital's rule for complex functions.

i Theorem 3.27: Let $f(z)$ and $g(z)$ be functions differentiable at the point z_0 such that $f(z_0) = g(z_0) = 0$. If $g'(z_0) \neq 0$, then

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

✓ **Proof:** This is an immediate consequence of the definition of the derivative:

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}}{\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0}} = \frac{\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)}}{\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{g(z) - g(z_0)}} = \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)}$$

Note that $g'(z) \neq 0$ in a neighborhood of z_0 . This follows from $g'(z_0) \neq 0$ and Proposition 3.20. ■

◆ **Example 22:** Theorem 3.27 allows us to compute a limit very easily.

$$\lim_{z \rightarrow 0} \frac{1 - \cos z}{z^2} = \lim_{z \rightarrow 0} \frac{\sin z}{2z} = \lim_{z \rightarrow 0} \frac{\cos z}{2} = \frac{1}{2}$$

➤

¹ and recall that when $w \rightarrow w_0$, we always have $w \neq w_0$



Exercises for chapter 3



1. Write a complete proof of Theorem [3.8](#) (Heine's theorem).

2. What are the domains of the functions $\tan z$ and $\cot z$?



3. Prove the identities [\(3.17\)](#), [\(3.19\)](#), [\(3.13\)](#), on pages 87 and 89.



4. Prove Theorem [3.7](#).

5. Prove that $e^{\bar{z}} = \overline{e^z}$, $\sin \bar{z} = \overline{\sin z}$.



6. Prove the identity: $|\sin z|^2 + |\cos z|^2 = 1 + 2 \sinh^2(\operatorname{Im}(z))$.



7. Prove the identities: $\cosh x = \cos ix$, $\sinh x = -i \sin ix$.



8. Convert the function $f(z)$ to Cartesian notation $f(x + iy) = u(x, y) + iv(x, y)$:

(a) $f(z) = 5z^2 - 3iz + 1$

(b) $f(z) = \frac{z+3i}{z^2+1}$

(c) $f(z) = e^z + e^{-z}$

(d) $f(z) = \frac{3iz^2-4}{|z-2|}$



9. Find the domain of each function in Exercise [8](#).



10. A function is given in Cartesian notation: $f(x + iy) = x^2 - y^2 - 2y + i(2x - 2xy)$. Convert it to ordinary notation in the variable z .

Hint: $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

11. Show that the inversion function $f(z) = \frac{1}{z}$ maps

(a) the circle $|z| = r$ to the circle $|z| = 1/r$.

(b) the ray $\operatorname{Arg}(z) = \theta$ ($-\pi < \theta < \pi$) to the ray $\operatorname{Arg}(z) = -\theta$.

(c) the circle $|z - r| = r$ to the vertical line $x = \frac{1}{2r}$ (for real $r > 0$).

Remark: Note that in part b, the ray does not include the point $z = 0$ (which in any case does not belong to the domain of the argument function Arg).



12. The Joukowski map is defined by $J(z) = \frac{1}{2} \left(z + \frac{1}{z} \right)$. Prove that J maps the unit circle $|z| = 1$ to the real interval $[-1, 1]$.

13. What is the image of the punctured disk $0 < |z| \leq 1$ under the function $f(z) = \frac{1}{z}$?



14. A function of the form $f(z) = az + b$ is called a linear map. Prove that f maps straight lines to straight lines and circles to circles.





15. Prove:

$$\operatorname{Log}(c^z) = z\operatorname{Log}(c) + 2\pi i \left[\frac{1}{2} - \frac{\operatorname{Im}(z\operatorname{Log}(c))}{2\pi} \right]$$



16. Let n be a given natural number, and let $f(z) = z^n$. Prove that f maps the sector

$$S = \left\{ z \in \mathbb{C} : -\frac{\pi}{n} < \operatorname{Arg}(z) \leq \frac{\pi}{n} \right\}$$

onto the entire complex plane $\mathbb{C}$. Is f one-to-one on S ?

17. Let n be a given natural number. Prove that the logarithm function $\log z$ maps the strip

$$S_n = \{ z \in \mathbb{C} : (2n-1)\pi < \operatorname{Im}(z) \leq (2n+1)\pi \}$$

one-to-one onto the entire complex plane $\mathbb{C}$.



18. Let $z \neq 0$, and let $c, d \in \mathbb{C}$. Prove that all values of the multivalued function z^{dc} are contained among the values of $(z^c)^d$. Are there conditions on c and d that guarantee equality between the two functions?



19. Let $z \neq 0$, and let $m, n \in \mathbb{N}$ be two positive natural numbers. Does the equality $(z^m)^{\frac{1}{n}} = (z^{\frac{1}{n}})^m$ always hold?



20. Prove that, for every complex number $c \in \mathbb{C}$, there exists $z \in \mathbb{C}$ such that $\sin z = c$. Prove the analogous assertion for $\cos z = c$.

21. Find all solutions of the equation $\cos z = 2$.

22. Prove that, for every $w \in \mathbb{C}$, the equation $\cosh(z) = w$ has infinitely many distinct solutions.

23. Find all solutions of the equation $z^{1-i} = 4$.

24. Find the domains of the functions $\arcsin$, $\arccos$, $\arctan$, arccot .



25. Define principal branches of the functions $\arccos$, $\arctan$, and arccot under the names Arccos , Arctan , and Arccot . Find the domain, slit, and region of continuity of each of these functions.

26. Determine whether each of the following limits exists, and compute it when it does:

$$(a) \lim_{z \rightarrow 3i} \frac{z^2+9}{z-3i} \quad (b) \lim_{z \rightarrow 2i} \frac{\bar{z}+z^2}{1-\bar{z}} \quad (c) \lim_{z \rightarrow \infty} \frac{z^2+35z+14}{2z^2-7} \quad (d) \lim_{z \rightarrow 1} \frac{1-z}{z^2+5z-6}$$



27. Let $\mathbb{H}$ be the upper complex half-plane $\operatorname{Im}(z) > 0$, and let $f(z) = e^{2\pi iz}$. Prove that

$$f(\mathbb{H}) = \dot{\mathbb{D}}(0, 1)$$

where $\dot{\mathbb{D}}(0, 1)$ is the punctured unit disk (see Definition 3.2 on page 91).



28. Find the image of each of the following sets under the function $f(z) = e^z$:

- (a) $S = \{z \in \mathbb{C} : -5 \leq \operatorname{Re}(z) \leq 5, \operatorname{Im}(z) = \frac{\pi}{4}\}$
- (b) $S = \{z \in \mathbb{C} : \operatorname{Re}(z) = 3, -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2}\}$
- (c) $S = \{z \in \mathbb{C} : -2 < \operatorname{Re}(z) < 1, 0 < \operatorname{Im}(z) < \pi\}$
- (d) $S = \{z \in \mathbb{C} : < \operatorname{Re}(z) < 1, -\frac{\pi}{3} < \operatorname{Im}(z) < \frac{2\pi}{3}\}$



29. Find the image of the half-strip

$$S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq \pi, \operatorname{Im}(z) \geq 0\}$$

under each of the following functions:

- (a) $f(z) = e^{iz}$
- (b) $f(z) = ie^{iz}$
- (c) $f(z) = ie^{-iz}$

30. Find the image of each of the following sets under the function $f(z) = \cos z$:

- (a) $S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) < \pi, -2 < \operatorname{Im}(z) < 2\}$
- (b) $S = \{z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{4}, \operatorname{Im}(z) > 0\}$

31. Prove that, for every $z_1, z_2 \in \mathbb{C} - \{0\}$, there exists an integer k such that

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2) + 2k\pi i$$

32. Find all complex numbers z for which $\sin z$ is real.

33. Compute all possible values of the following expressions:

- (a) $(-1)^{\sqrt{2}}$
- (b) 2^{1-i}
- (c) $(1+i)^{\sqrt{3}}$
- (d) $(-1)^{\frac{1}{3}}$
- (e) $(\cos i)^i$
- (f) $i^{\sin i}$
- (g) $\log(\log(-1))$
- (h) $i^{\log(i)}$



34. Let $f(z)$ be an analytic branch of z^i on the slit plane $D = \mathbb{C} - \{iy : y \leq 0\}$. It is given that $f(1) = e^{2\pi}$. Compute $f(i)$.

Answer: $f(i) = e^{\frac{3\pi}{2}}$

35. Let $U = \mathbb{C} - (-\infty, 0]$ be the usual slit plane, and let y be a positive real number. Compute



the limit

$$\lim_{y \rightarrow 0^+} [\operatorname{Log}(a + iy) - \operatorname{Log}(a - iy)]$$

for the case $a > 0$ and the case $a < 0$.

Hint: The expression $y \rightarrow 0^+$ means that y is a positive real number tending to zero from the right.



- 36.** Let n be a positive natural number. Give as simple and general a topological characterization as possible of the kinds of open sets on which an analytic branch of the multivalued function $z^{1/n}$ exists.

Hint: An analytic branch of $z^{1/n}$ depends on the existence of an analytic branch of $\log$. An analytic branch of $\log$ exists only on an open, simply connected set that does not contain the origin.



- 37.** Let n be a positive natural number. Find all analytic branches of the multivalued function $z^{1/n}$ on the slit plane $D = \mathbb{C} - (-\infty, 0]$.

- 38.** Find the images of lines parallel to the x -axis or the y -axis under each of the following functions:

(a) $\operatorname{Log}(iz)$ (b) $\operatorname{Log}(-iz) + 1$ (c) $\operatorname{Log}(z^2)$



- 39.** Compute the following limits:

(a) $\lim_{z \rightarrow 0} (1 + z)^{1/z}$ (b) $\lim_{z \rightarrow 2} \frac{\sqrt{z}-2}{z-2}$ (c) $\lim_{z \rightarrow i} z \operatorname{Arg}(\bar{z})$
 (d) $\lim_{z \rightarrow 0} \frac{z^4}{|z|^4}$ (e) $\lim_{z \rightarrow 0} \frac{|\bar{z}|^2 + z}{z}$ (f) $\lim_{z \rightarrow 0} z \frac{\operatorname{Im}(z^2)}{z^2}$
 (g) $\lim_{z \rightarrow \infty} \frac{3z^2 - 2z}{z^2 - iz + 8}$ (h) $\lim_{z \rightarrow 5} \frac{3z}{z^2 - (5-i)z - 5i}$ (i) $\lim_{z \rightarrow \infty} (5z^3 + 7z + 4)$



- 40.** Find all points of discontinuity of the following functions:

(a) $f(z) = \operatorname{Log}(z^2 - 1)$ (b) $f(z) = \operatorname{Arg}(z^2)$ (c) $f(z) = \operatorname{Log}(z^3 - 1)$
 (d) $f(z) = \operatorname{Arg}(z^3)$ (e) $f(z) = \sqrt{z^2 + 1}$ (f) $f(z) = \sqrt{z^2 - 1}$

- 41.** Find the region of continuity of the function $f(z) = \frac{\operatorname{Re}(z)}{1+|z|}$.



- 42.** Find the region of continuity of the function

$$f(z) = \begin{cases} \frac{\operatorname{Re}(z^2)}{z}, & z \neq 0 \\ 0 & z = 0 \end{cases}$$



- 43.** Let $f: \mathbb{C} \rightarrow \mathbb{C}$, $g: \mathbb{C} \rightarrow \mathbb{C}$, be two continuous functions. Prove that the set

$$S = \{z \in \mathbb{C} : f(z) = g(z)\}$$

is closed.

Hint: Show that if $z_n \in S$ and $z_n \rightarrow z$, then $z \in S$. Alternatively, define $h(z) = f(z) - g(z)$, and then $S = h^{-1}(0)$.





44. Let f be a continuous function on a connected set S . Suppose that f takes only integer values on S . Prove that f is constant on S .

Hint: If f is not constant, then there exist two points $z_1, z_2 \in S$ such that $f(z_1) = m$ and $f(z_2) = n$, for two integers $m < n$. Let $\sigma : [a, b] \rightarrow S$ be a path joining z_1 to z_2 . By definition, σ is continuous, and hence the composite function $f(\sigma(t))$ is also continuous. We have $f(\sigma(a)) = m$ and $f(\sigma(b)) = n$, so by the intermediate value theorem (Calculus 1), there exists a point $a < t < b$ such that $f(\sigma(t)) = m + 0.5$, a contradiction.

45. Prove directly from the definition of the derivative that $f(z) = \frac{1}{z}$ is differentiable at every point $z \neq 0$, and find its derivative.



46. Determine whether the following function is differentiable at the point $z = 0$:

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

47. Give three examples of branches of $\log(z)$ on the slit plane $\mathbb{C} - \{ri : r \geq 0\}$.



48. Prove that no analytic branch of the logarithm function $\log$ exists on the domain $\mathbb{C} - \{0\}$.



[Link to solutions](#)



Chapter 4

Analytic Functions

In the present chapter, we study in depth the structure, properties, and uses of the various kinds of analytic functions, and encounter important families of such functions.

4.1 The Cauchy–Riemann Equations



Fig. 4.1: Augustin-Louis Cauchy
1789–1857



Fig. 4.2: Bernhard Riemann
1826–1866

- At the end of the preceding chapter, we computed several derivatives of elementary functions. Computing derivatives of additional functions requires a deeper understanding of the complex derivative, which will enable us to develop rules and formulas for their calculation. The work of the mathematicians Augustin-Louis Cauchy and Bernhard Riemann on this subject produced the theory of the Cauchy–Riemann equations, which makes this possible.
- We note that, for a complex function, the requirement of differentiability presents the function with a much greater challenge than in the real case, because in the complex plane one may approach the point z_0 along infinitely many different paths. Along each of these paths, the limit $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ must have the same value.
- We begin by examining the two simplest paths along which one may approach the point z_0 in the complex plane (see Figure 4.3).
 - (1) A horizontal straight line passing through z_0 : $\sigma_1(t) = z_0 + t$
 - (2) A vertical straight line passing through z_0 : $\sigma_2(t) = z_0 + ti$

Computing the limit $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ along each of these simple paths yields several surprising results. The calculation will, of course, be carried out using the Cartesian form of f

$$f(x + iy) = u(x, y) + iv(x, y)$$

Assumption: the function f is differentiable at the point $z_0 = x_0 + iy_0$. Therefore, there exists $r > 0$ such that f is defined on the neighborhood $D(z_0, r)$ (as shown in Figure 4.3).

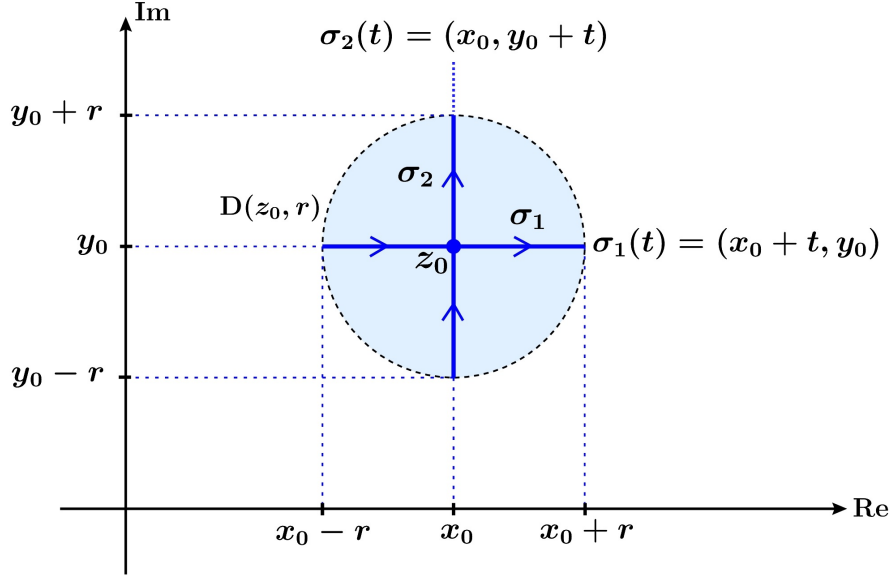


Fig. 4.3: Paths used to differentiate $f(z)$ in deriving the Cauchy–Riemann equations

Differentiation along the path σ_1

$$\begin{aligned} f'(z_0) &= \lim_{t \rightarrow 0} \frac{f(\sigma_1(t)) - f(z_0)}{\sigma_1(t) - z_0} = \lim_{t \rightarrow 0} \frac{f((x_0 + t) + iy_0) - f(x_0 + iy_0)}{(x_0 + t, y_0) - (x_0, y_0)} \\ &= \lim_{t \rightarrow 0} \frac{[u(x_0 + t, y_0) + iv(x_0 + t, y_0)] - [u(x_0, y_0) + iv(x_0, y_0)]}{(t, 0)} \\ &= \lim_{t \rightarrow 0} \frac{u(x_0 + t, y_0) - u(x_0, y_0)}{t} + i \lim_{t \rightarrow 0} \frac{v(x_0 + t, y_0) - v(x_0, y_0)}{t} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

Along the path $\sigma_2(t)$

$$\begin{aligned} f'(z_0) &= \lim_{t \rightarrow 0} \frac{f(\sigma_2(t)) - f(z_0)}{\sigma_2(t) - z_0} = \lim_{t \rightarrow 0} \frac{f(x_0 + i(y_0 + t)) - f(x_0 + iy_0)}{(x_0, y_0 + t) - (x_0, y_0)} \\ &= \lim_{t \rightarrow 0} \frac{[u(x_0, y_0 + t) + iv(x_0, y_0 + t)] - [u(x_0, y_0) + iv(x_0, y_0)]}{(0, t)} \end{aligned}$$

$$\begin{aligned}
&= \lim_{t \rightarrow 0} \frac{u(x_0 + t, y_0) - u(x_0, y_0)}{it} + i \lim_{t \rightarrow 0} \frac{v(x_0 + t, y_0) - v(x_0, y_0)}{it} \\
&= \lim_{t \rightarrow 0} \frac{v(x_0 + t, y_0) - v(x_0, y_0)}{t} - i \lim_{t \rightarrow 0} \frac{u(x_0 + t, y_0) - u(x_0, y_0)}{t} \\
&= v_y(x_0, y_0) - iu_y(x_0, y_0)
\end{aligned}$$

In summary, we have obtained the following theorem.

i Theorem 4.1: If $f(x + iy) = u(x, y) + iv(x, y)$ is differentiable at the point $z_0 = x_0 + iy_0$, then the two (real) functions $u(x, y)$ and $v(x, y)$ have first-order partial derivatives u_x , u_y , v_x , and v_y at the point (x_0, y_0) , and they satisfy the pair of **Cauchy–Riemann equations** (Cauchy–Riemann)

$$(CR) \quad \begin{cases} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0) \end{cases}$$

This pair of equations is denoted by (CR). In addition, we have obtained two different formulas for computing the derivative:

$$\begin{aligned}
f'(z_0) &= u_x(x_0, y_0) + iv_x(x_0, y_0) \\
&= v_y(x_0, y_0) - iu_y(x_0, y_0)
\end{aligned}$$

i Definition 4.1: The **Cauchy–Riemann equations** for a pair of real functions of two variables $u(x, y)$ and $v(x, y)$ are given by

$$\begin{cases} u_x &= v_y \\ u_y &= -v_x \end{cases} \quad (CR)$$

We say that a pair of functions (u, v) satisfies the **Cauchy–Riemann equations** on a set $D \subseteq \mathbb{R}^2$ if it has first-order partial derivatives satisfying the pair of equations (CR) at every point $(x, y) \in D$.

- ❖ **Example 1:** The functions $u = x^2 - y^2$ and $v = 2xy$ satisfy the Cauchy–Riemann equations throughout the real plane $\mathbb{R}^2$.
- Theorem 4.1 states that if $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$, then its components u and v satisfy the Cauchy–Riemann equations at that point. The question is whether the converse is also true. That is, if $u(x, y)$ and $v(x, y)$ are known to satisfy the Cauchy–Riemann equations at the point (x_0, y_0) , is $u(x, y) + iv(x, y)$ differentiable at that point?
- Since we have examined only two simple paths among infinitely many possible paths along which



the limit may fail to exist, it would be surprising if the answer were affirmative. Indeed, when a single point is involved, the answer is negative (see the counterexample below). For an open set, however, the answer is affirmative.

- In contrast to most theorems studied in this course, which were proved in the 18th and 19th centuries, the [Looman–Menchoff theorem](#) is a relatively recent theorem from the beginning of the 20th century. Its complete proof lies beyond the scope of the course, but we present a proof of a weaker version, together with links to the complete proof.

i Theorem 4.2: The Looman–Menchoff Theorem (Looman–Menchoff)

Let $f(z) = u(x, y) + iv(x, y)$ be a complex function on an open set S . Then $f(z)$ is analytic on S if and only if $f(z)$ is continuous and satisfies the Cauchy–Riemann equations on S .

- We proved the first direction of the theorem at the beginning of the section: if $f(z)$ is analytic on the set S , then by Proposition [3.21](#), f is continuous on S , and the calculation performed above proves that it satisfies the Cauchy–Riemann equations.
- The second direction of the theorem is much more complicated and was proved in 1936 using more advanced tools from mathematical analysis that lie beyond the scope of the course (links to the proof appear below).
- In this direction, one assumes that $f(z)$ is continuous and satisfies the Cauchy–Riemann equations on the set S , and must prove that $f(z)$ is analytic on S , that is, differentiable at every point of S (since S is open, differentiability implies analyticity by Proposition [3.23](#)).
- Instead, we prove the classical theorem taught in an academic course on complex functions, which uses an assumption stronger than continuity: **differentiability as a function of two real variables**. This notion of differentiability is discussed in various [Calculus 2](#) courses.

i Theorem 4.3: Let $f(x + iy) = u(x, y) + iv(x, y)$ be a complex function in a neighborhood of the point $z_0 = x_0 + iy_0$. If $u(x, y)$ and $v(x, y)$ are differentiable at the point (x_0, y_0) and satisfy the Cauchy–Riemann equations there, then f is differentiable at the point z_0 .

- ✓ **Proof:** By differentiability, there exist functions $\varepsilon_u(t, s)$ and $\varepsilon_v(t, s)$ such that

$$\begin{aligned}\Delta u &= u(x_0 + t, y_0 + s) - u(x_0, y_0) = u_x(x_0, y_0)t + u_y(x_0, y_0)s + \varepsilon_u(t, s)\sqrt{t^2 + s^2} \\ \Delta v &= v(x_0 + t, y_0 + s) - v(x_0, y_0) = v_x(x_0, y_0)t + v_y(x_0, y_0)s + \varepsilon_v(t, s)\sqrt{t^2 + s^2}\end{aligned}$$

and such that $\lim_{h \rightarrow 0} \varepsilon_u(t, s) = \lim_{h \rightarrow 0} \varepsilon_v(t, s) = 0$.

Recall that $\sqrt{t^2 + s^2} = |h|$. We compute the derivative.



$$\begin{aligned}
f'(z_0) &= \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{[u(x_0 + t, y_0 + s) + iv(x_0 + t, y_0 + s)] - [u(x_0, y_0) + iv(x_0, y_0)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{[u(x_0 + t, y_0 + s) - u(x_0, y_0)] + i[v(x_0 + t, y_0 + s) - v(x_0, y_0)]}{h} \\
&= \lim_{h \rightarrow 0} \frac{[u_x(x_0, y_0)t + u_y(x_0, y_0)s + \varepsilon_u(t, s)|h|] + i[v_x(x_0, y_0)t + v_y(x_0, y_0)s + \varepsilon_v(t, s)|h|]}{h} \\
&= \lim_{h \rightarrow 0} \frac{u_x(x_0, y_0)t + iv_y(x_0, y_0)s + u_y(x_0, y_0)s + iv_x(x_0, y_0)t + \varepsilon_u(t, s)|h| + i\varepsilon_v(t, s)|h|}{h} \\
&\quad \downarrow \text{(CR)} \quad \downarrow \text{(CR)} \\
&= \lim_{h \rightarrow 0} \frac{u_x(x_0, y_0)t + iu_x(x_0, y_0)s - v_x(x_0, y_0)s + iv_x(x_0, y_0)t + \varepsilon_u(t, s)|h| + i\varepsilon_v(t, s)|h|}{h} \\
&= \lim_{h \rightarrow 0} \frac{u_x(x_0, y_0)t + iu_x(x_0, y_0)s + i[v_x(x_0, y_0)t + iv_x(x_0, y_0)s] + [\varepsilon_u(t, s) + i\varepsilon_v(t, s)] \cdot |h|}{h} \\
&= \lim_{h \rightarrow 0} \frac{u_x(x_0, y_0) \cdot (t + is) + iv_x(x_0, y_0) \cdot (s + it) + [\varepsilon_u(t, s) + i\varepsilon_v(t, s)] \cdot |h|}{h} \\
&= \lim_{h \rightarrow 0} \frac{u_x(x_0, y_0) \cdot h + iv_x(x_0, y_0) \cdot h + [\varepsilon_u(t, s) + i\varepsilon_v(t, s)] \cdot |h|}{h} \\
&= u_x(x_0, y_0) + iv_x(x_0, y_0) + \lim_{h \rightarrow 0} [\varepsilon_u(t, s) + i\varepsilon_v(t, s)] \cdot \frac{|h|}{h} \\
&= u_x(x_0, y_0) + iv_x(x_0, y_0)
\end{aligned}$$

The last limit tends to zero because $\varepsilon_u(t, s) + i\varepsilon_v(t, s)$ tends to zero and the expression $\frac{|h|}{h}$ is bounded. We have therefore shown that if $u(x, y)$ and $v(x, y)$ are differentiable at the point (x_0, y_0) and satisfy the Cauchy–Riemann equations there, then f is differentiable at the point z_0 .

Finally, if u and v are differentiable at every point of an open set S and satisfy the Cauchy–Riemann equations at every point of S , then f is differentiable at every point of S , and hence f is analytic on the set S . ■

i Corollary 4.4: Let $f(x + iy) = u(x, y) + iv(x, y)$ be a complex function on an open set S . If the functions u and v have continuous first-order partial derivatives on S , and if u and v satisfy the Cauchy–Riemann equations on S , then f is analytic on S .

✓ Proof: In the [Calculus 2](#) course, we learned¹ that continuity of the first-order partial derivatives implies differentiability. ■

◇ Example 2: The function $f(x + iy) = (x^2 + 5x - y^2) + (2xy + 5y)i$ satisfies the conditions of the corollary throughout the complex plane $\mathbb{C}$: $u = x^2 + 5x - y^2$, $v = 2xy + 5y$, $u_x = 2x + 5 = v_y$, $u_y = -2y = -v_x$. Of course, all the partial derivatives are continuous throughout the complex plane $\mathbb{C}$, and therefore, by the corollary, f is analytic on all of $\mathbb{C}$. It is

¹ See Euler's theorem at the end of Chapter 3 in the [Calculus 2](#) text



not difficult to discover that $f(z) = z^2 + 5z$.

➤ Links to a complete proof of the Looman–Menchoff theorem:



[Looman–Menchoff theorem](#)



[Cauchy–Riemann](#)



[When is a Function that Satisfies the Cauchy–Riemann Equations Analytic?](#)

Remarks

1. The Looman–Menchoff theorem is a local theorem. That is, its conditions are formulated on an open set S and do not apply to a single point! Satisfying the continuity and Cauchy–Riemann conditions at a single point does not guarantee analyticity at that point. For example, the function $f(z) = |z|^2$ is continuous throughout the complex plane $\mathbb{C}$:

$$\begin{cases} f(z) = f(x + iy) = |z|^2 = (x^2 + y^2) + 0 \cdot i \\ u(x, y) = x^2 + y^2 \\ v(x, y) = 0 \end{cases}$$

because $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$ are continuous throughout the plane $\mathbb{R}^2$. It is easy to check that the functions $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations only at the point $z = 0$! Therefore, $f(z)$ is not differentiable on the punctured plane $\mathbb{C} - \{0\}$. Thus, f is not analytic at the point $z = 0$, even though it satisfies the Looman–Menchoff conditions at that point. We check differentiability at the point $z = 0$:

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{|h|^2 - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{h \cdot \bar{h}}{h} = \lim_{h \rightarrow 0} \bar{h} = 0 \end{aligned}$$

In this example, f turns out to be differentiable at the point (but not analytic!); however, even this is not guaranteed!

2. In fact, the Looman–Menchoff conditions at a single point do not even guarantee differentiability at the point where they hold. For example, the function

$$f(z) = \begin{cases} \frac{z^5}{|z|^4} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous throughout the complex plane $\mathbb{C}$ and satisfies the Cauchy–Riemann equations at $z = 0$, but is not differentiable at $z = 0$! Further details are available at the following link:





how-to-show-that-fz-fracz5z4-satisfies-the-cauchy-riemann-equation

3. The stronger condition that u and v are differentiable at the point z_0 and satisfy the Cauchy–Riemann equations guarantees differentiability at z_0 (but not analyticity!), as we saw in the second part of the proof above.
4. It is therefore possible for $f(z)$ to be continuous and satisfy the Cauchy–Riemann equations at the point z_0 and nevertheless fail to be differentiable at the point z_0 . The example in Remark 2 has this property. This situation proves that u and v are not differentiable at the point (x_0, y_0) .

Consequences

i Proposition 4.5: If $f'(z) = 0$ on a domain S , then f is constant on S .

✓ Proof: By Theorem 4.1, for every $z = x + iy \in S$,

$$\begin{aligned} f'(z) &= u_x(x, y) + iv_x(x, y) = 0 \\ f'(z) &= v_y(x, y) - iu_y(x, y) = 0 \end{aligned}$$

and therefore necessarily $u_x = u_y = 0$ throughout the domain S , so $u(x, y)$ must be a constant function. Similarly, we conclude that $v(x, y)$ is constant on S . Therefore, f is constant on S . ■

- Remark: The proposition is not true if S is merely an open set. For example, if $f'(z) = -1$ for $\operatorname{Re}(z) < -1$ and $f(z) = 1$ for $\operatorname{Re}(z) > 1$, but is undefined on the strip $-1 \leq \operatorname{Re}(z) \leq 1$, then clearly $f'(z) = 0$, yet f is not constant on its domain.
- In the proof, we used the following result from Calculus 2: if the partial derivatives of $u(x, y)$ vanish on a connected domain D , then u is constant on D . Here is a short proof of this result.

Proof: Let $(x_1, y_1), (x_2, y_2) \in D$ be two points such that the straight path joining them is contained in the domain D :

$$r(t) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)), \quad 0 \leq t \leq 1$$

By Lagrange's mean value theorem, there exists a point $t_0 \in [0, 1]$ such that $r(1) - r(0) = r'(t_0)$. That is,

$$u(x_2, y_2) - u(x_1, y_1) = \vec{\nabla} u(r(t_0)) \cdot r'(t_0) = 0$$

By hypothesis, the gradient of u vanishes, and we obtain $u(x_2, y_2) = u(x_1, y_1)$. Since the domain D is connected, any two points in it can be joined by a polygonal path, and hence the equality holds for every pair of distinct points, so the function is constant on D . ■

i Proposition 4.6: If a function f on a domain S is analytic and its conjugate $\bar{f}$ is also analytic, then f is constant.

Hint: Write $f(x + iy) = u(x, y) + iv(x, y)$. Then $\bar{f}(x + iy) = u(x, y) - iv(x, y)$. Use the Cauchy–Riemann equations to show that $u_x = u_y = v_x = v_y = 0$ at every point of S .



i Proposition 4.7: If a function $|f(z)| = C$ has constant modulus on a domain S , then f is constant on S .

✓ Proof: If $C = 0$, then clearly, for every z , $f(z) = 0$, and we are done. We may therefore assume that $C > 0$. The hypothesis $|f(z)| = C$ implies that

$$f(z)\overline{f(z)} = |f(z)|^2 = C^2$$

Thus, $|\overline{f(z)}| = C$ as well, and therefore, by the preceding proposition, $\overline{f(z)}$ is constant. ■

◇ Exercise 4.1: If $f: S \rightarrow \mathbb{C}$ is an analytic function on a domain S that takes only real values, then f is constant.

Hint: Write $f(z) = u(x, y) + iv(x, y)$. By hypothesis, $v(x, y) = 0$ at every point $x + iy \in S$. Therefore, $v_x = v_y = 0$. The Cauchy–Riemann equations then also give $u_x = u_y = 0$.

◇ Exercise 4.2: Let $f: S \rightarrow \mathbb{C}$ be an analytic function on a domain S . If any one of the functions $\operatorname{Re}(f(z))$, $\operatorname{Im}(f(z))$, or $\arg(f(z))$ is constant, then $f(z)$ is constant.

i Proposition 4.8: If $f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z = x + iy$, then

$$|f'(z)|^2 = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

✓ Proof: The last expression is familiar as the **Jacobian** of u and v . By Theorem 4.1,

$$\begin{aligned} f'(z) &= u_x(x, y) + iv_x(x, y) \\ f'(z) &= v_y(x, y) - iu_y(x, y) \end{aligned}$$

Therefore,

$$\begin{aligned} |f'(z)|^2 &= f'(z)\overline{f'(z)} = [u_x + iv_x] \cdot [v_y + iu_y] \\ &= (u_x v_y - u_y v_x) + i(u_x u_y + v_x v_y) = (u_x v_y - u_y v_x) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \end{aligned}$$

The expression $i(u_x u_y + v_x v_y)$ must vanish because $|f'(z)|^2$ is a real number. ■

i Corollary 4.9: If $f(z)$ is analytic at the point z_0 and $f'(z_0) \neq 0$, then f is invertible in a neighborhood of z_0 . If $w_0 = f(z_0)$, then

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)} \quad (4.1)$$



✓ **Proof:** Write $f(z) = u(x, y) + iv(x, y)$. By the preceding proposition, the Jacobian of $u(x, y)$ and $v(x, y)$ is $|f'(z_0)|^2 \neq 0$. By the inverse function theorem studied in Calculus 2, if the Jacobian of the system $u(x, y), v(x, y)$ does not vanish at a given point, then there exists a neighborhood in which the system is invertible. For us, this means that the complex function f is invertible at the point z_0 . The inverse function theorem also guarantees that the inverse function f^{-1} is continuous, and therefore the conditions of Theorem 3.26 are satisfied, guaranteeing that f^{-1} is differentiable and that equality (4.1) also holds. ■

4.2 Derivatives of Elementary Functions

➤ We compute the derivatives of the basic functions so that we can subsequently calculate the derivatives of elementary functions using the differentiation rules for addition, subtraction, multiplication, division, and composition.

The Function e^z

➤ Recall that the definition of e^z is

$$e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

This is also the Cartesian form of e^z , whose components are

$$\begin{aligned} u(x, y) &= e^x \cos y \\ v(x, y) &= e^x \sin y \end{aligned}$$

By Theorem 4.1 (Cauchy–Riemann),

$$(e^z)' = v_y - iu_y = e^x \cos y + ie^x \sin y = e^z$$

As expected, we have obtained

$$(e^z)' = e^z \quad (4.2)$$

The Trigonometric Functions

➤ Recall that the functions **sin** and **cos** were defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

From the formula $(e^z)' = e^z$ and the chain rule, we easily obtain

$$\begin{aligned} (\sin z)' &= \cos z \\ (\cos z)' &= -\sin z \end{aligned} \tag{4.3}$$

➤ The functions **tan** and **cot** were defined by

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} = \frac{i(e^{-iz} - e^{iz})}{e^{iz} + e^{-iz}} \\ \cot z &= \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}} \end{aligned}$$

Using the quotient rule, it is easy to show that

$$\begin{aligned} (\tan z)' &= \frac{1}{\cos^2 z} \\ (\cot z)' &= -\frac{1}{\sin^2 z} \end{aligned} \tag{4.4}$$

The Logarithm Function $\log z$

➤ In Proposition [3.1](#), we encountered the basic identity

$$e^{\log z} = z \quad (z \neq 0)$$

The identity follows from the definition of the logarithm function, but it may also be proved as follows:

$$e^{\log z} = e^{\log |z| + i \arg(z)} = e^{\log |z|} e^{i \arg(z)} = |z| e^{i \arg(z)} = z$$

➤ Computing the derivative:

$$\begin{aligned} e^{\log z} = z &\implies [e^{\log z}]' = (z)' = 1 \\ &\implies [\log z]' \cdot e^{\log z} = 1 \\ &\implies [\log z]' = \frac{1}{e^{\log z}} = \frac{1}{z} \end{aligned}$$

We therefore obtain

$$\frac{d}{dz} \log z = \frac{1}{z} \quad (4.5)$$

Inverse Trigonometric Functions

- In the preceding chapter, we found formulas for the inverse trigonometric functions: formulas (3.20) and (3.21). We leave it for the student to compute the derivatives of these functions using the appropriate differentiation rules. Verify that the same formulas are also valid for the real versions of these functions. Also state the domains on which the formulas are valid.

$$\begin{aligned} \frac{d}{dz} \arcsin z &= \frac{1}{\sqrt{1-z^2}} & \frac{d}{dz} \arctan z &= \frac{1}{1+z^2} \\ \frac{d}{dz} \arccos z &= -\frac{1}{\sqrt{1-z^2}} & \frac{d}{dz} \operatorname{arccot} z &= -\frac{1}{1+z^2} \end{aligned} \quad (4.6)$$

The Power Functions z^c

- The power function z^c , for a complex constant $c \in \mathbb{C} - \{0\}$, was defined by²

$$z^c = e^{c \log z}$$

Therefore, by the chain rule,

$$\frac{d}{dz} z^c = e^{c \log z} \cdot (c \log z)' = cz^c \cdot \frac{1}{z} = cz^{c-1}$$

and hence we obtain the formula

$$(z^c)' = cz^{c-1} \quad (4.7)$$

The Exponential Functions c^z

- In the preceding chapter, we defined the exponential function by

$$c^z = e^{z \log c}$$

² For $c = 0$, $z^c = 1$ for every $z \neq 0$.



Therefore, by the chain rule,

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log c} = e^{z \log c} \cdot (z \log c)' = c^z \cdot \log c$$

and we obtain

$$(c^z)' = c^z \cdot \log c \quad (4.8)$$

➤ We may therefore summarize the discussion in the following theorem.

i Theorem 4.10: Every basic function is analytic (except at slit points).

➤ The elementary functions are formed from the basic functions by addition, subtraction, multiplication, division, and composition. We may therefore summarize the section by stating that every elementary function is analytic on the common domain of analyticity of the basic functions from which it was constructed.

Analytic Branches

➤ In the preceding chapter, we briefly defined the concept of an **analytic branch**. A multivalued function such as $\log z$ has infinitely many different branches. Most of these branches are uninteresting because they fail to be continuous or differentiable on the desired domain. The question that will interest us in most cases is: **Given a multivalued function $f(z)$ and a domain D , does there exist at least one branch of f that is analytic on the domain D ?**

i Definition 4.2: Let $f(z)$ be a multivalued function defined on a domain D . Every branch of f that is analytic on D is called an **analytic branch of f on D** .

➤ **Example 3:** Let $f(z) = (z^2 - 1)^{\frac{1}{2}}$, and let D be the domain $|z| < 1$ (the open unit disk). Does an analytic branch of $f(z)$ exist on the domain D ?

➤ **Solution:** By definition, $(z^2 - 1)^{\frac{1}{2}} = e^{\frac{1}{2} \log(z^2 - 1)}$. We must find a branch of $\log$ such that $\log(z^2 - 1)$ is analytic in the disk D . First, we find the image E of D under the function $z^2 - 1$:

$$E = \{z^2 - 1 : z \in D\}$$

The function z^2 maps the unit disk D onto itself, and therefore $z^2 - 1$ shifts D to the disk $|z + 1| = 1$.

The principal branch of the logarithm is unsuitable because it is not analytic on the slit $(-\infty, 0]$, which crosses the disk E . But if we choose the branch of the logarithm corresponding to the interval $(0, 2\pi]$,

$$L(z) = \ln |z| + i\theta, \quad 0 < \theta \leq 2\pi, \quad z = re^{i\theta}$$

then its slit is the ray $[0, \infty)$, which does not intersect the disk E , and therefore $L(z^2 - 1)$ is



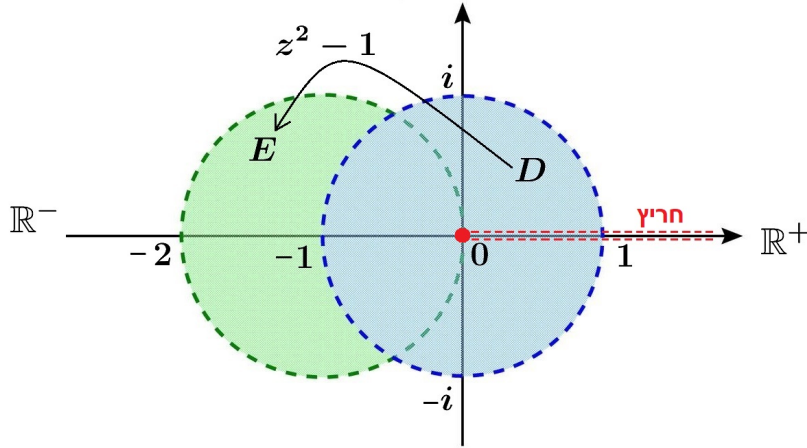


Fig. 4.5: The function $z^2 - 1$ maps the unit disk D to the disk $E: |z + 1| < 1$

an analytic branch of $\log(z^2 - 1)$ in the disk E . Thus,

$$F(z) = (z^2 - 1)^{\frac{1}{2}} = e^{\frac{1}{2}L(z^2-1)}$$

is an answer to the question. We leave it for the student to determine whether there are additional analytic branches.

❖ **Exercise 4.3:** What is the derivative of the analytic branch found in the preceding example?

❖ **Example 4:** Let $f(z) = (z^2 - 1)^{\frac{1}{2}}$, and let D be the domain $|z| > 1$ (a plane with a hole). Does an analytic branch of $f(z)$ exist on the domain D ?

🔴 **Solution:** Again, by the definition of the power function, $f(z) = e^{\frac{1}{2}\log(z^2-1)}$. The function $\log$ is multivalued, and among its many branches we must find one that is analytic on the domain D . Consider, for example, the principal branch of $\log$:

$$F(z) = e^{\frac{1}{2}\text{Log}(z^2-1)}$$

Recall that the slit of the principal branch Log is $(-\infty, 0]$, and therefore $F(z)$ is analytic only at points z satisfying $z^2 - 1 \notin (-\infty, 0]$. Substituting $z = x + iy$ into this condition shows that the slit points of $F(z)$ are all the points $z = x + iy$ satisfying the equivalent condition

$$z^2 - 1 = (x^2 - y^2 - 1) + 2xyi \in (-\infty, 0]$$

That is, $2xy = 0$ and $x^2 - y^2 - 1 \leq 0$. After simplifying the conditions, we find that the slit of F is the set

$$\begin{aligned} S &= \{x + iy : (x = 0 \vee y = 0) \wedge |x| \leq 1\} \\ &= (-\infty i, \infty i) \cup [-1, 1] \end{aligned}$$

where the expression $(-\infty i, \infty i)$ denotes the imaginary axis. Therefore, the domain of analyt-

icity of $F(z)$ is

$$D = \mathbb{C} - [(-\infty i, \infty i) \cup [-1, 1]]$$

See Figure 4.6. It follows that F is not analytic on the entire imaginary axis and hence is not an analytic branch on the domain $|z| > 1$, because the slit of F crosses this domain along its entire length.

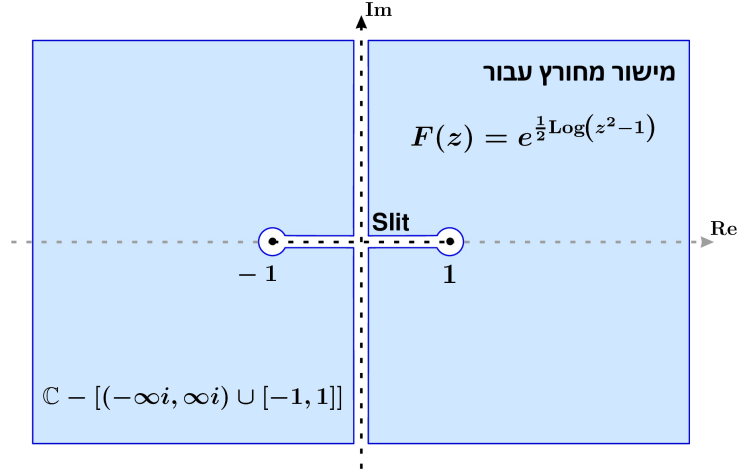


Fig. 4.6: A slit plane for the function $F(z) = e^{\frac{1}{2}\text{Log}(z^2-1)}$. The slit is the union of the imaginary axis and the segment $[-1, 1]$

It is not difficult to check that every branch of $\log(z^2 - 1)$ leads to a similar situation in which the slit intersects the domain $|z| > 1$, and therefore this approach fails completely. In situations of this kind, it is advisable to try factoring one of the expressions in the function into a product (or quotient) of other expressions. In our case, we use the following factorization:

$$z^2 - 1 = z^2 \left(1 - \frac{1}{z^2}\right)$$

Therefore,

$$(z^2 - 1)^{\frac{1}{2}} = \pm z \left(1 - \frac{1}{z^2}\right)^{\frac{1}{2}}$$

We may therefore try the following branch:

$$G_1(z) = ze^{\frac{1}{2}\text{Log}\left(1 - \frac{1}{z^2}\right)}$$

When $|z| > 1$, the expression $1 - \frac{1}{z^2}$ never lies³ on the slit $(-\infty, 0]$ of Log , and therefore the function $\text{Log}\left(1 - \frac{1}{z^2}\right)$ is analytic on the domain $|z| > 1$. We have thus found that $G_1(z)$ is an analytic branch of $f(z)$. It is easy to check that

$$G_2(z) = -ze^{\frac{1}{2}\text{Log}\left(1 - \frac{1}{z^2}\right)}$$

³ $1 - \frac{1}{z^2}$ is a negative real number only if $z = x$ is real. But $|x| > 1$ implies $1 - \frac{1}{x^2} > 0$

is also an analytic branch of $f(z)$ on the domain $|z| > 1$.

❖ **Exercise 4.4:** What are the derivatives of the analytic branches found in the preceding example?

4.3 Harmonic Functions

➤ The theory of Cauchy–Riemann is the first building block in understanding the nature and applications of analytic functions. The concept of a harmonic function is another important building block in this direction.

❗ **Definition 4.3:** Let $u(x, y)$ be a real function on a planar domain $D \subseteq \mathbb{R}^2$. The function u is called **harmonic on the domain D** if it is twice continuously differentiable on D and satisfies the equation

$$u_{xx} + u_{yy} = 0 \quad (4.9)$$

at every point of the domain D .

➤ Of course, equations equivalent to (4.9) are $u_{xx} = -u_{yy}$ and $u_{yy} = -u_{xx}$.

❗ **Proposition 4.11:** Let $f(z)$ be an analytic function on a domain $D \subseteq \mathbb{C}$, and let $u(x, y) + iv(x, y)$ be the Cartesian representation of f . Then both u and v are harmonic functions on the domain D .

✔ **Proof:** This follows immediately from the Cauchy–Riemann equations (Theorem 4.1):

$$(CR) \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Therefore,

$$\begin{aligned} u_{xx} &= \partial_x u_x = \partial_x v_y = v_{yx} \\ u_{yy} &= \partial_y u_y = \partial_y (-v_x) = -v_{xy} \end{aligned}$$

In [Calculus 2](#), we learned that $v_{xy} = v_{yx}$ when the partial derivatives are continuous. Therefore, $u_{xx} = -u_{yy}$, and hence u is harmonic on the domain D . In the same way, we prove that v is also harmonic on D . ■

❖ **Example 5:** The Cartesian form of the function $f(z) = z^2$ is $(x^2 - y^2) + 2xyi$. Therefore, by Proposition 4.11, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are harmonic functions, as may be checked easily.



❖ **Example 6:** The function $f(z) = x^2 + y^2i$ is not analytic because $u(x, y) = x^2$ is not harmonic! The function $v(x, y) = y^2$ is not harmonic either. In fact, Proposition 4.11 tells us that no analytic function can have x^2 as its real part!

❗ **Definition 4.4:** Let $u(x, y)$ and $v(x, y)$ be harmonic functions on a domain D . The function $v(x, y)$ is called a **harmonic conjugate of $u(x, y)$** if the two functions satisfy the Cauchy–Riemann equations:

$$(CR) \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

A pair of harmonic functions (u, v) in which v is a harmonic conjugate of u is called a **harmonic pair**.

❖ **Example 7:** The function $v(x, y) = 2xy$ is the harmonic conjugate of $u(x, y) = x^2 - y^2$ on the entire complex plane $\mathbb{C}$. In a preceding example, we saw that these two functions are the components in the Cartesian representation of $f(z) = z^2$. In the following theorem, we prove that this is not a coincidence.

❖ **Exercise 4.5:** Prove that $u(x, y) = \ln \sqrt{x^2 + y^2}$ is harmonic on $\mathbb{C} - \{0\}$ and find its harmonic conjugate. What is the domain of the harmonic conjugate you found?

Hint: Checking harmonicity is easy. The harmonic conjugate $v(x, y)$ is a solution of the system $\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$

That is,

$$\begin{cases} v_x = -\frac{y}{x^2 + y^2} \\ v_y = \frac{x}{x^2 + y^2} \end{cases}$$

By simple integration, we easily obtain

$$v(x, y) = \int \frac{-y dx}{x^2 + y^2} = \arctan \frac{y}{x} + C$$

One must verify that $v(x, y)$ also satisfies the second equation $v_y = \frac{x}{x^2 + y^2}$; otherwise, there is no harmonic conjugate! The domain has become slightly smaller: it does not include the y -axis.

❗ The harmonic-conjugacy relation is not symmetric. If v is a harmonic conjugate of u , then in general u is not a harmonic conjugate of v .

❗ **Theorem 4.12:** Let $u(x, y)$ and $v(x, y)$ be functions on a domain D . Then the function $f(x + iy) = u(x, y) + iv(x, y)$ is analytic on the domain D if and only if (u, v) is a harmonic pair on D .

✓ **Proof:** Direction 1: Suppose that f is analytic on the domain D . In Proposition 4.11, we proved that both functions u and v are harmonic on D . By definition, saying that v is a harmonic conjugate of u is precisely to say that the pair (u, v) satisfies the Cauchy–Riemann equations, and this follows, of course, from Theorem 4.1. We have shown that (u, v) is a harmonic pair. Direction 2: By the definitions of a harmonic function and harmonic conjugacy, all the partial derivatives of $u(x, y)$ and $v(x, y)$ exist and are continuous on the domain D , and, in addition,



satisfy the Cauchy–Riemann equations. Therefore, by Theorem 4.3, f is analytic on the domain D . ■

- A screenshot from Chapter 7 of the Calculus 2 course notes, which may be downloaded by clicking it. This recalls one of the fundamental theorems of vector analysis, on which we shall rely in order to prove a fundamental theorem about harmonic functions (Theorem 4.13).

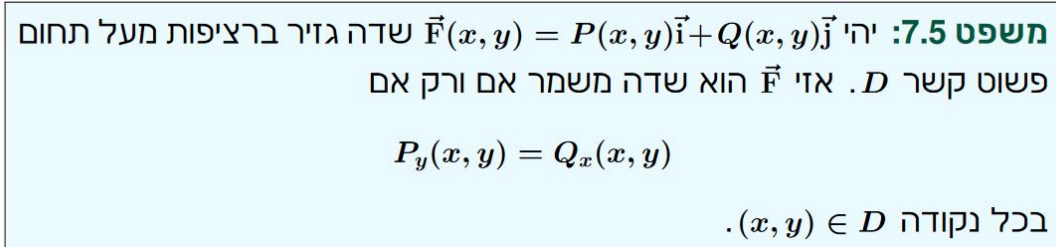


Fig. 4.7: Link for downloading the Calculus 2 lecture notes: <https://samyzaf.com/technion/hedva2t/hedva2.pdf>

- Recall that a conservative field $\vec{F} = P\vec{i} + Q\vec{j}$ means that there exists a potential function $v(x, y)$ satisfying $v_x = P$ and $v_y = Q$.

i Theorem 4.13: Let $u(x, y)$ be a harmonic function on a simply connected domain D . Then there exists a unique harmonic function $v(x, y)$ (up to an additive constant) on the domain D such that (u, v) is a harmonic pair on D .

- ✓ **Proof:** Set

$$P = -u_y, \quad Q = u_x$$

Then the fact that u is harmonic ($-u_{yy} = u_{xx}$) implies that

$$P_y = -u_{yy} = u_{xx} = Q_x$$

Thus, $\vec{F} = (P, Q)$ is a conservative planar field (or, equivalently, $Pdx + Qdy$ is an exact differential), and therefore there exists a potential function $v(x, y)$ on the domain D satisfying

$$v_x = P = -u_y \quad ; \quad v_y = Q = u_x$$

We prove that v is harmonic:

$$v_{xx} = \partial_x v_x = \partial_x [-u_y] = -u_{yx} = -u_{xy} = -\partial_y u_x = -\partial_y v_y = -v_{yy}$$

We have shown that v is a harmonic conjugate of the function u .

The converse is also true: if $w(x, y)$ is a potential function for the differential $Pdx + Qdy$, then w is a harmonic conjugate of u . The potential function is known to be unique up to an additive constant; hence $w - v = C$, so v is unique up to an additive constant. ■

i Corollary 4.14: If $u(x, y)$ is a harmonic function on a simply connected domain D , then there exists a function $v(x, y)$ (a harmonic conjugate) such that $u + iv$ is analytic on the domain D .

✓ Proof: By the preceding theorem, there exists a harmonic conjugate v . The algorithm for finding v is simple: $v(x, y)$ is a potential function of the conservative field $\vec{F} = (-u_y, u_x)$. The functions u and v satisfy the Cauchy–Riemann equations at the point (x_0, y_0) . The differentiability of u and v at the point (x_0, y_0) follows from the continuity of the partial derivatives and an appropriate theorem from Calculus 2 and therefore, by Theorem 4.3, the function $u + iv$ is analytic. ■

i Corollary 4.15: If $u(x, y)$ is a harmonic function on a simply connected domain D , then there exists an analytic function $f: D \rightarrow \mathbb{C}$ whose real part is $u(x, y)$.

i Corollary 4.16: If $v(x, y)$ is a harmonic function on a simply connected domain D , then there exists an analytic function $f: D \rightarrow \mathbb{C}$ whose imaginary part is $v(x, y)$.

✓ Proof: This assertion is not symmetric with the preceding assertion because the conjugacy relation is not symmetric! By the preceding corollary, there exists an analytic function $g(z)$ on the domain D such that $\operatorname{Re}(g(z)) = v$. If we define $f(z) = ig(z)$, we obtain the required function. ■

➤ Another way to prove the uniqueness of the harmonic conjugate is as follows: if $w(x, y)$ is a harmonic conjugate of $u(x, y)$, then w must satisfy $w_y = u_x = v_y$ and $w_x = -u_y = v_x$. Therefore, $w_x = v_x$ and $w_y = v_y$, so v and w differ by a constant.

4.4 Conformal Mappings

- The present section is based on the subject of paths, and it is therefore advisable to review [the section on paths](#) in Chapter 2.
- The graph of a complex function is a four-dimensional manifold that cannot be represented simply by means of two-dimensional drawings. One of the customary ways to analyze and visualize analytic functions is by examining how they act on paths in the plane.
- The simplest paths are straight lines, particularly horizontal and vertical lines, from which planar grids are constructed. The way in which an analytic function acts on a grid is one of the simplest ways to visualize and understand the function.



❖ **Example 8:** On the left side of Figure 4.8 is a rectangular subdivision (a 15×15 grid) of the rectangle $D = [1, 4] \times [1, 4]$. Note the red color of the horizontal lines and the blue color of the vertical lines. On the right side of the figure is the image of the grid under the function $f(z) = \frac{1}{z}$. The shades of blue and red show where the horizontal and vertical lines of the grid are mapped.

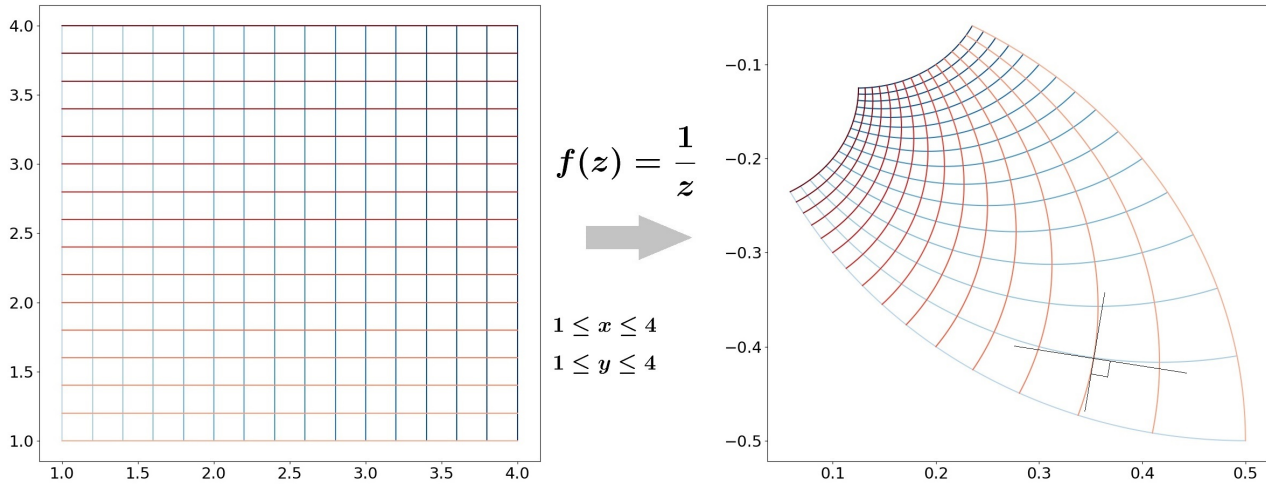


Fig. 4.8: The image of the rectangle $D = [1, 4] \times [1, 4]$ under the function $f(z) = \frac{1}{z}$

We mention in advance that, despite the distorting effect of the function, the angles between the lines are preserved under the mapping (90°). This is the fundamental property of a **conformal mapping**, which we study in this section. We begin by defining the basic concepts.

❗ **Definition 4.5:** A path $\sigma(t) = x(t) + iy(t)$, $t \in I$, is called **smooth** if the two real functions $x(t)$ and $y(t)$ are continuously differentiable at every interior point $t \in I$, and, in addition, $x'(t)$ and $y'(t)$ do not vanish simultaneously at any point.

➤ The last condition is sometimes expressed as $x'(t)^2 + y'(t)^2 > 0$ for every interior point $t \in I$, or simply as $\sigma'(t) \neq \vec{0}$, where $\sigma'(t) = x'(t) + iy'(t)$. Recall from Calculus 2 (or Infinitesimal Calculus 2) that the vector $\sigma'(t)$ is the velocity vector of a particle whose law of motion is given by $\sigma(t)$. Thus, a smooth path is one for which the direction of motion is clear at every instant, because the velocity vector exists and does not vanish. The vector $\sigma'(t)$ is also a direction vector of the line tangent to the path at the point t .

❗ **Definition 4.6:** Let

$$\sigma_1(t) = x_1(t) + iy_1(t), \quad t \in [a, b]$$

$$\sigma_2(t) = x_2(t) + iy_2(t), \quad t \in [b, c]$$

be two paths such that $\sigma_1(b) = \sigma_2(a)$ (the endpoint of σ_1 is the initial point of σ_2). The **concatenation** $\sigma: [a, c] \rightarrow \mathbb{C}$ of σ_1 and σ_2 is defined by

$$\sigma(t) = \begin{cases} \sigma_1(t), & a \leq t \leq b \\ \sigma_2(t), & b \leq t \leq c \end{cases}$$

and is denoted by $\sigma = \sigma_1 + \sigma_2$ (see Figure 4.9).

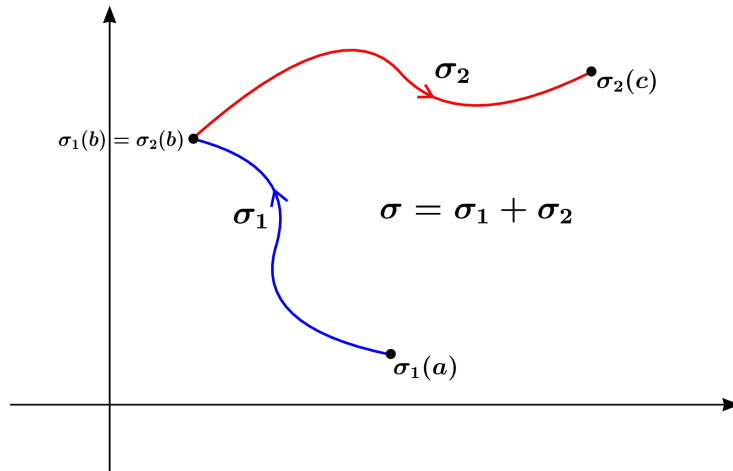


Fig. 4.9: The concatenation $\sigma = \sigma_1 + \sigma_2$ of two successive paths $\sigma_1: [a, b] \rightarrow \mathbb{C}$ and $\sigma_2: [b, c] \rightarrow \mathbb{C}$

i **Definition 4.7:** A path $\sigma(t) = x(t) + iy(t)$, $t \in I$, is called **piecewise smooth** if it can be represented as the concatenation of finitely many smooth paths:

$$\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_n$$

Equivalently, a path σ is piecewise smooth if σ is continuously differentiable except possibly at finitely many points t at which it is not differentiable or is differentiable but $\sigma'(t) = 0$. Most cases we encounter are better suited to this definition.

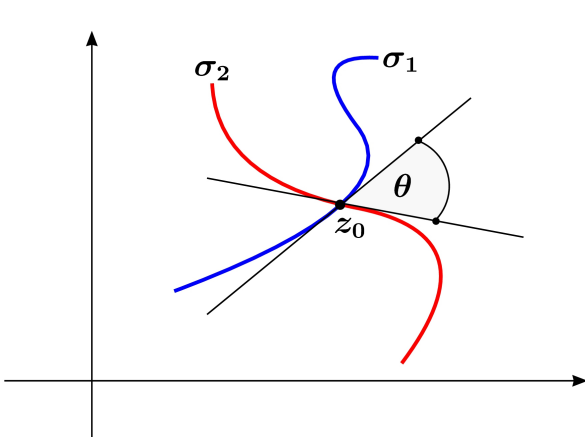


Fig. 4.10: An angle θ between two paths σ_1 and σ_2

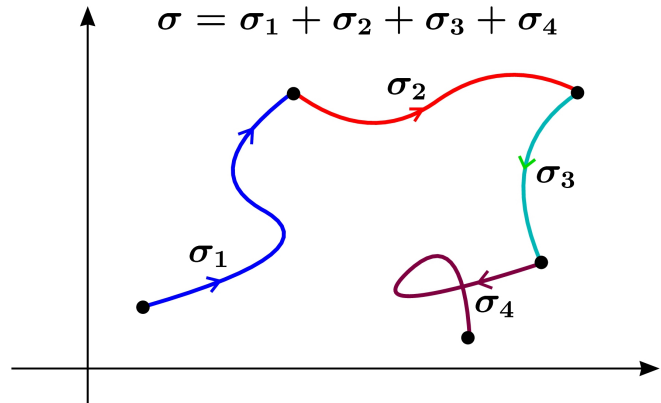


Fig. 4.11: A piecewise smooth path formed by concatenating four smooth paths

Example 9: The path $\sigma(t) = t^5 + t^3i$, $t \in [-1, 1]$, is not smooth even though it is continuously differentiable throughout the interval $(-1, 1)$, because its derivative $\sigma'(t) = 5t^4 + 3t^2i$ vanishes at the point $t = 0$. But if we divide the parameter domain into the two intervals $[-1, 0]$ and $[0, 1]$, we obtain two smooth paths σ_1 and σ_2 whose concatenation is σ .

❖ **Example 10:** The path $\sigma(t) = \sqrt[3]{t^5} + ti$ is smooth on the interval $[-1, 1]$ because the derivative $\sigma'(t) = \frac{5}{3}\sqrt[3]{t^2} + i$ is continuous throughout the interval $(-1, 1)$ and does not vanish at any point. But the curve corresponding to this path is identical to the curve in the preceding example! Thus, by using an improved parametrization, one can obtain a smooth parametric description of the curve. Consequently, smoothness is sometimes a property of the parametrization rather than of the curve being described!

❗ **Definition 4.8:** Let $\sigma(t) = x(t) + iy(t)$, $t \in I$, be a given path. The **reverse of σ** , denoted by $-\sigma$ and defined by $-\sigma(t) = x(a + b - t) + iy(a + b - t)$, is the same path as σ , except that the direction of traversal is reversed.

➤ The next step we need is to determine how a continuous function $f: S \rightarrow \mathbb{C}$ acts on a path $\sigma(t) = x(t) + iy(t)$ contained in its domain S . This operation is the composition of f with σ , denoted by $f \circ \sigma$ or $f(\sigma)$ and defined by

$$(f \circ \sigma)(t) = f(\sigma)(t) = f(x(t) + y(t)i)$$

The continuity of f guarantees that the composition $f(x(t) + y(t)i)$ produces a new path γ on the same interval I as σ . The following proposition gives a formula for computing the slope of the tangent line at every point $\gamma(t)$. This is a special case of the general chain rule for functions of several variables.

❗ **Proposition 4.17:** Let $f: D \rightarrow \mathbb{C}$ be an analytic function with a continuous derivative on a domain D , and let

$$\sigma(t) = x(t) + iy(t), \quad t \in I$$

be a differentiable path on a real interval I . Then the path $\gamma(t) = f(\sigma(t))$ is also differentiable on the interval I , and the following chain rule holds:

$$\gamma'(t) = f'(\sigma(t)) \cdot \sigma'(t) \tag{4.10}$$

✔ **Proof:** Write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. The continuous differentiability of f guarantees that $u(x, y)$ and $v(x, y)$ are continuously differentiable and satisfy the Cauchy–Riemann equations, and therefore the chain rule from Infinitesimal Calculus 2 may be applied to the functions $u(x(t), y(t))$ and $v(x(t), y(t))$.

$$\begin{aligned}
\gamma'(t) &= \frac{d}{dt} [u(x(t), y(t)) + iv(x(t), y(t))] \\
&= u_x x'(t) + u_y y'(t) + iv_x x'(t) + iv_y y'(t) \\
&\quad \downarrow \text{(CR)} \qquad \qquad \qquad \downarrow \text{(CR)} \\
&= u_x x'(t) - v_x y'(t) + iv_x x'(t) + iu_x y'(t) \\
&\quad \swarrow \qquad \qquad \searrow \qquad \qquad \swarrow \qquad \qquad \searrow \\
&= u_x x'(t) + iu_x y'(t) - v_x y'(t) + iv_x x'(t) \\
&\quad \swarrow \qquad \qquad \searrow \qquad \qquad \swarrow \qquad \qquad \searrow \\
&= u_x [x'(t) + iy'(t)] + iv_x [x'(t) + iy'(t)] \\
&= [u_x + iv_x] \cdot [x'(t) + iy'(t)] \\
&= [u_x(x(t), y(t)) + iv_x(x(t), y(t))] \cdot [x'(t) + iy'(t)] \\
&= f'(\sigma(t)) \cdot \sigma'(t)
\end{aligned}$$

We used the Cauchy–Riemann equations (CR) in passing from the second line to the third. ■

Remarks

1. Later, we prove (Cauchy's theorem) that if $f(z)$ is analytic on a domain D , then in fact $f(z)$ has derivatives of every order! That is, a function analytic on a domain is in fact infinitely differentiable on the same domain. Therefore, the continuity assumption on the derivative in the preceding proposition is unnecessary.
2. The continuity of the derivative $f'(z)$ is needed for applying the chain rule to $u(x(t), y(t))$ and $v(x(t), y(t))$.
3. The rule, of course, applies only to interior time points t of the interval I . If I is a closed interval $[a, b]$, then the chain rule we proved applies only at the points $a < t < b$.
4. For convenience and brevity in the preceding calculation, we wrote u_x instead of $u_x(x(t), y(t))$, and v_x instead of $v_x(x(t), y(t))$.

📌 Reminder from Calculus 2:

If $\sigma(t) = x(t) + iy(t)$ is a smooth path on a real interval I , then for every interior point $t_0 \in I$,

a. the vector $(x'(t_0), y'(t_0))$ is a tangent vector to the path σ at the point $\sigma(t_0) = (x(t_0), y(t_0))$.

b. The equation of the line tangent to the path at the point $\sigma(t_0) = (x(t_0), y(t_0))$ is given by the straight path

$$\ell(t) = \sigma(t_0) + \sigma'(t_0) \cdot t$$

$$\ell(t) = (x(t_0), y(t_0)) + t(x'(t_0), y'(t_0)) = x(t_0) + iy(t_0) + (x'(t_0) + iy'(t_0))t$$



c. The angle θ between the tangent line ℓ and the x -axis is given by the formula

$$\theta = \text{Arg}(\sigma'(t_0)) = \text{Arg}(x'(t_0) + iy'(t_0)) \quad (4.11)$$

Example: The path $\sigma_1(t) = t^5 + it^3$, $-1 < t < 1$, is smooth on the interval $(-1, 1)$ except at the point $t = 0$. The equation of the tangent line at every point $t_0 \neq 0$ is

$$\ell(t) = (t_0^5 + it_0^3) + (5t_0^4 + 3it_0^2)t$$

See Figure 4.13.

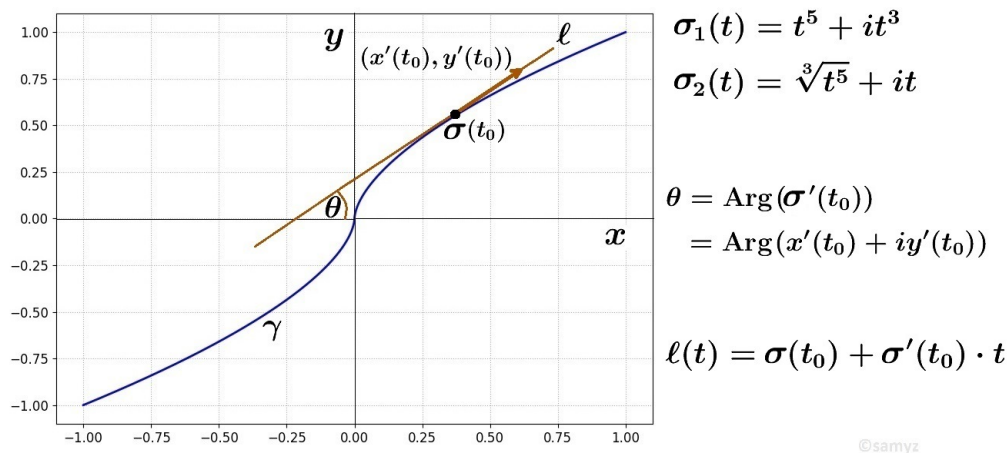


Fig. 4.13: A parametrization of the curve $y = \sqrt[5]{x^3}$: $\sigma_1(t) = (t^5, t^3)$; a smooth parametrization: $\sigma_2(t) = \sqrt[3]{t^5} + it$

The formula is not valid at the point $t_0 = 0$. An equivalent smooth parametrization can be found:

$$\begin{cases} \sigma_2(t) = \sqrt[3]{t^5} + it \\ -1 < t < 1 \end{cases}$$

It is easy to check that the two paths σ_1 and σ_2 are equivalent (they describe the same set of points) on the interval $(-1, 1)$. Additional background on paths may be found in the Calculus 2 notes (Chapter 3, Section 2), which may be downloaded from the following link:

Differential and Integral Calculus 2 – Lecture Notes

i Definition 4.9: The angle between two paths that intersect at a common point is defined to be the acute angle formed by the two tangents to the paths at their point of intersection.

Remark: See Figures 4.10 and 4.14 for a visual illustration.

➤ Denote by $\angle_{z_0}(\sigma_1, \sigma_2)$ the acute angle between two paths σ_1 and σ_2 at their point of intersection z_0 . Let t_1 and t_2 be the parameter values at which $z_0 = \sigma_1(t_1) = \sigma_2(t_2)$. Denote by ℓ_1 and ℓ_2 the lines tangent to the paths at the point z_0 (see Figure 4.14). The direction vectors of ℓ_1 and ℓ_2 are $\sigma'_1(t_1)$ and $\sigma'_2(t_2)$, respectively. The angle between the line ℓ_1 and

the positive real axis is $\text{Arg}(\sigma'_1(t_1))$, and the angle between the line ℓ_2 and the positive real axis is $\text{Arg}(\sigma'_2(t_2))$. Therefore, we obtain

$$\angle_{z_0}(\sigma_1, \sigma_2) = |\text{Arg}(\sigma'_2(t_2)) - \text{Arg}(\sigma'_1(t_1))| \mod \pi/2 \quad (4.12)$$

Remark: We reduced modulo $\frac{\pi}{2}$ in order to obtain the acute angle when the result is greater than $\frac{\pi}{2}$.

i Definition 4.10: A function $f(z)$ is called a **conformal mapping** on a domain $D \subseteq \mathbb{C}$ if it satisfies the following two conditions:

- a. f preserves angles between paths. That is, for any two smooth paths σ_1 and σ_2 in the domain D , and every point of intersection z_0 of the two paths, the following equality holds:

$$\angle_{z_0}(\sigma_1, \sigma_2) = \angle_{f(z_0)}(f(\sigma_1), f(\sigma_2))$$

- b. For every point z_0 in the domain D , the limit $\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$ exists.

¹ See Figure 4.14 for a visual illustration.

² If the function $f(z)$ is analytic, then the second condition holds automatically.

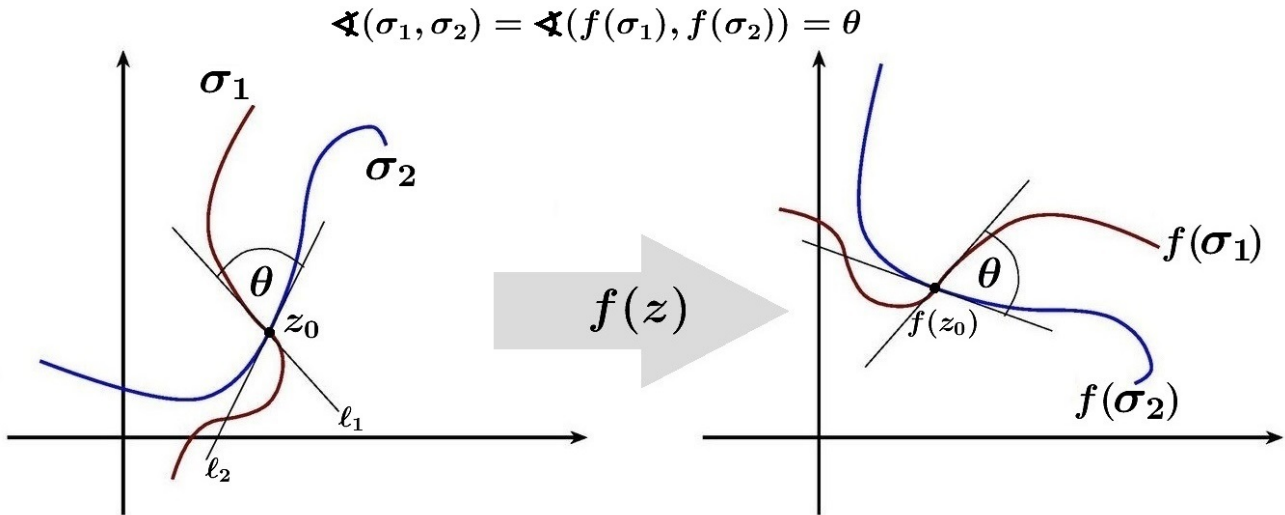


Fig. 4.14: A conformal mapping $f(z)$ preserves the angle between two paths σ_1 and σ_2

Example 11: If we restrict ourselves to straight paths only, Figure 4.15 gives the impression that the function $f(z) = \sin z$ preserves the right angles between the grid lines shown on the left side of the figure. Recall that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

- o A parametrization of a horizontal line is $\sigma_1(t) = t + y_0i$, $-\infty < t < \infty$.

- A parametrization of a vertical line is $\sigma_2(t) = x_0 + ti$, $-\infty < t < \infty$.
- The paths $\sigma_1(t)$ and $\sigma_2(t)$ intersect at the point $z_0 = x_0 + iy_0$.

We begin by examining the image of the horizontal lines (red):

$$\begin{aligned}
 f(\sigma_1(t)) &= \frac{e^{i\sigma_1(t)} - e^{-i\sigma_1(t)}}{2i} = \frac{e^{it-y_0} - e^{-it+y_0}}{2i} \\
 &= \frac{e^{-y_0}(\cos t + i \sin t) - e^{y_0}(\cos t - i \sin t)}{2i} = \frac{e^{y_0} + e^{-y_0}}{2} \sin t + i \cdot \frac{e^{y_0} - e^{-y_0}}{2} \cos t \\
 &= \cosh(y_0) \sin t + i \sinh(y_0) \cos t
 \end{aligned}$$

We have obtained an ellipse whose semiaxes are $a = \cosh(y_0)$ and $b = \sinh(y_0)$.

Similarly, examining the image of the vertical lines (blue) gives

$$f(\sigma_2(t)) = \sin(x_0) \cosh(t) + i \cos(x_0) \sinh(t)$$

This is a hyperbola⁴ whose semimajor axis is a and whose focal distance is 1.

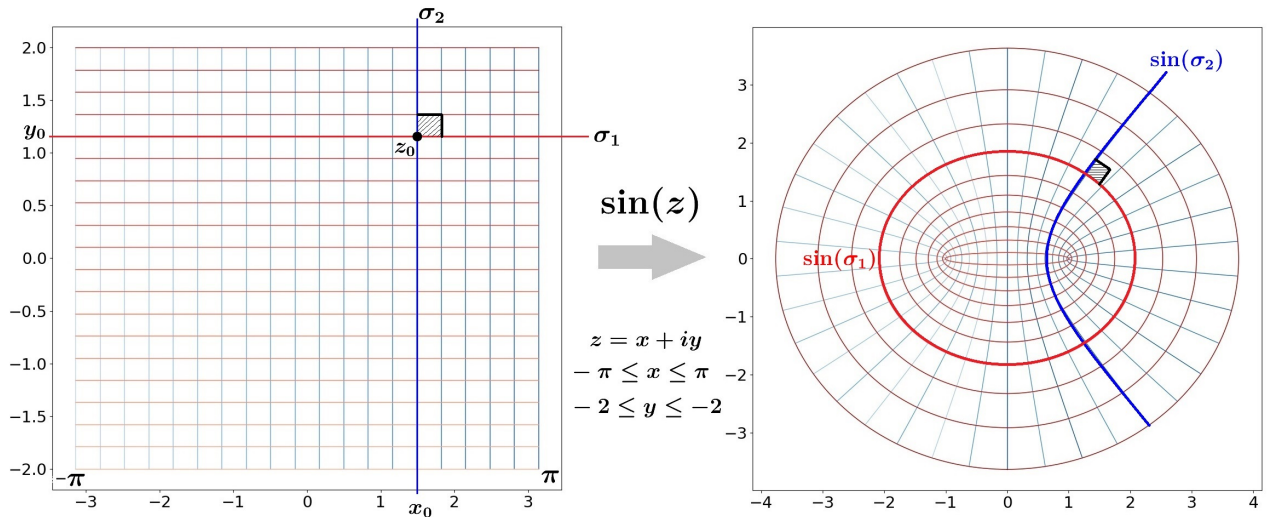


Fig. 4.15: The image of the rectangle $D = [-\pi, \pi] \times [-2, 2]$ under the function $f(z) = \sin(z)$

Computing the angle $\angle_{z_0}(\sigma_1, \sigma_2) = \frac{\pi}{2}$ is straightforward. To compute the angle between the paths $f(\sigma_1)$ and $f(\sigma_2)$, we must calculate the corresponding derivatives:

$$\begin{aligned}
 [f(\sigma_1(t))]' &= \cosh(y_0) \cos t - i \sinh(y_0) \sin t \\
 [f(\sigma_2(t))]' &= \sin(x_0) \sinh(t) + i \cos(x_0) \cosh(t)
 \end{aligned}$$

The paths σ_1 and σ_2 intersect at the point $z_0 = x_0 + iy_0$, and therefore we substitute $t = x_0$

⁴ If we set $u = \sin(x_0) \cosh(t)$, $v = \cos(x_0) \sinh(t)$, $a = \sin(x_0)$ and $b = \cos(x_0)$, then it is easy to verify that $\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1$. The semimajor axis of the hyperbola is $\sin(x_0)$, and its focal distance is $\sqrt{a^2 + b^2} = 1$.

in the first equation and $t = y_0$ in the second:

$$\begin{aligned} [f(\sigma_1(x_0))]' &= \cosh(y_0) \cos x_0 - i \sinh(y_0) \sin x_0 \\ [f(\sigma_2(y_0))]' &= \sin(x_0) \sinh(y_0) + i \cos(x_0) \cosh(y_0) \end{aligned}$$

It is easy to check that the inner product (not the complex product!) of these two vectors is zero. That is, the two vectors are perpendicular, and hence

$$\angle_{z_0}(f(\sigma_1), f(\sigma_2)) = \frac{\pi}{2}$$

We have therefore proved that f preserves right angles between horizontal and vertical lines, but this is still far from proving that f is a conformal mapping! For this purpose, we must show that f preserves angles of every size between two general paths that are not necessarily straight.

i Proposition 4.18: Let $f(z)$ be a function analytic at the point z_0 . If $f'(z_0) \neq 0$, then f preserves the angle between any two smooth paths passing through z_0 .

✓ Proof: Let $\sigma_1: I_1 \rightarrow \mathbb{C}$ and $\sigma_2: I_2 \rightarrow \mathbb{C}$ be two smooth paths intersecting at the point z_0 , and let $\gamma_1 = f(\sigma_1)$, $\gamma_2 = f(\sigma_2)$, be the images of σ_1 and σ_2 under the function f . The assumption $f'(z_0) \neq 0$ guarantees that the paths γ_1 and γ_2 are smooth in a neighborhood of $f(z_0)$. By formula (4.10),

$$\begin{aligned} \gamma_1'(t) &= f'(\sigma_1(t)) \cdot \sigma_1'(t) \\ \gamma_2'(t) &= f'(\sigma_2(t)) \cdot \sigma_2'(t) \end{aligned}$$

By formula (??), $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, and hence

$$\begin{aligned} \arg(\gamma_1'(t)) &= \arg(f'(\sigma_1(t))) + \arg(\sigma_1'(t)) \\ \arg(\gamma_2'(t)) &= \arg(f'(\sigma_2(t))) + \arg(\sigma_2'(t)) \end{aligned}$$

Let $t_1 \in I_1$ and $t_2 \in I_2$ be the parameter values at which $z_0 = \sigma_1(t_1) = \sigma_2(t_2)$. Then

$$\begin{aligned} \arg(\gamma_1'(t_1)) &= \arg(f'(\sigma_1(t_1))) + \arg(\sigma_1'(t_1)) = \arg(f'(z_0)) + \arg(\sigma_1'(t_1)) \\ \arg(\gamma_2'(t_2)) &= \arg(f'(\sigma_2(t_2))) + \arg(\sigma_2'(t_2)) = \arg(f'(z_0)) + \arg(\sigma_2'(t_2)) \end{aligned}$$

Therefore,

$$\text{Arg}(\gamma_1'(t_1)) - \text{Arg}(\gamma_2'(t_2)) = \text{Arg}(\sigma_1'(t_1)) - \text{Arg}(\sigma_2'(t_2))$$



By formula (4.12),

$$\begin{aligned}\angle_{z_0}(\sigma_1, \sigma_2) &= \left| \operatorname{Arg}(\sigma'_2(t_2)) - \operatorname{Arg}(\sigma'_1(t_1)) \right| \bmod \pi/2 \\ &= \left| \operatorname{Arg}(\gamma'_2(t_2)) - \operatorname{Arg}(\gamma'_1(t_1)) \right| \bmod \pi/2 \\ &= \angle_{f(z_0)}(\gamma_1, \gamma_2)\end{aligned}$$

We have shown that f preserves angles between paths intersecting at the point z_0 . ■

❖ **Example 12:** The condition $f'(z) \neq 0$ in Proposition 4.18 is necessary, as the following example shows. The derivative of $f(z) = z^3$ vanishes at the point $z_0 = 0$. Figure 4.16 shows two straight paths $\sigma_1(t) = t$, $\sigma_2(t) = \sqrt{3}t + ti$, the angle between which is $\frac{\pi}{6}$, whereas the angle between their images expands to $\frac{\pi}{2}$.

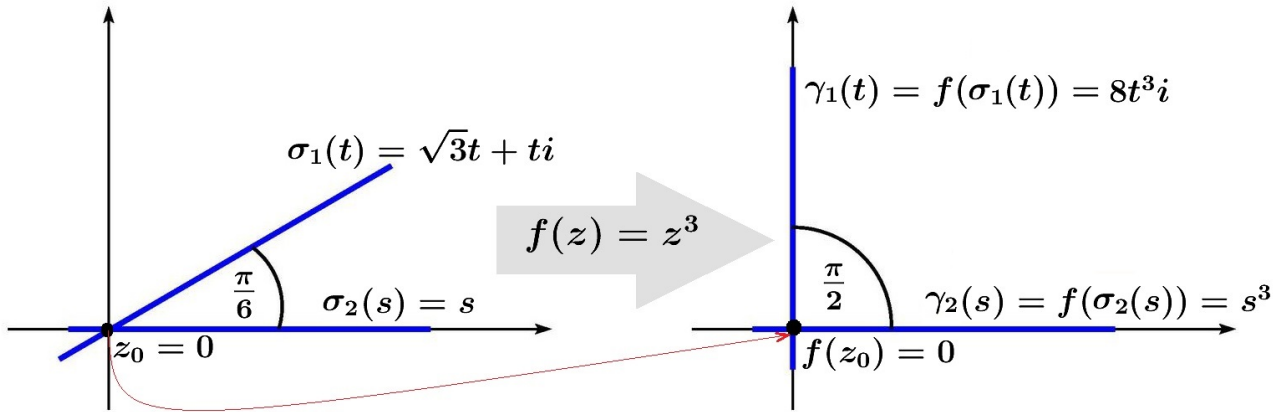


Fig. 4.16: The angle between the paths $\gamma_1(t) = \sqrt{3}t + ti$ and $\gamma_2(s) = s$ at the point $z_0 = 0$ is $\theta = \frac{\pi}{6}$. The angle between the paths $f(\gamma_1)$ and $f(\gamma_2)$ is $\frac{\pi}{2}$. The angle is not preserved under the mapping $f(z) = z^3$ because the function $f(z)$ is not conformal at the point $z_0 = 0$ ($f'(z_0) = 0$).

❖ The following theorem is an immediate consequence of Proposition 4.18.

❗ **Theorem 4.19:** If $f(z)$ is analytic on a domain D , and if $f'(z) \neq 0$ for every $z \in D$, then f is a conformal mapping on D .

✓ **Proof:** Recall that, to prove conformality, Definition 4.10 requires us to prove two things: preservation of angles between paths, and existence of the limit $\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$. Preservation of angles between paths is an immediate consequence of Proposition 4.18. The existence of the limit follows from the analyticity of f :

$$\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \lim_{z \rightarrow z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \left| \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \right| = |f'(z_0)|$$

In the last step, we used Corollary 3.11. ■

❗ In many contexts, an angle also has an orientation. We note that a conformal mapping may reverse the orientation of the angle between paths σ_1 and σ_2 , as shown in Figure 4.17.

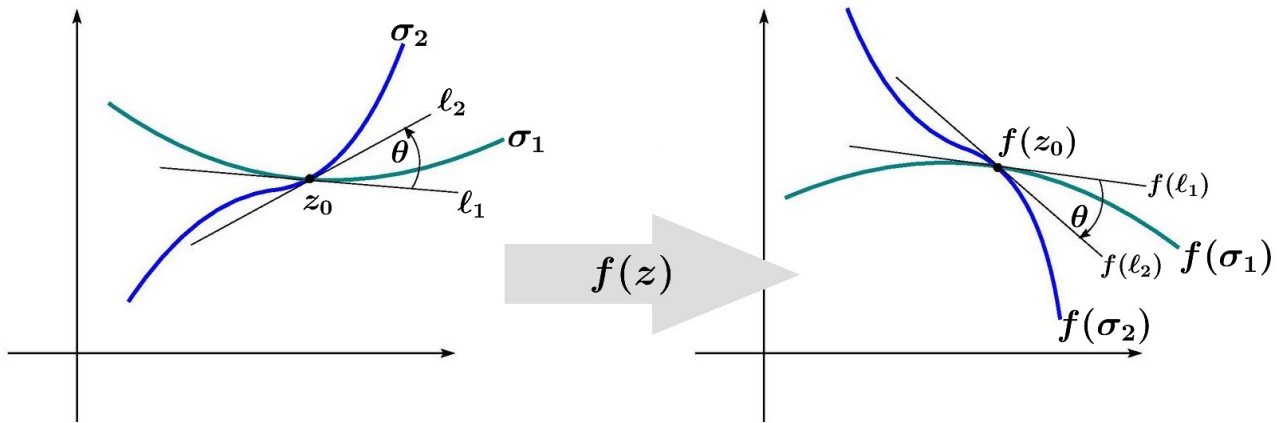


Fig. 4.17: Under a conformal mapping $f(z)$, the orientation of the angle between paths σ_1 and σ_2 is reversed (but the mapping still preserves the angle!)

Existence of the Limit $\lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$

- ❶ The first condition for conformality has been discussed and illustrated extensively. The second condition, which may also be called “the existence of a local stretching/contraction factor”, is related to the earliest uses of conformal mappings in cartography and graphics. It resembles the definition of the derivative of the function $f(z)$ at the point $f(z_0)$, but is in fact a weaker condition: the existence of condition b does not imply differentiability at the point z_0 .
- ❷ The geometric meaning of this condition concerns the stretching (or contraction) properties of the function. The ratio $\frac{|f(z) - f(z_0)|}{|z - z_0|}$ is the stretching/contraction factor between the size of the original displacement $z - z_0$ and that of its image $f(z) - f(z_0)$. This condition requires that this factor tend to a constant as the segment $z - z_0$ becomes smaller. The following diagram illustrates what happens to a family of concentric circles about the point z_0 under a conformal mapping.

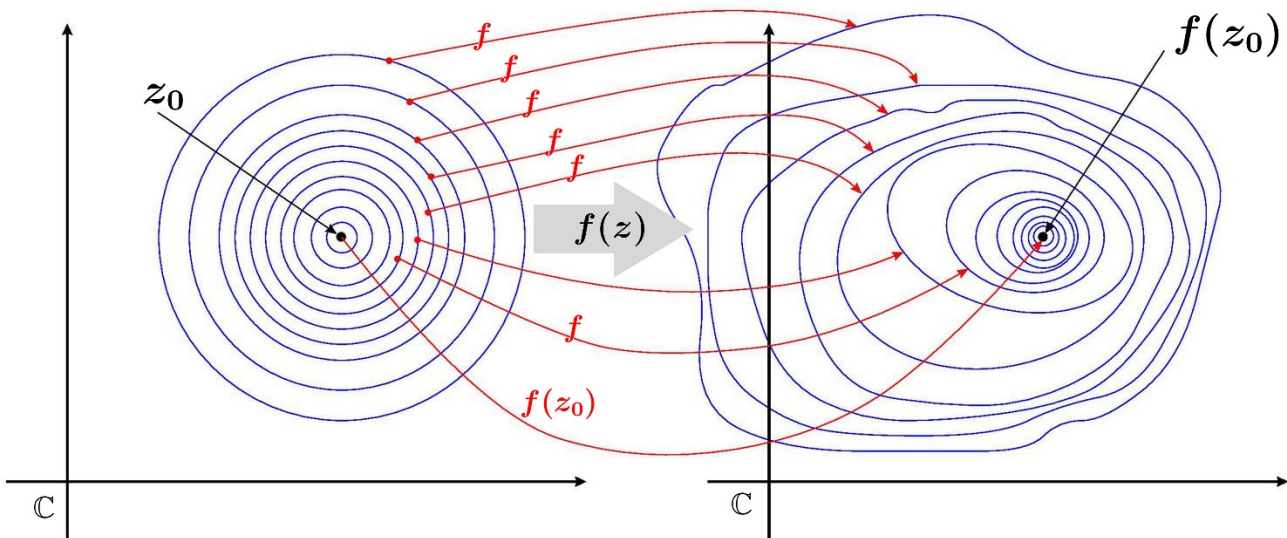


Fig. 4.18: Under a conformal mapping $f(z)$, a family of concentric circles is mapped to a family of closed paths about $f(z_0)$, with the paths becoming increasingly circular near $f(z_0)$

As the diagram suggests, the stretching/contraction action of $f(z)$ becomes increasingly uni-

form in all directions about the point z_0 , and therefore small circles about z_0 are mapped approximately to circles (possibly stretched or contracted) about $f(z_0)$.

❖ **Exercise 4.6:** By Theorem 4.19, the mapping $f(z) = e^z$ is conformal on the entire complex plane $\mathbb{C}$ because $f'(z) = e^z \neq 0$ for every $z \in \mathbb{C}$. Examine Figure 4.19 and prove that $f(z)$ maps vertical lines to circles centered at the origin and maps horizontal lines to straight lines passing through the origin. Note the shades of red and blue in order to see how the rectangle on the left is mapped to its image on the right.

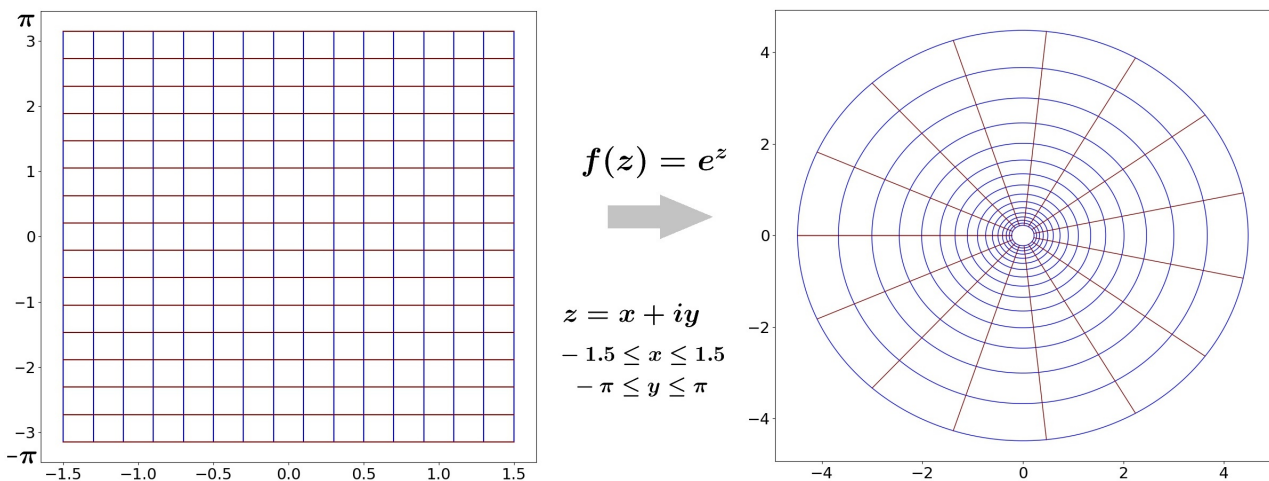


Fig. 4.19: The image of the rectangle $D = [-1.5, 1.5] \times [-\pi, \pi]$ under the function $f(z) = e^z$

4.5 The Riemann Sphere

- ❖ In the following section, we introduce Möbius transformations, an important and useful family of analytic functions. For this purpose, we need to understand an extended model of the set of complex numbers called the **The Riemann Sphere**. This model is based on the topological equivalence between the surface of the three-dimensional sphere (called a the **Riemann Sphere**) and the complex plane with one additional point adjoined: ∞ .
- ❖ Before introducing the Riemann sphere, we present the two-dimensional counterpart of the idea through the topological equivalence between the punctured unit circle and the set of real numbers $\mathbb{R}$.
- ❖ The punctured circle is defined to be the unit circle $x^2 + y^2 = 1$ with the point $(0, 1)$ removed (which is sometimes called the "north pole")

$$C = \{ (x, y) : x^2 + y^2 = 1 \} - \{(0, 1)\}$$

We denote the north pole $(0, 1)$ by N . To every point $A(x, y)$ on the punctured circle C , we associate a point a on the real axis as follows: draw a straight line through the two points N

and (x, y) . Clearly, the resulting line must intersect the real axis at a real point a .

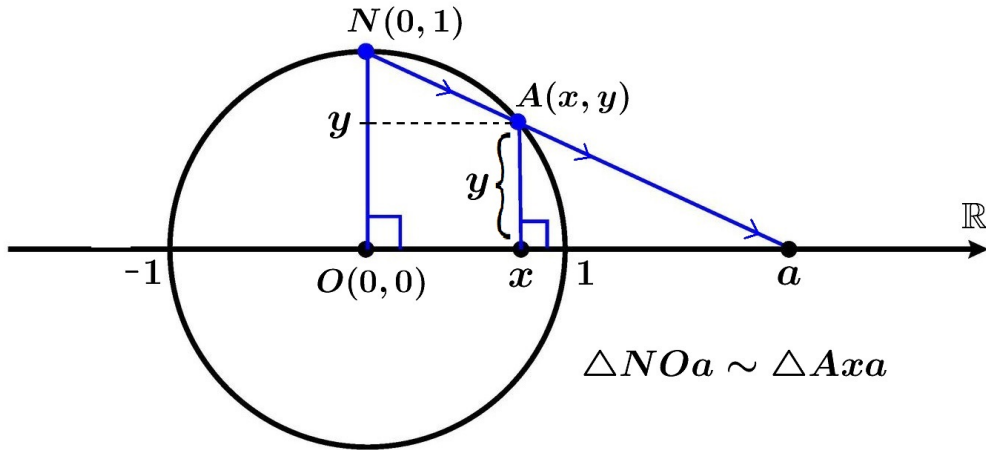


Fig. 4.20: A topological equivalence between a punctured circle and the real line

➤ Geometric considerations involving two similar triangles give

$$\frac{y}{a-x} = \frac{1}{a}$$

and from this we obtain our smooth isomorphism (a differentiable, one-to-one, and onto mapping) from the circle C to the real line $\mathbb{R}$:

$$a = \frac{x}{1-y}$$

➤ Note that the lower semicircle is mapped to the closed interval $[-1, 1]$, whereas the upper semicircle (without the pole $(0, 1)$!) is mapped to all the remaining points of the real line (see Figure 4.21).

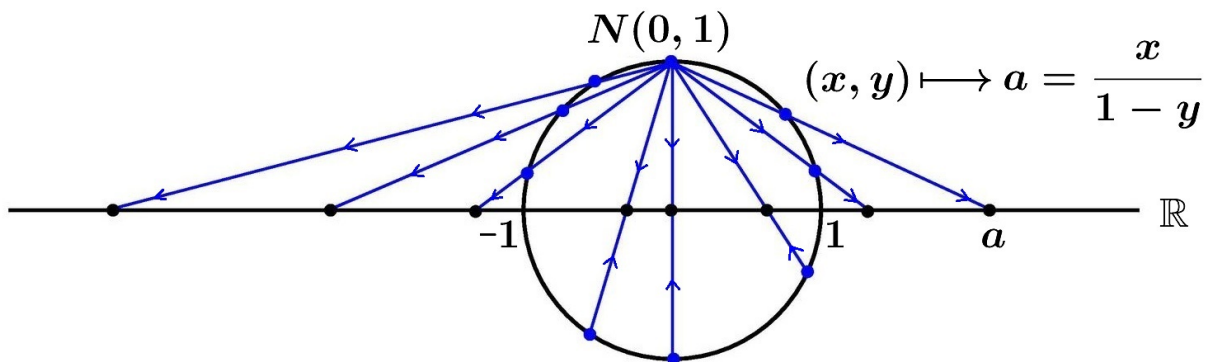


Fig. 4.21: A topological equivalence between a punctured disk and the real line

➤ The inverse mapping from the set of real numbers $\mathbb{R}$ to the punctured circle C is obtained by solving the system

$$\begin{cases} a = \frac{x}{1-y} \\ x^2 + y^2 = 1 \end{cases}$$

The first equation gives $x = a(1 - y)$, and substituting this into the second equation yields a

quadratic equation in the variable y :

$$(a^2 + 1)y^2 - 2a^2y + (a^2 - 1) = 0$$

its solution is $y = \frac{a^2 - 1}{a^2 + 1}$, and therefore every $a \in \mathbb{R}$ corresponds to a point $(x, y) \in C$:

$$(x, y) = \left(\frac{2a}{a^2 + 1}, \frac{a^2 - 1}{a^2 + 1} \right)$$

It is not difficult to check that this mapping is one-to-one and onto.

- Similar geometric considerations lead to the idea of the equivalence between the Riemann sphere (Riemann's Sphere) and the complex plane $\mathbb{C}$.

i Definition 4.11: The **Riemann Sphere** $\mathbb{S}$ is the surface of the unit sphere in the space $\mathbb{R}^3$:

$$\mathbb{S} = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \}$$

The point $N(0, 0, 1)$ is called the **north pole** of the sphere.

- For every point $(x, y, z) \in \mathbb{S}$ distinct from N , draw the line through N and the point (x, y, z) until it meets the xy -plane (that is, $\mathbb{C}$) at the point $a + bi$. See Figure 4.22.

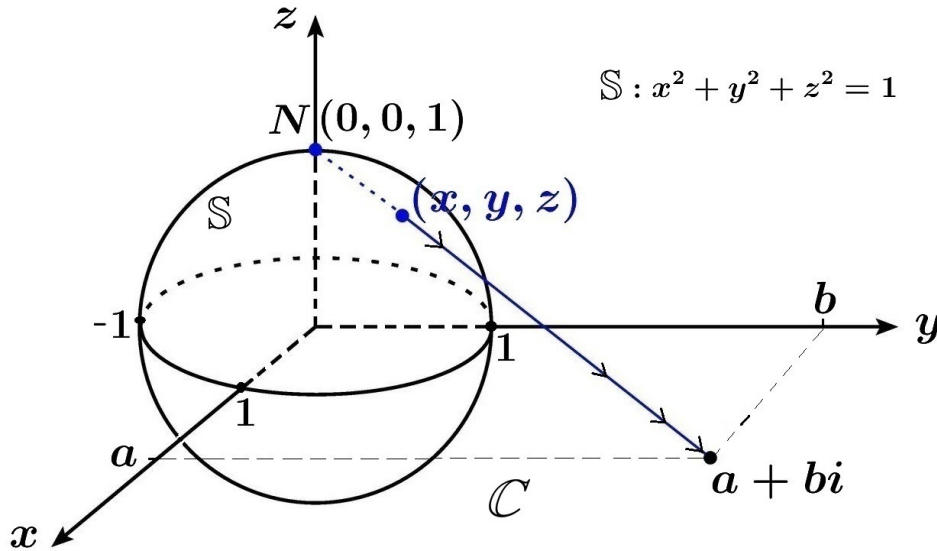


Fig. 4.22: Stereographic projection from the Riemann sphere to the complex plane

If we consider the projections onto the xz -plane and the yz -plane (as shown in Figure 4.23), we obtain the formulas appearing in Figure 4.23:

from which the following pair of equalities is readily obtained:

$$a = \frac{x}{1 - z}, \quad b = \frac{y}{1 - z}$$

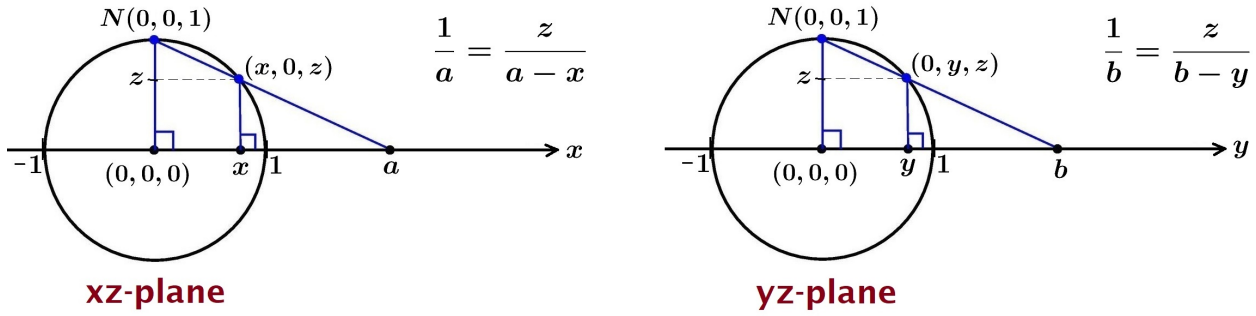


Fig. 4.23: The projections of the preceding geometric configuration onto the xz -plane and the yz -plane

In other words, the mapping

$$a + bi = \frac{x}{1 - z} + \frac{y}{1 - z}i$$

from the punctured sphere

$$\dot{\mathbb{S}} = \{ (x, y, z) : x^2 + y^2 + z^2 = 1 \} - \{ (0, 0, 1) \}$$

to the complex plane $\mathbb{C}$ is one-to-one and onto. Therefore, $\dot{\mathbb{S}}$ is another model of the set of complex numbers. This mapping is also called the **stereographic projection**.

- The inverse mapping from the complex plane $\mathbb{C}$ to the punctured sphere $\dot{\mathbb{S}}$ is obtained by solving the system

$$\begin{cases} a = \frac{x}{1 - z} \\ b = \frac{y}{1 - z} \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

The first two equations allow us to express the variables x and y in terms of z :

$$\begin{cases} x = a(1 - z) \\ y = b(1 - z) \end{cases}$$

Substituting these into the third equality yields a quadratic equation in the variable z :

$$(a^2 + b^2 + 1)z^2 - 2(a^2 + b^2)z + (a^2 + b^2 - 1) = 0$$

its solution is $z = \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1}$, and therefore every $a + bi \in \mathbb{C}$ corresponds to a point (x, y, z) on the sphere $\mathbb{S}$:

$$(x, y, z) = \left(\frac{2a}{a^2 + b^2 + 1}, \frac{2b}{a^2 + b^2 + 1}, \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1} \right)$$

Again, it is not difficult to check that this mapping is one-to-one, onto, and continuously differ-

entiable.

- We have thus proved the topological equivalence between $\mathbb{C}$ and the punctured Riemann sphere. If we add to $\dot{\mathbb{S}}$ the north pole $N(0, 0, 1)$ and to the complex plane the point at infinity ∞ , we obtain a topological equivalence between the **extended complex plane**

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$$

and the unit sphere $\mathbb{S}$.

- The correspondence $T: \mathbb{S} \rightarrow \mathbb{C}_\infty$ between the sphere and the extended complex plane is

$$T(x, y, z) = \begin{cases} \frac{x}{1-z} + \frac{y}{1-z}i, & z < 1 \\ \infty, & z = 1 \end{cases}$$

The inverse correspondence $T^{-1}: \mathbb{C}_\infty \rightarrow \mathbb{S}$ from the extended complex plane to the sphere is

$$\begin{cases} T^{-1}(a + bi) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1} \right) \\ T^{-1}(\infty) = N(0, 0, 1) \end{cases}$$

Thus, after the point ∞ is added to the complex plane, the plane becomes a sphere (more precisely, the surface of a sphere).

- Every new model enables us to learn new things about the field of complex numbers. The Riemann-sphere model reveals interesting connections between circles and straight lines. We examine how the topological mapping between the Riemann sphere and the complex plane acts on various sets. For example: what is the image in the plane $\mathbb{C}_\infty$ of a circle on the sphere? What is the image on the Riemann sphere of a straight line in the plane $\mathbb{C}_\infty$?
- A circle on the Riemann sphere is obtained by intersecting a plane with the sphere:

$$\begin{cases} Ax + By + Cz = D \\ x^2 + y^2 + z^2 = 1 \end{cases} \quad (4.13)$$

Denote by σ the set of points (x, y, z) satisfying the system (4.13). Then a complex number $a + bi$ belongs to the inverse image $T^{-1}[\sigma]$ if and only if its inverse image

$$T^{-1}(a + bi) = \left(\frac{2a}{a^2 + b^2 + 1}, \frac{2b}{a^2 + b^2 + 1}, \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1} \right)$$



satisfies the system (4.13). That is,

$$\frac{2Aa}{a^2 + b^2 + 1} + \frac{2Bb}{a^2 + b^2 + 1} + \frac{C(a^2 + b^2 - 1)}{a^2 + b^2 + 1} = D$$

Multiplying both sides by $a^2 + b^2 + 1$ and collecting terms gives

$$(C - D)(a^2 + b^2) + 2Aa + 2Bb = C + D \quad (4.14)$$

We consider the following possibilities:

1. If $C = D$, then equation (4.14) becomes the general equation of a straight line in the complex plane:

$$Aa + Bb = 2C, \quad (a + bi \in \mathbb{C}) \quad (4.15)$$

Note that the condition $C = D$ implies that the north pole $N(0, 0, 1)$ satisfies the equation of the circle (4.13)! Thus, every circle on the Riemann sphere passing through N becomes a straight line in the complex plane. In the extended plane, the image of the circle becomes a line passing through the origin and also containing the point ∞ (the image of the pole $N(0, 0, 1)$).

2. If $C \neq D$, then equation (4.14) is the equation of a circle in the complex plane. In this case, it is clear that the circle (4.13) does not pass through the pole $N(0, 0, 1)$. Thus, every circle on the Riemann sphere that does not pass through the pole N becomes a circle in the complex plane (which, of course, does not contain the point ∞).
3. If $C = D = 0$, then equation (4.14) becomes the equation of a straight line passing through the origin:

$$Aa + Bb = 0, \quad (a + bi \in \mathbb{C}) \quad (4.16)$$

The condition $C = D = 0$ implies that the north and south poles satisfy the equation of the circle (4.13)! Thus, every circle on the Riemann sphere passing through both poles becomes a straight line in the complex plane passing through the origin (including ∞).

- In both cases, we found that circles on the Riemann sphere are mapped to circles or straight lines in the complex plane, and therefore a straight line in the complex plane is sometimes regarded as a **circle in the broad sense** or a **generalized circle**. We summarize these results in the following proposition.

i **Definition 4.12:** A **generalized line** in the extended plane is any ordinary straight line with the point at infinity ∞ adjoined. A **generalized circle** in the extended plane is any ordinary circle or generalized line.

- ⚠ Note that Definition 4.12 implies that every generalized line in $\mathbb{C}_\infty$ is also a generalized circle! One may imagine the two rays of the line meeting abstractly at the point at infinity and forming



an infinite circle. But it is more accurate to understand this topologically: applying T^{-1} to a generalized line produces a circle on the Riemann sphere that passes through the north pole. As noted above, the north pole is the point on the sphere corresponding to the point at infinity of $\mathbb{C}_\infty$.

- i Proposition 4.20:** With respect to the topological equivalence $T: \mathbb{S} \rightarrow \mathbb{C}_\infty$, $T^{-1}: \mathbb{C}_\infty \rightarrow \mathbb{S}$
- a. Every circle on the sphere passing through both poles is mapped to a generalized line in $\mathbb{C}_\infty$ passing through the origin.
 - b. Every circle on the sphere passing through the north pole is mapped to a generalized line in $\mathbb{C}_\infty$.
 - c. Every circle on the sphere that does not pass through the north pole is mapped to an ordinary circle in $\mathbb{C}_\infty$.
 - d. Each of the preceding assertions is also true in the reverse direction.

4.6 Möbius Transformations (Möbius)

- Now that we have studied the Riemann sphere, we can proceed to Möbius transformations, named after the mathematician **August Ferdinand Möbius**.



WIKIPEDIA **August Ferdinand Möbius 1790-1868**



- Möbius transformations have important applications in image processing and cartography. The best way to understand them is to define them on the extended complex plane $\mathbb{C}_\infty$.

- i Definition 4.13:** The general form of a Möbius transformation $T: \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is

$$T(z) = \frac{az + b}{cz + d}, \quad z \neq -\frac{d}{c}, \infty$$

where $a, b, c, d \in \mathbb{C}$ are complex constants satisfying $ad \neq bc$.

- a. If $c = 0$, then T is a linear polynomial on $\mathbb{C}$, and at the point at infinity $z = \infty$ we define

$$T(\infty) = \lim_{z \rightarrow \infty} T(z) = \infty$$

- b. If $c \neq 0$, then at the point $z = -\frac{d}{c}$ the denominator vanishes, and therefore we define separately

$$T\left(-\frac{d}{c}\right) = \lim_{z \rightarrow -\frac{d}{c}} T(z) = \infty$$



and at the point $z = \infty$, the definition is

$$T(\infty) = \lim_{z \rightarrow \infty} T(z) = \frac{a}{c}$$

- The definitions of T at the point $z = -\frac{d}{c}$ and at the point $z = \infty$ agree with the definitions of infinite limits surveyed at the end of Chapter 3. See Definitions 3.6 and 3.8 on pages 103-101.
- If $c \neq 0$, then T is a rational function (a quotient of linear polynomials). If $c = 0$, then T is a linear polynomial. We already computed the limits in the definition in Propositions 3.18 and 3.19 at the end of Chapter 3.
- The condition $ad \neq bc$ guarantees that this is not a constant mapping (which is not of interest)⁵. Another reason for this condition may be seen in the proof of the following proposition.

i Proposition 4.21: Every Möbius transformation T is a conformal mapping on $\mathbb{C}$.

✓ **Proof:** For every $z \in \mathbb{C}$,

$$T'(z) = \frac{a(cz + d) - (az + b)c}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} \neq 0$$

We have obtained $T'(z) \neq 0$ for every $z \in \mathbb{C}$. Therefore, by Theorem 4.19, T is a conformal mapping on $\mathbb{C}$. ■

- Multiplying the constants a , b , c , and d by a constant $r \neq 0$ produces the same Möbius transformation:

$$\frac{az + b}{cz + d} \equiv \frac{r(az + b)}{r(cz + d)}$$

It is therefore customary to "normalize" a Möbius transformation by imposing the requirement $ad - bc = 1$. A Möbius transformation whose coefficients satisfy $ad - bc = 1$ is called **normalized** (or said to have **normal form**).

i Proposition 4.22: Every Möbius transformation T is a one-to-one and onto mapping from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$, and the inverse mapping T^{-1} is also a Möbius transformation.

- ✓ **Proof:** Let $z \in \mathbb{C}$. Solving for the unknown w in the equality $z = \frac{aw+b}{cw+d}$ shows that every Möbius transformation $T(z) = \frac{az+b}{cz+d}$ has an inverse mapping that is also a Möbius transformation:

$$T^{-1}(z) = \frac{dz - b}{-cz + a} \tag{4.17}$$

⁵ This occurs when $\frac{az+b}{cz+d} \equiv \lambda$. Therefore, $az + b = \lambda(cz + d) = \lambda cz + \lambda d$. This is an equality between two polynomials of degree 1, so their coefficients must be equal: $a = \lambda c$, $b = \lambda d$, that is, $\frac{a}{c} = \frac{b}{d}$, and hence $ad = bc$.



If $c \neq 0$, then $T^{-1}(a/c) = \infty$ and $T^{-1}(\infty) = -d/c$. If $c = 0$, then $T^{-1}(\infty) = \infty$. We leave the details to the student. ■

- A conformal mapping that is one-to-one and onto from a space to itself is called an **automorphism**. The preceding two propositions imply that every Möbius transformation is an automorphism of $\mathbb{C}_\infty$.

i Proposition 4.23: The composition of two Möbius transformations is a Möbius transformation.

✓ **Proof:** Let

$$T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, \quad T_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

be two given Möbius transformations.

$$\begin{aligned} T_1(T_2(z)) &= \frac{a_1 \cdot \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \cdot \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} = \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2)z + a_1 b_2 + b_1 d_2}{(c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2} \end{aligned}$$

The last expression is clearly of the form of a Möbius transformation. We leave it for the student to verify that $T_1(T_2)$ takes the correct values at the points $z = -\frac{d_2}{c_2}$ and $z = \infty$. ■

- The result of the composition $T_1(T_2)$ looks suspiciously like matrix multiplication:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}$$

In addition, the inverse matrix of the coefficients of a Möbius transformation,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

yields the coefficients of the inverse Möbius transformation (whose determinant is 1).

- Denote by **Auto**($\mathbb{C}_\infty$) the set of all Möbius transformations on $\mathbb{C}_\infty$. It is not difficult to prove that this set has the algebraic structure of a group under composition and inversion. The identity element of **Auto**($\mathbb{C}_\infty$) is the identity mapping $T(z)$, which is, of course, the Möbius transformation with coefficients $a = 1$, $b = c = 0$, and $d = 1$, or, in matrix form, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

- ❗ A reminder of the concept of a group as an algebraic structure is shown in Figure 4.24 (click the figure to open the corresponding Wikipedia page).



חבורה G היא מבנה אלגברי בסיסי הכולל קבוצה עם פעולה בינארית $\cdot$ ("סגורה": לכל $a, b \in G$ מתקיים $a \cdot b \in G$), אשר מקיימת את התכונות הבאות:

- **אסוציאטיביות** (קיבוציות): לכל $a, b, c \in G$ מתקיים $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- **קיום איבר יחידה**: קיים איבר $e \in G$ כך שלכל $a \in G$ מתקיים $a \cdot e = e \cdot a = a$.
- **קיום איבר הופכי**: לכל $a \in G$ קיים $b \in G$ כך ש- $a \cdot b = b \cdot a = e$.

(מהאקסיומות נובע שיש רק איבר יחידה אחד, ושלכל איבר יש הפכי אחד).

חבורה אבלית (חילופית) היא חבורה שבה מתקיים, בנוסף, תנאי הקומוטטיביות (חילופיות) לכל $a, b \in G$: $a \cdot b = b \cdot a$.

Fig. 4.24: Definition of a group as an algebraic structure (Wikipedia)

- ❖ If each Möbius transformation $T(z) = \frac{az+b}{cz+d}$ with normalized coefficients ($ad - bc = 1$) is associated with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we obtain an isomorphism between the special linear group (Special Linear Group)

$$\mathrm{SL}(2, \mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{C}^{2 \times 2} \mid ad - bc = 1 \right\}$$

and the automorphism group $\mathrm{Auto}(\mathbb{C}_\infty)$. This is an immediate consequence of the three propositions proved above. We leave the details to the student.

- ❖ **Example 13:** As initial examples, we present the four **basic Möbius transformations**, from which all other Möbius transformations can be constructed (by composition).

Table 4.1: Basic Möbius Transformations

Formula	Action	Operation
$z \mapsto z + b$	translation	translation
$z \mapsto az$	rotation, $a = e^{i\alpha}$	rotation
$z \mapsto rz$	dilation, $r \in \mathbb{R}, r > 0$	dilation or contraction
$z \mapsto \frac{1}{z}$	inversion	inversion (reciprocal)

i Proposition 4.24: Every Möbius transformation can be obtained as a composition of basic Möbius transformations.

- ✔ **Proof:** Let $T(z) = \frac{az+b}{cz+d}$ be a Möbius transformation ($ad \neq bc$). We consider two different cases.

(a) If $c = 0$, then necessarily $d \neq 0$, and therefore we may divide the numerator and denomi-

nator by d :

$$T(z) = \frac{az + b}{d} = \frac{a}{d}z + \frac{b}{d}$$

Let $re^{i\theta} = \frac{a}{d}$ be the polar representation of $\frac{a}{d}$. Then the mapping T is obtained by the following sequence of operations:

$$z \xrightarrow{\text{rotation}} e^{i\theta}z \xrightarrow{\text{dilation}} re^{i\theta}z = \frac{a}{d}z \xrightarrow{\text{translation}} \frac{a}{d}z + \frac{b}{d} = T(z)$$

(b) If $c \neq 0$, then we may write

$$\begin{aligned} T(z) &= \frac{az + b}{cz + d} = \frac{\frac{a}{c}z + \frac{b}{c}}{z + \frac{d}{c}} \\ &= \frac{\frac{a}{c} \cdot \left(z + \frac{d}{c}\right) + \frac{b}{c} - \frac{ad}{c^2}}{z + \frac{d}{c}} = \frac{a}{c} + \frac{\frac{bc-ad}{c^2}}{z + \frac{d}{c}} \\ &= p + \frac{q}{z + r} \end{aligned}$$

In the last step, we set $p = \frac{a}{c}$, $q = \frac{bc-ad}{c^2}$, and $r = \frac{d}{c}$. Therefore, we may write

$$T(z) = \frac{az + b}{cz + d} = p + \frac{q}{z + r} = p + \frac{|q|e^{i\theta}}{z + r}$$

where $\theta = \text{Arg}(q)$. It is not difficult to identify the elementary operations involved in the last expression:

$$\begin{aligned} z &\xrightarrow{\text{translation}} z + r \xrightarrow{\text{inversion}} \frac{1}{z + r} \xrightarrow{\text{rotation}} \frac{e^{i\theta}}{z + r} \\ &\xrightarrow{\text{dilation}} \frac{|q|e^{i\theta}}{z + r} = \frac{q}{z + r} \xrightarrow{\text{translation}} p + \frac{q}{z + r} = T(z) \end{aligned}$$

i Corollary 4.25: A mapping $T: \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is a Möbius transformation if and only if it is linear, $T(z) = az + b$, $a \neq 0$, or it has the form

$$T(z) = p + \frac{q}{z + r} \tag{4.18}$$

for constants $p, q, r \in \mathbb{C}$, $q \neq 0$.

¹ Of course, one must add that $T(-r) = \infty$ and $T(\infty) = p$.

² In some books, the definition of a Möbius transformation is based on formula (4.18).

➤ The next property of a Möbius transformation that we discuss is how it acts on straight lines and

circles. It is easy to prove that translations, rotations, and dilations or contractions map lines to lines and circles to circles. We leave this fact as a homework exercise for the student. For the inversion (Inversion) $\frac{1}{z}$, the situation is slightly more delicate:

- i Proposition 4.26:** Under the mapping $T(z) = \frac{1}{z}$,
- a. every straight line passing through the origin is mapped to another straight line passing through the origin, whose slope has the opposite sign.
 - b. Every straight line not passing through the origin is mapped to a circle passing through the origin.
 - c. Every circle passing through the origin is mapped to a straight line (not passing through the origin).
 - d. Every circle not passing through the origin is mapped to another circle that also does not pass through the origin.

✓ Proof: Part a is clear, so we focus only on the other parts. This proposition can be proved in several ways. Here is a proof from the book *Complex Variables* by R.B. Ash and W.P. Novinger, pages 79-80. The authors have granted permission to download it from the following link:



Complex Variables, R.B. Ash and W.P. Novinger

We shall render the proof by Ash and Novinger in simpler language:

Auxiliary proposition (canonical equation of a circle in the plane $\mathbb{C}$): For every circle L in the plane $\mathbb{C}$, there exist a complex constant α and a real constant p such that L is the set of points in the plane $\mathbb{C}$ satisfying the equation

$$z\bar{z} - \alpha\bar{z} - \bar{\alpha}z = p$$

Proof of the auxiliary proposition: Let L be a general circle with center α and radius r :

$$|z - \alpha|^2 = r^2$$

We may write

$$(z - \alpha)(\bar{z} - \bar{\alpha}) = r^2$$

Expanding the parentheses and collecting terms gives

$$z\bar{z} - \alpha\bar{z} - \bar{\alpha}z = r^2 - |\alpha|^2$$

Set $p = r^2 - |\alpha|^2$. Then p is real, and we obtain

$$|z - \alpha|^2 = r^2 \iff z\bar{z} - \alpha\bar{z} - \bar{\alpha}z = p$$



Therefore, the equation $z\bar{z} - \alpha\bar{z} - \bar{\alpha}z = p$ describes the circle L .

The converse is clear: for every complex α and real $p \geq 0$, the equation $z\bar{z} - \alpha\bar{z} - \bar{\alpha}z = p$ describes a circle of radius $r = \sqrt{p + |\alpha|^2}$ and center α . One need only reverse the preceding steps to obtain the equation $|z - \alpha|^2 = r^2$. This completes the proof of the auxiliary proposition.

We return to the proof of the main proposition: A circle L passes through the origin if and only if $r = |\alpha|$. That is, if and only if $p = 0$. Under the mapping $w = \frac{1}{z}$, the circle L is mapped to a set in the w -plane whose equation is

$$\frac{1}{w} \cdot \frac{1}{\bar{w}} - \alpha \frac{1}{\bar{w}} - \frac{\bar{\alpha}}{w} = p \quad (*)$$

Suppose that $p \neq 0$ (that is, the circle L does not pass through the origin). Multiply both sides of the equality by $\frac{w\bar{w}}{p}$:

$$1 - \frac{\alpha}{p}w - \frac{\bar{\alpha}}{p}\bar{w} = w\bar{w}$$

Set $\beta = -\frac{\bar{\alpha}}{p}$ (recall that p is real); then

$$w\bar{w} - \beta\bar{w} - \bar{\beta}w = 1$$

We have obtained a canonical equation of a circle in the w -plane whose center is β and whose radius is $r = \sqrt{1 + \beta^2}$ (and which does not pass through the origin).

If $p = 0$ (L passes through the origin), then multiplying equality (*) by $w\bar{w}$ yields the equation

$$1 - \alpha w - \bar{\alpha}w = 0$$

That is, $\operatorname{Re}(\alpha w) = \frac{1}{2}$. If we set $\alpha = a + bi$ and $w = x + iy$, then $\operatorname{Re}(\alpha w) = ax - by = \frac{1}{2}$. We have obtained the equation of a line $ax - by = \frac{1}{2}$ that does not pass through the origin. ■

i Theorem 4.27: A Möbius transformation maps generalized circles to generalized circles.

✓ **Proof:** By Proposition 4.24, every Möbius transformation is obtained by composing the basic transformations: translation, rotation, dilation, and inversion. It is easy to prove that translation, rotation, and dilation map circles to circles and straight lines to straight lines (see the corresponding exercise at the end of the chapter). By the preceding proposition, inversion maps lines and circles to lines and circles. Therefore, the composition of all these operations maps generalized circles to generalized circles. ■

◆ **Example 14:** Let $T(z) = \frac{z-i}{z+i}$.

(a) Prove that T maps the x -axis to the unit circle $|z| = 1$.

(b) Prove that T maps the upper half-plane $\operatorname{Im}(z) > 0$ onto the unit disk $|z| < 1$.



🔴 **Solution: (a)** It is easy to verify that for every real x ,

$$|T(x)| = \left| \frac{x-i}{x+i} \right| = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1$$

and therefore the x -axis is mapped to the unit circle $|z| = 1$. The point at infinity $x = \infty$ is also mapped to the point $z = 1$. This calculation does not prove that points not on the x -axis are not mapped to the unit circle. It is therefore advisable to know another solution method.

Second solution: since T is a Möbius transformation, by Theorem 4.27, the image of the x -axis must be a straight line or a circle. Therefore, examining the images of three points on the x -axis suffices to determine its image under T . Choose, for example, the points $x = -1, 0, 1$:

$$\begin{aligned} T(-1) &= \frac{-1-i}{-1+i} = \frac{(-1-i)(-1-i)}{2} = \frac{2i}{2} = i \\ T(0) &= \frac{0-i}{0+i} = -1 \\ T(1) &= \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{2} = \frac{-2i}{2} = -i \end{aligned}$$

The three points $z = i, -1, -i$ lie on the unit circle $|z| = 1$, and therefore the image of the x -axis must be the unit circle⁶. The calculation also proves that the x -axis is mapped to the unit circle in the counterclockwise direction. One may verify that the interval $(-\infty, \infty)$ is mapped to the set $e^{i\theta}$, $0 < \theta < 2\pi$, in the same order. The point $x = \infty$ is mapped to the point $z = 1$.

(b) It suffices to prove that the positive part of the imaginary axis is mapped inside the unit disk $|z| < 1$. Let yi , $y > 0$, be a point on the positive imaginary axis. Then

$$|T(yi)| = \left| \frac{yi-i}{yi+i} \right| = \left| \frac{y-1}{y+1} \right| < 1$$

The reason this calculation suffices is that, for a point $x + iy$, $y > 0$, Proposition 3.16 implies that the image of the upper half-plane must be connected and therefore must be contained in the open unit disk (because the positive imaginary axis has already been mapped there).

Invariance of the Cross Ratio

🔴 One of the important properties of a Möbius transformation is the **invariance of the cross ratio**. The concept of the cross ratio comes from [projective geometry](#), which deals with properties of projections. Figure 4.25 illustrates one of the fundamental results of projective geometry: the cross ratio is invariant under projective projection. The cross ratio of four points A , B , C , and D lying on a straight line in the plane (in that order) is defined by $\frac{AC \cdot BD}{BC \cdot AD}$ and is denoted by $(A, B; C, D)$. If these four points are projected onto a second line, as shown in Figure 4.25, the ratio is preserved.

⁶ Strictly speaking, one must include in the x -axis the point at infinity $x = \infty$, which is mapped to the point $T(\infty) = \frac{\infty-i}{\infty+i} = 1$



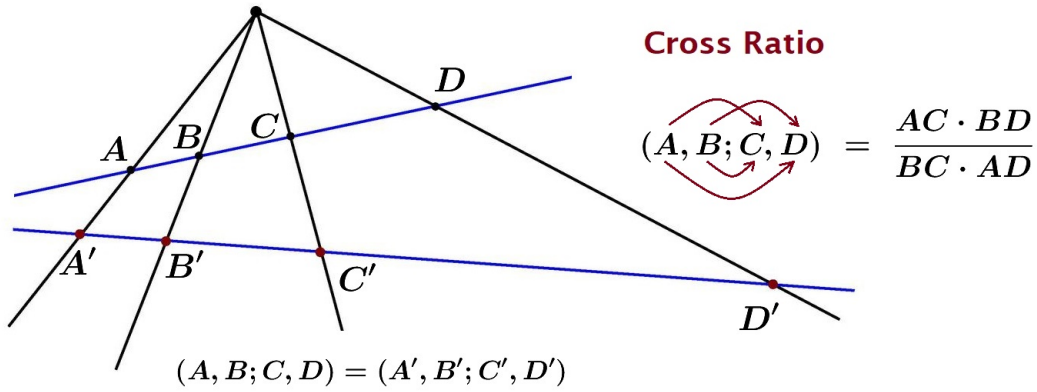


Fig. 4.25: The origin of the cross ratio in projective geometry, based on the geometric theorem shown in the diagram

If we replace the points A , B , C , and D by four distinct complex numbers z_1 , z_2 , z_3 , and z_4 , then we obtain

$$AC = z_1 - z_3, \quad BD = z_2 - z_4, \quad BC = z_2 - z_3, \quad AD = z_1 - z_4$$

and obtain the definition of the cross ratio in the context of Möbius transformations:

$$(A, B; C, D) = \frac{AC \cdot BD}{BC \cdot AD} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = (z_1, z_2; z_3, z_4)$$

In the complex case, the points z_1 , z_2 , z_3 , and z_4 need not lie on the same straight line. In addition, one of the points may be the point at infinity ∞ . In that case, cancellation occurs between the numerator and denominator. For example, if $z_1 = \infty$, then

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = \frac{(\infty - z_3)(z_2 - z_4)}{(z_2 - z_3)(\infty - z_4)} = \frac{z_2 - z_4}{z_2 - z_3}$$

i Proposition 4.28: Let $T: \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ be a given Möbius transformation. For any four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$, the following equality holds:

$$(T(z_1), T(z_2); T(z_3), T(z_4)) = (z_1, z_2; z_3, z_4)$$

That is, a Möbius transformation **preserves the cross ratio**.

✓ Proof: By Proposition 4.24, it suffices to prove that every basic Möbius transformation preserves the cross ratio. Intuitively, it is clear that translations and rotations do not change the distances between the points z_1 , z_2 , z_3 , z_4 , and therefore their cross ratio is preserved. For the dilation (or contraction) $T(z) = rz$, $r \in \mathbb{R}$, $r \neq 0$, the ratios of distances are preserved, and hence

the cross ratio is preserved:

$$\begin{aligned}
 (T(z_1), T(z_2); T(z_3), T(z_4)) &= \frac{(T(z_1) - T(z_3))(T(z_2) - T(z_4))}{(T(z_2) - T(z_3))(T(z_1) - T(z_4))} \\
 &= \frac{(rz_1 - rz_3)(rz_2 - rz_4)}{(rz_2 - rz_3)(rz_1 - rz_4)} \\
 &= \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = (z_1, z_2; z_3, z_4)
 \end{aligned}$$

It remains to prove the proposition for the inversion $T(z) = \frac{1}{z}$.

$$\begin{aligned}
 (T(z_1), T(z_2); T(z_3), T(z_4)) &= \frac{(T(z_1) - T(z_3))(T(z_2) - T(z_4))}{(T(z_2) - T(z_3))(T(z_1) - T(z_4))} \\
 &= \frac{(\frac{1}{z_1} - \frac{1}{z_3})(\frac{1}{z_2} - \frac{1}{z_4})}{(\frac{1}{z_2} - \frac{1}{z_3})(\frac{1}{z_1} - \frac{1}{z_4})} = \frac{\frac{z_3 - z_1}{z_1 z_3} \cdot \frac{z_4 - z_2}{z_2 z_4}}{\frac{z_3 - z_2}{z_2 z_3} \cdot \frac{z_4 - z_1}{z_1 z_4}} \\
 &= \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} = (z_1, z_2; z_3, z_4)
 \end{aligned}$$

Since every Möbius transformation is obtained as a composition of the four basic transformations, it too preserves the cross ratio. ■

i Theorem 4.29: Given any three distinct points $z_2, z_3, z_4 \in \mathbb{C}_\infty$, and any three distinct points $w_2, w_3, w_4 \in \mathbb{C}_\infty$, there exists a unique Möbius transformation $T: \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ satisfying $T(z_2) = w_2$, $T(z_3) = w_3$, $T(z_4) = w_4$.

✓ Proof: Assuming such a mapping T exists, it must satisfy Proposition 4.28. For every $z \in \mathbb{C}_\infty$,

$$(z, z_2; z_3, z_4) = (T(z), T(z_2); T(z_3), T(z_4)) = (T(z), w_2; w_3, w_4)$$

The equality gives an equation from which $T(z)$ can be isolated:

$$(T(z), w_2; w_3, w_4) = \frac{(T(z) - w_3)(w_2 - w_4)}{(w_2 - w_3)(w_1 - w_4)} = \frac{(z - z_3)(z_2 - z_4)}{(z_2 - z_3)(z - z_4)}$$

From this, we readily obtain

$$T(z) = w_3 + \frac{(z - z_3)(z_2 - z_4)(w_2 - w_3)(w_1 - w_4)}{(z_2 - z_3)(z - z_4)(w_2 - w_4)}$$

Set

$$p = \frac{(z_2 - z_4)(w_2 - w_3)(w_1 - w_4)}{(z_2 - z_3)(w_2 - w_4)}$$



This is a constant independent of the variable z , and therefore

$$T(z) = w_3 + \frac{p(z - z_3)}{z - z_4} = \frac{(p + w_3)z - (pz_3 + w_3z_4)}{z - z_4}$$

The form of the right-hand side makes it clear that this is a Möbius transformation. ■

- ✎ In practice, the simplest and easiest way to remember how to find T is to solve for the variable w in the following formula:

$$(z, z_2; z_3, z_4) = (w, w_2; w_3, w_4) \quad (4.19)$$

or, more explicitly,

$$\frac{(z - z_3)(z_2 - z_4)}{(z_2 - z_3)(z - z_4)} = \frac{(w - w_3)(w_2 - w_4)}{(w_2 - w_3)(w - w_4)} \quad (4.20)$$

- ✎ **Example 15:** Find a Möbius transformation that maps the imaginary axis to the circle $|z - 5| = 1$.

- ✎ **Solution:** Choose three points on the imaginary axis: $z_2 = -i$, $z_3 = 0$, $z_4 = i$, and three points on the circle $|z - 5| = 1$: $w_2 = 4$, $w_3 = 5 + i$, $w_4 = 6$, and use formula (4.20):

$$\frac{(z - 0)(-i - i)}{(-i - 0)(z - i)} = \frac{(w - (5 + i))(4 - 6)}{(4 - (5 + i))(w - 6)}$$

After simplification, we obtain

$$\frac{2iz}{iz + 1} = \frac{-2w + 10 + 2i}{-(1 + i)w + 6 + 6i}$$

and solving for w gives

$$w = T(z) = \frac{(5 - i)z + 5 + i}{z + 1}$$

It is advisable to verify that $T(-i) = 4$, $T(0) = 5 + i$, and $T(i) = 6$.

- ✎ Users of the Python programming language are advised to use the [SymPy](https://sympy.org/) package to verify that the result is correct:

```
from sympy import *
z = Symbol("z", complex=True)
```



```
w = Symbol("w", complex=True)
eqn = (-2*w+10+2*I)/(-(1+I)*w+6+6*I) - (2*I*z)/(z*I +1)
solve(eqn, w)
=> [(5*z - I*z + 5 + I)/(z + 1)]
```

Of course, Matlab or Mathematica may also be used for this purpose.

❖ **Example 16:** Another way to solve the preceding problem is to choose points more “intelligently”, possibly including the point at infinity. Choose, for example, three points on the imaginary axis: $z_2 = 0$, $z_3 = i$, $z_4 = \infty$, and three points on the circle $|z - 5| = 1$: $w_2 = 4$, $w_3 = 5 + i$, $w_4 = 6$, and again use formula (4.20):

$$\frac{(z - i)(0 - \infty)}{(0 - i)(z - \infty)} = \frac{(w - (5 + i))(4 - 6)}{(4 - (5 + i))(w - 6)}$$

After simplification, we obtain

$$iz + 1 = \frac{-2w + 10 + 2i}{-(1 + i)w + 6 + 6i}$$

which is easier to manipulate than the equation in the preceding example. Solving for w gives

$$w = T(z) = \frac{6z + 4}{z + 1} = 6 - \frac{2}{z + 1}$$

This is, of course, a different mapping from the one obtained in the preceding example because the chosen corresponding points are different, but it accomplishes the required task: T maps the imaginary axis to the circle $|z - 5| = 1$.

❖ **Example 17:** What is the locus of all points satisfying the equation

$$\left| \frac{z - 3}{z + 3} \right| = 2$$

🌟 **Solution:** This is the inverse image under the Möbius transformation $T(z) = \frac{z-3}{z+3}$ of the circle $|z| = 2$. By Proposition 4.22, T is one-to-one, and its inverse is also a Möbius transformation. By formula (4.17),

$$T(z) = \frac{az + b}{cz + d} \implies T^{-1}(z) = \frac{dz - b}{-cz + a}$$

Therefore, in our case, the inverse mapping is

$$T^{-1}(z) = \frac{3z + 3}{-z + 1}$$

By Theorem 4.27, the image of the circle $|z| = 2$ under T^{-1} must be a circle or a generalized

line. It therefore suffices to examine three distinct points on the circle $|z| = 2$ in order to determine it. Substitute, for example, the points $z = 2, -2, 2i$, and obtain

$$T^{-1}(2) = -9, \quad T^{-1}(-2) = -1, \quad T^{-1}(2i) = -1.8 + 2.4i$$

It is immediately clear that this is a circle. The three points form a right triangle whose hypotenuse is the segment $(-9, -1)$. Therefore, the equation of the circle is $|z + 5| = 4$.

4.7 Links to Online Resources

Links to Resources on Differentiability and Conformal Mappings

- Conformal mappings have classical applications in cartography (geographic maps and the mapping of regions). Recall that the geometric representation of complex numbers by means of the real plane was proposed by the Belgian mathematician and cartographer Caspar Wessel (1745-1818). Conformal mappings have various applications in physics, computer graphics, and, more recently, in areas such as pattern recognition and facial recognition (the first link is to a lecture by Professor Yaron Lipman).

-  [What does it mean to take a complex derivative? \(visually explained\)](#)
-  [But what is the Riemann zeta function? Visualizing analytic continuation](#)
-  [A Few Applications of Conformal Mappings for Surface Comparison — Prof. Yaron Lipman](#)
-  [Obama deformed by holomorphic complex functions \(conformal map\)](#)
-  [Complex Analysis 02: Mappings](#)
-  [232 Application of conformal mapping](#)
-  [Mercator projection](#)
-  [What is the best way to explain conformal mapping?](#)
-  [Conformal Mapping and its Applications](#)

Links to Resources on the Riemann Sphere

-  [Stereographic projection of Riemann sphere](#)
 [Mapping Lines and Circles onto the Riemann Sphere 1](#)
 [Mapping Lines and Circles onto the Riemann Sphere 2](#)
 [Möbius Transformations Revealed \[HD\]](#)
 [1/z map, part 1/2](#)
 [1/z map, part 2/2](#)

Exercises for chapter 4

1. Prove that if $f(z)$ and $g(z)$ are analytic on a domain S , and if $\operatorname{Re}(f(z)) = \operatorname{Re}(g(z))$ at every point of S , then there exists a constant C such that $f(z) = g(z) + C$ throughout S . In other words, an analytic function is determined up to an additive constant by its real part.
-  2. Find a function $f(z)$ that is continuous throughout $\mathbb{C}$ but differentiable only on the line $y = x$.
Hint: $u = x^2, v = y^2$
-  3. If $f: S \rightarrow \mathbb{C}$ is an analytic function on a domain S that takes only real values, then f is constant.
Hint: Write $f(z) = u(x, y) + iv(x, y)$. By hypothesis, $v(x, y) = 0$ at every point $x + iy \in S$. Therefore, $v_x = v_y = 0$. The Cauchy-Riemann equations then also give $u_x = u_y = 0$.
4. Compute the derivative of the function $f(z) = \operatorname{Log}(iz^2 + 3)$ and find its domain of analyticity.
5. Find the maximal domain on which the function $f(z) = \operatorname{Log}(1 - z^n)$ is analytic, where $n \geq 1$ is a natural number.
6. Let $f(z) = u(x, y) + iv(x, y)$ be an entire function satisfying $u_y - v_x = -2$ for every $z \in \mathbb{C}$. Prove that f is a linear function (that is, of the form $az + b$).
-  7. For which values of a , b , and c is the function $f(z) = a(x^2 + y^2) + ibxy + c$ analytic throughout the complex plane?
8. Using the Cauchy-Riemann equations, characterize all entire functions satisfying $\operatorname{Re}(f'(z)) = 0$.



9. Prove that the function

$$f(z) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} + \frac{xyi}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

satisfies the Cauchy-Riemann equations at the point $z = 0$, but f is not differentiable at that point.

10. Let $f(z)$ be an entire function. Prove that the function $g(z) = \bar{z}f(z)$ is differentiable at the point z if and only if $f(z) = 0$.

11. Determine whether the following function is differentiable at the point $z = 0$:

$$f(z) = \begin{cases} z \sin \frac{1}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$



12. For each of the following functions, find all points at which it is analytic and all points at which it is differentiable but not analytic:

$$\begin{array}{lll} \text{(a)} f(z) = |z|^2 & \text{(b)} f(z) = z^2 + \bar{z} & \text{(c)} f(z) = 2xy + i(x^2 - y^2) \\ \text{(d)} f(z) = 2|z|^2 + i\bar{z}^2 & \text{(e)} f(z) = \cos |z| + i \sin |z| & \text{(f)} f(z) = \frac{z}{1+|z|^2} \end{array}$$



13. Prove that there is no analytic function $f(z)$ satisfying $\operatorname{Re}(f) = xy^2$.

14. Find all analytic functions $f(z)$ satisfying $[\operatorname{Re}(f)]^2 = \operatorname{Im}(f)$.

15. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function satisfying $v(x, y) = \frac{e^x - e^{-x}}{2} \sin y$, $f(0) = 0$. Find f .

16. Find an analytic function $f(z)$ satisfying $\operatorname{Re}(f) = xy + \frac{x}{x^2+y^2}$.

17. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function whose real part is $u = e^{\lambda x} \cos y$. Find all values of λ for which u is harmonic. What is the harmonic conjugate of u ?



18. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function on a simply connected domain D . Prove that the family of level curves of u is orthogonal to the family of level curves of v .

Hint: The level curves of u are the curves given implicitly by $u(x, y) = C$, for every real constant C . One must show that every curve $u(x, y) = C_1$ is perpendicular to every curve $v(x, y) = C_2$ at each point where they intersect. It suffices to prove that the product of the gradients (the inner product!) vanishes. Recall that the gradient is $\vec{\nabla}u = (u_x, u_y)$.



19. Let $f(z)$ be an analytic function on a domain D . It is known that its image $f(D)$ is contained in a straight line. Prove that f is constant.





20. Find the most general harmonic function of the form $ax^3 + bx^2y + cxy^2 + dy^3$.

21. Determine whether the following real function is harmonic and, if so, find its harmonic conjugate:

$$u(x, y) = \sin x \cdot \cosh y + 2 \cos x \cdot \sinh y + x^2 - y^2 + 2xy$$

Hint: A standard formula for computing a harmonic conjugate: $v(x, y) = \int_0^y u_x(x, t) dt - \int_0^x u_y(s, 0) ds$. Alternatively, the harmonic conjugate may be found using the Cauchy-Riemann equations.

22. Review the table of basic Möbius transformations (Table 4.1 on page 152) and prove formally that translation, rotation, and dilation map straight lines to straight lines and circles to circles.



23. Let $T(z) = \frac{az+b}{cz+d}$ be a given Möbius transformation. A point z is called a **fixed point** (**Fixed Point**) of T if $T(z) = z$. Prove that every Möbius transformation has at most two distinct fixed points, and find a formula for computing them.

24. Find a Möbius transformation that maps the real line to the unit circle $|z| = 1$.

25. Find a Möbius transformation that maps the unit circle to itself and the point 0 to $\frac{1}{2}$.

26. Find a Möbius transformation that maps the unit disk $|z| < 1$ onto the following half-plane:

$$S = \{x + iy : x + y > 0\}$$

27. The Möbius transformation $T(z) = \frac{z+1}{z-1}$ is given. Find the images of the real axis, the imaginary axis, and the unit circle $|z| = 1$ under T . What is the image of the unit disk $|z| < 1$ under T ?

Hint: First determine the image of the unit circle $|z| = 1$, and then determine where the point $z = 0$ is mapped.

28. Find a Möbius transformation that maps the circle $|z| = 2$ to the circle $|z+1| = 1$, the point $z = -2$ to the point $z = 0$, and the point $z = 0$ to the point $z = i$.



29. Find the simplest possible algebraic characterization of all Möbius transformations that map the real line onto itself. Prove that every Möbius transformation mapping the real axis onto itself can be expressed using real coefficients only.

Hint: For precision, one should require $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, where $\mathbb{R}_\infty$ is the real axis together with the point at infinity. But this follows from the continuity of T .



30. Find the simplest possible algebraic characterization of all Möbius transformations that map the upper half-plane onto itself.



31. Find the simplest possible algebraic characterization of all Möbius transformations that map the unit circle onto itself.



Hint: T is characterized by the condition $z\bar{z} = 1 \implies T(z)\overline{T(z)} = 1$.

32. Prove that the function $f(z) = z|z|$ is a conformal mapping at the point $z = 0$, even though $f'(0) = 0$. Does this contradict Proposition 4.18?



33. Find a conformal mapping $f(z)$ that maps the set

$$U = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0 \text{ and } |z| > 1\}$$

onto the upper half-plane $H : \operatorname{Im}(z) > 0$ (see Figure 4.26).

Suggestion: Try to construct a Möbius transformation that maps the real axis onto itself and the unit circle to the imaginary axis.

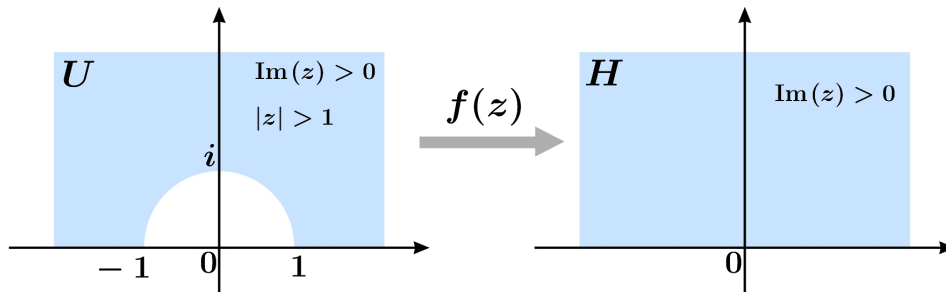


Fig. 4.26: An auxiliary figure for Exercise 33



34. Let n be a positive natural number. Find a conformal mapping that maps the sector $0 < \operatorname{Arg}(z) < \frac{\pi}{n}$ onto the unit disk $|z| < 1$.



35. Construct an analytic one-to-one mapping from the right half of the unit disk $|z| < 1$, $\operatorname{Re}(z) > 0$, onto the entire unit disk $|z| < 1$.

Hint: Proceed in stages. First, recall that the mapping $T_1(z) = iz$ maps the right half of the disk to the upper half. (Recall from Chapter 1 that multiplication by i is equivalent to a rotation through 90° !). In the second stage, show that the Möbius transformation $T_2(z) = \frac{1+z}{1-z}$ maps the upper half-disk to the first quadrant of the complex plane (first solve Exercise 27).

The mapping $T_3(z) = z^2$ maps the first quadrant to the upper half-plane, and finally, the mapping $T_4(z) = \frac{z-i}{z+i}$ is an isomorphism from the upper half-plane to the unit disk (see Example 14).



36. Prove or disprove:

- If $f(z)$ is analytic on a domain D , then $\overline{f(z)}$ is also analytic on D .
- There exists a Möbius transformation that maps the open first quadrant onto the disk $|z| < 1$.
- The function $e^{i\operatorname{Re}(z)}$ is nowhere differentiable.
- There exists a conformal mapping from the strip $0 < \operatorname{Re}(z) < \pi$ onto the upper half-plane $\operatorname{Im}(z) > 0$.
- There exists a conformal mapping from the strip $0 < \operatorname{Im}(z) < \pi$ onto the unit disk $|z| < 1$.
- If $f(z)$ is analytic on a domain D , then $\overline{f(\bar{z})}$ is also analytic on D .
- If $f(z)$ is entire and $f'(z) \neq 0$ for every $z \in \mathbb{C}$, then f is one-to-one.





37. An analytic function $f(z)$ on a domain D is called an **analytic branch of $\log$ on D** if $e^{f(z)} = z$ for every $z \in D$. Prove the following four assertions:

- a. If D is any open disk not containing the point $z = 0$, then an analytic branch of **$\log$** exists on D .
- b. If L is a ray emanating from the point $z = 0$, then an analytic branch of **$\log$** exists on $\mathbb{C} - L$.
- c. If D is a domain containing the circle $|z| = 1$, then no analytic branch of **$\log$** exists on D .
- d. If $g(z)$ is a continuous function on a domain D satisfying $e^{g(z)} = z$ for every $z \in D$, then g is necessarily analytic on D , and hence g is necessarily an analytic branch of **$\log$** on D .



38. Let D be a simply connected domain, let $f(z)$ be analytic on D , and let $r > 0$ be a given real number.

- (a) Prove that the set $S_r = \{z \in D : |f(z)| < r\}$ is open.
- (b) Prove that every connected component of S_r is a simply connected domain.



39. Let $f(z)$ and $g(z)$ be two analytic functions on a domain D . It is given that, for every $z \in D$,

$$|f(z)|^2 + |g(z)|^2 = 1$$

Prove that both functions $f(z)$ and $g(z)$ are constant on D .

Hint: Use the Laplacian identity for complex functions: $\Delta|f|^2 = 4|f'(z)|^2$.

The Laplacian refers to the Cartesian representation $f(z) = u(x, y) + iv(x, y)$. (It is advisable to prove the identity as an additional exercise).



[Link to Solutions](#)



Chapter 5

Integration

As in the differential and integral calculus of real numbers, complex functions also have an integration operation and several deep, elegant theorems with important applications in many diverse fields.

5.1 Integration of a Complex Function over a Real Interval

- We begin with a simple generalization of the definite integral over a real interval, familiar from Calculus 1: $\int_a^b f(t) dt$, where $f: [a, b] \rightarrow \mathbb{R}$ is a real-valued function.
- If $f: [a, b] \rightarrow \mathbb{C}$ is a complex-valued function, then of course it has a Cartesian representation

$$f(t) = x(t) + iy(t)$$

where $x(t)$ and $y(t)$ are real-valued functions on the interval $[a, b]$.

i Definition 5.1: Let $f: [a, b] \rightarrow \mathbb{C}$, $f(t) = x(t) + iy(t)$. We say that f is *integrable*¹ over the interval $[a, b]$, if the real-valued functions $x(t)$ and $y(t)$ are integrable over the interval $[a, b]$. The integral of f over $[a, b]$ is defined by

$$\int_a^b f(t) dt = \int_a^b x(t) dt + i \int_a^b y(t) dt$$

◆ **Example 1:** Compute the value of $\int_0^{\frac{\pi}{2}} e^{it} dt$

🔧 **Solution:**

$$\int_0^{\frac{\pi}{2}} e^{it} dt = \int_0^{\frac{\pi}{2}} [\cos t + i \sin t] dt = \int_0^{\frac{\pi}{2}} \cos t dt + i \int_0^{\frac{\pi}{2}} \sin t dt = 1 + i$$

- In view of the simple reduction to two real integrals, most theorems concerning the definite integral familiar from Calculus 1 also hold automatically for the complex integral over an interval. We will not bother to prove the following formulas, and leave the details for the student to verify.

¹ In the Riemann sense (Riemann). There is also [integrability in the Lebesgue sense \(Lebesgue\)](#) which is studied in more advanced courses in mathematical analysis.

i Theorem 5.1: If $f(t)$ and $g(t)$ are two functions integrable over the interval $[a, b]$, then

$$(5.1) \quad \int_a^b [f(t) \pm g(t)] dt = \int_a^b f(t) dt \pm \int_a^b g(t) dt$$

$$(5.2) \quad \int_a^b c f(t) dt = c \int_a^b f(t) dt$$

$$(5.3) \quad F'(t) = f(t) \implies \int_a^b f(t) dt = F(b) - F(a)$$

$$(5.4) \quad F(t) = \int_a^t f(s) ds \implies F'(t) = f(t)$$

◆ Formula (5.3) is the *fundamental theorem of differential and integral calculus*.

◆ Formula (5.4) is the *Newton-Leibniz formula*.

✓ **Proof:** We shall prove only formula (5.2). The proofs of the other formulas are entirely similar. By assumption, $f(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are integrable real-valued functions over the interval $[a, b]$. Therefore, by the integration theorems from Calculus 1,

$$\begin{aligned} \int_a^b c f(t) dt &= \int_a^b c [x(t) + iy(t)] dt = \int_a^b cx(t) dt + i \int_a^b cy(t) dt \\ &= c \int_a^b x(t) dt + ic \int_a^b y(t) dt = c \left[\int_a^b x(t) dt + i \int_a^b y(t) dt \right] \\ &= c \int_a^b f(t) dt \end{aligned}$$

➤ The following inequality will be very useful for estimating the magnitude of the integral of a complex function. It does not follow from its real counterpart and therefore requires a suitable verification and proof.

i Theorem 5.2: (Integral triangle inequality) If f is integrable on the interval $[a, b]$, then

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt \leq (b - a) \cdot \text{Max}_{a \leq t \leq b} |f(t)| \quad (5.5)$$

✓ **Proof:** Write $f(t) = x(t) + iy(t)$, where $f: [a, b] \rightarrow \mathbb{C}$ is a given continuous function. Let $\mathcal{P}$ be a partition of the interval $[a, b]$

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

Let $\Delta(\mathcal{P})$ be the parameter of the partition $\mathcal{P}$,

$$\Delta(\mathcal{P}) = \text{Max}\{\Delta t_0, \Delta t_1, \Delta t_2, \dots, \Delta t_{n-1}\}$$

For every $k = 0, 1, 2, \dots, n-1$, let $\hat{t}_k$ be an arbitrary point in the interval $\Delta t_k = [t_k, t_{k+1}]$

$$t_k \leq \hat{t}_k \leq t_{k+1}$$

The Riemann sum associated with the partition $\mathcal{P}$ is defined by

$$\mathcal{S}(\mathcal{P}) = \sum_{k=0}^{n-1} f(\hat{t}_k) \cdot \Delta t_k = \sum_{k=0}^{n-1} [x(\hat{t}_k) + iy(\hat{t}_k)] \cdot \Delta t_k$$

By decomposing $\mathcal{S}(\mathcal{P})$ into two integral sums, it can be proved that $f(t)$ is integrable on the interval $[a, b]$ if and only if the following limit exists:

$$\int_a^b f(t) dt = \lim_{\Delta(\mathcal{P}) \rightarrow 0} \mathcal{S}(\mathcal{P}) = \lim_{\Delta(\mathcal{P}) \rightarrow 0} \sum_{k=0}^{n-1} f(\hat{t}_k) \cdot \Delta t_k$$

By the triangle inequality,

$$\begin{aligned} \left| \sum_{k=0}^{n-1} f(\hat{t}_k) \cdot \Delta t_k \right| &= \left| f(\hat{t}_0) \cdot \Delta t_1 + f(\hat{t}_1) \cdot \Delta t_1 + \dots + f(\hat{t}_{n-1}) \cdot \Delta t_{n-1} \right| \\ &\leq \left| f(\hat{t}_0) \right| \cdot \Delta t_1 + \left| f(\hat{t}_1) \right| \cdot \Delta t_1 + \dots + \left| f(\hat{t}_{n-1}) \right| \cdot \Delta t_{n-1} \\ &= \sum_{k=0}^{n-1} \left| f(\hat{t}_k) \right| \cdot \Delta t_k \end{aligned}$$

On the right-hand side we have obtained the Riemann sum of the real-valued function $|f(t)|$ over the interval $[a, b]$. Since $f(t)$ is continuous, $|f(t)|$ is also continuous, and hence integrable. Therefore,

$$\left| \int_a^b f(t) dt \right| = \lim_{\Delta(\mathcal{P}) \rightarrow 0} \left| \sum_{k=0}^{n-1} f(\hat{t}_k) \cdot \Delta t_k \right| \leq \lim_{\Delta(\mathcal{P}) \rightarrow 0} \sum_{k=0}^{n-1} \left| f(\hat{t}_k) \right| \cdot \Delta t_k = \int_a^b |f(t)| dt$$

Denote $M = \text{Max}_{a \leq t \leq b} |f(t)|$. Then M is an upper bound for the real-valued function $|f(t)|$ on the interval $[a, b]$, and therefore, by an appropriate theorem from Calculus 1, $\int_a^b |f(t)| dt \leq M(b-a)$. ■

- Two additional proofs based on the Cauchy-Schwarz inequality and the modulus technique can be found at the following two links:

 [Absolute Value of Complex Integral](#)

 [Modulus of Complex Integral](#)

Length of a Path

- One of the first formulas learned in Calculus 2 is the length formula for a smooth path $\sigma: [a, b] \rightarrow \mathbb{R}^2$, $\sigma(t) = (x(t), y(t))$

$$\ell(\sigma) = \int_a^b |\sigma'(t)| dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt$$

where $\ell(\sigma)$ denotes the length of the path σ . Since, for us, $\mathbb{R}^2$ is a model for the set of complex numbers, there is in fact no essential difference between the path $\sigma(t) = (x(t), y(t))$ and the path $\sigma(t) = x(t) + iy(t)$. If σ is a piecewise smooth path, the lengths of the smooth paths composing it must be added.

- Example 2:** Compute the length of the spiral $\sigma(t) = te^{it}$, $0 \leq t \leq 2\pi$.

Solution:

$$\begin{aligned} \ell(\sigma) &= \int_0^{2\pi} |\sigma'(t)| dt = \int_0^{2\pi} |e^{it} + ite^{it}| dt \\ &= \int_0^{2\pi} |e^{it}| |1 + it| dt = \int_0^{2\pi} |1 + it| dt \\ &= \int_0^{2\pi} \sqrt{1 + t^2} dt = \frac{t\sqrt{1+t^2}}{2} + \frac{\ln|t + \sqrt{1+t^2}|}{2} \Big|_0^{2\pi} \\ &= \frac{\sqrt{1+4\pi^2}}{2} + \frac{\ln|2\pi + \sqrt{1+4\pi^2}|}{2} \end{aligned}$$

We leave it to the student to verify that $\int \sqrt{1+t^2} dt = \frac{\sqrt{1+4t^2}}{2} + \frac{\ln|2t + \sqrt{1+4t^2}|}{2}$.

5.2 Integration of Complex Functions along a Path

- We now proceed to generalize the definite integral $\int_a^b f(x) dx$ from real integral calculus to a complex definite integral. To do this, the real interval $[a, b]$ connecting the point a to the point b must be replaced by a path σ connecting a point $z_1 = \sigma(a)$ to a point $z_2 = \sigma(b)$, where

$$\sigma: [a, b] \rightarrow \mathbb{C}$$

is a path in the complex plane.

- In the real case, there is one and only one path connecting a point a to a point b , whereas in the complex plane there are infinitely many such paths. Therefore, at the first stage, in the

complex case we are required to define a **line integral** $\int_{\sigma} f(z)dz$, because an integral between two points z_1, z_2 has no unambiguous meaning (for now). Recall the notions of a partition and a Riemann sum, familiar from differential and integral calculus.

i **Definition 5.2:** A **partition** $\mathcal{P}$ of a real interval $[a, b]$ consists of two components:

1. A finite sequence of points in the interval $[a, b]$

$$a = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots < t_n = b$$

2. A sequence of intermediate points $\hat{t}_k$, one from each subinterval $\Delta t_k = [t_k, t_{k+1}]$:

$$t_k \leq \hat{t}_k \leq t_{k+1}$$

i **Definition 5.3:** The **Riemann sum** of a function $f(z)$ over a path

$$\sigma: [a, b] \rightarrow \mathbb{C}$$

associated with a given partition $\mathcal{P}$ is defined by

$$\mathcal{S}(\mathcal{P}) = \sum_{k=0}^n f(\sigma(\hat{t}_k)) [\sigma(t_{k+1}) - \sigma(t_k)] \quad (5.6)$$

Set $z_k = \sigma(t_k)$, $\Delta z_k = z_{k+1} - z_k$, and obtain the shorter formula

$$\mathcal{S}(\mathcal{P}) = \sum_{k=0}^n f(\sigma(\hat{t}_k)) \cdot \Delta z_k \quad (5.7)$$

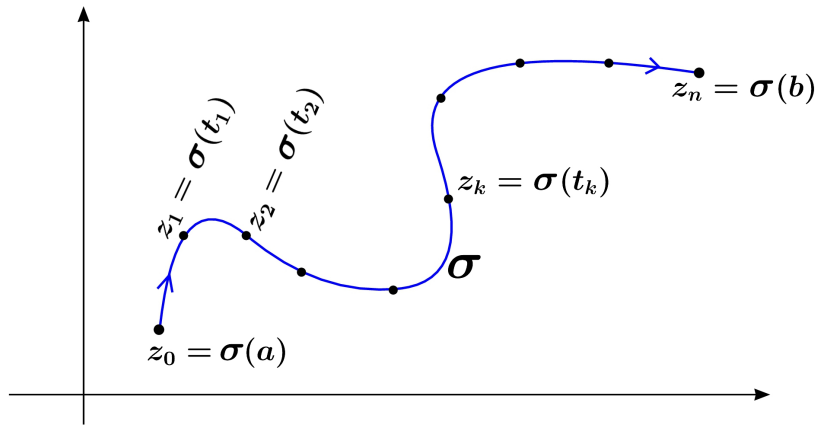


Fig. 5.1: Mapping the partition points $\mathcal{P}$ onto the path σ

o **The parameter of the partition** $\Delta(\mathcal{P})$ is defined by

$$\Delta(\mathcal{P}) = \max_{0 \leq k \leq n-1} (t_{k+1} - t_k)$$

This is the maximum length of the intervals in the partition. When this quantity $\Delta(\mathcal{P})$ tends to zero, the lengths of all the intervals in the partition tend to zero, and then the Riemann sums

$\mathcal{S}(\mathcal{P})$ should converge to the value of the integral (provided it exists).

i Definition 5.4: Let $f(z)$ be a complex function defined on a domain that contains a path

$$\sigma: [a, b] \rightarrow \mathbb{C}$$

The integral of f along the path σ , denoted by $\int_{\sigma} f(z) dz$, is defined to be the limit of all the Riemann sums $\mathcal{S}(\mathcal{P})$, as the parameter of the partition $\Delta(\mathcal{P})$ tends to zero:

$$I = \int_{\sigma} f(z) dz = \lim_{\Delta(\mathcal{P}) \rightarrow 0} \mathcal{S}(\mathcal{P})$$

The last statement is to be understood as follows: For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every partition $\mathcal{P}$, if $\Delta(\mathcal{P}) < \delta$, then $|I - \mathcal{S}(\mathcal{P})| < \varepsilon$. If this limit exists, we say that f is **integrable along the path σ** .

- If we interpret the function $f(z)$ as a (complex) scalar physical quantity, such as pressure or temperature at a point z (measured using complex values), then the integral of f along the path σ is the continuous sum of the values along the path. For example, in the case of temperature, the integral computes the amount of heat accumulated along the path.
- Of course, the above limit may fail to exist, in which case f is not integrable along the path σ . Computing the integral from the definition is clearly impractical, and practical methods must be found for this purpose. The considerable effort invested in the theory of Riemann sums in real differential and integral calculus bears fruit also in the complex case and yields the following result.

i Theorem 5.3: Let $f(z)$ be a continuous complex function on a domain containing a smooth path $\sigma: [a, b] \rightarrow \mathbb{C}$. Then f is integrable along the path σ , and moreover

$$\int_{\sigma} f(z) dz = \int_a^b f(\sigma(t)) \sigma'(t) dt \quad (5.8)$$

If σ is a piecewise smooth path, that is, $\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_n$ is a concatenation of n smooth paths, then the integral of f along σ is given by

$$\int_{\sigma} f(z) dz = \int_{\sigma_1} f(z) dz + \int_{\sigma_2} f(z) dz + \cdots + \int_{\sigma_n} f(z) dz \quad (5.9)$$

If σ is a closed path, we indicate this by the notation: $\oint_{\sigma} f(z) dz$.

- **Proof:** We present the core idea of the proof without entering into the details and subtleties of limits of Riemann sums. Let $\mathcal{P}$ be a partition of the path $\sigma(t) = x(t) + iy(t)$, $a \leq t \leq b$,

as described in Definitions [5.2](#), [5.3](#). Expand formula [\(5.6\)](#) somewhat:

$$\begin{aligned}
 \mathcal{S}(\mathcal{P}) &= \sum_{k=0}^n f(\sigma(\hat{t}_k)) [\sigma(t_{k+1}) - \sigma(t_k)] \\
 &= \sum_{k=0}^n f(\sigma(\hat{t}_k)) \cdot \{ [x(t_{k+1}) + iy(t_{k+1})] - [x(t_k) + iy(t_k)] \} \\
 &= \sum_{k=0}^n f(\sigma(\hat{t}_k)) \cdot \{ [x(t_{k+1}) - x(t_k)] + i[y(t_{k+1}) - y(t_k)] \} \\
 &= \sum_{k=0}^n f(\sigma(\hat{t}_k)) \cdot \left[\frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k} + i \frac{y(t_{k+1}) - y(t_k)}{t_{k+1} - t_k} \right] \cdot (t_{k+1} - t_k) \\
 &\approx \sum_{k=0}^n f(\sigma(\hat{t}_k)) \cdot [x'(\hat{t}_k) + iy'(\hat{t}_k)] \cdot (t_{k+1} - t_k)
 \end{aligned}$$

By Lagrange's mean value theorem, there is an intermediate point $t_k < \bar{t} < t_{k+1}$ such that

$$\frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k} = x'(\bar{t})$$

A similar formula applies to $y(t)$. By assumption, the path σ is smooth, and therefore both its components $x(t)$ and $y(t)$ are continuously differentiable. By the continuity of $x'(t)$, and since the partition parameter $\Delta(\mathcal{P})$ tends to zero, we may assume that $x'(\bar{t}) \approx x'(\hat{t})$. Therefore, in the last line of the expansion above we used the approximations

$$\frac{x(t_{k+1}) - x(t_k)}{t_{k+1} - t_k} \approx x'(\hat{t}_k) \quad \frac{y(t_{k+1}) - y(t_k)}{t_{k+1} - t_k} \approx y'(\hat{t}_k)$$

Thus the Riemann sum $\mathcal{S}(\mathcal{P})$ can be approximated by

$$\begin{aligned}
 \mathcal{S}(\mathcal{P}) &\approx \sum_{k=0}^n f(\sigma(\hat{t}_k)) \cdot [x'(\hat{t}_k) + iy'(\hat{t}_k)] \cdot (t_{k+1} - t_k) \\
 &= \sum_{k=0}^n f(\sigma(\hat{t}_k)) \cdot \sigma'(\hat{t}_k) \cdot \Delta t_k
 \end{aligned}$$

It is easy to see that the expression on the right-hand side is in fact a Riemann sum for the



integral $\int_a^b f(\sigma(t)) \cdot \sigma'(t) dt$, Therefore,

$$\begin{aligned} \int_{\sigma} f(z) dz &= \lim_{\Delta(\mathcal{P}) \rightarrow 0} \mathcal{S}(\mathcal{P}) \\ &= \lim_{\Delta(\mathcal{P}) \rightarrow 0} \sum_{k=0}^n f(\sigma(\hat{t}_k)) \cdot \sigma'(\hat{t}_k) \cdot \Delta t_k \\ &= \int_a^b f(\sigma(t)) \sigma'(t) dt \end{aligned}$$

The integral $\int_a^b f(\sigma(t)) \sigma'(t) dt$ exists because $f(z)$ is continuous, $\sigma(t)$ and $\sigma'(t)$ are also continuous, and therefore the function $f(\sigma(t)) \sigma'(t)$ is integrable over the interval $[a, b]$, and thus we obtain formula (5.8), on which the assertion of the theorem is based. ■

- In most cases, the path σ is an open or closed ($\sigma(a) = \sigma(b)$) piecewise smooth path. When the path is open, the integral is denoted by $\int_{\sigma} f(z) dz$, and when the path is closed, the notation is $\oint_{\sigma} f(z) dz$.
- In most books, the line integral $\int_{\sigma} f(z) dz$ is defined directly by formula (5.8):

$$\int_{\sigma} f(z) dz = \int_a^b f(\sigma(t)) \sigma'(t) dt$$

This of course avoids the discussion of Riemann sums and the proof of the last theorem, but it conceals the reasons for defining it in this way (for example, where did the path derivative $\sigma'(t)$ come from?). In any event, for practical purposes one may skip the discussion preceding formula (5.8) and regard it as the official definition of the line integral.

- Another way to interpret the integral of a complex function $f(z)$ along a path is to interpret $f(z)$ as a vector force field, where, for every planar point z , the number $f(z)$ represents the force vector acting at the point z . Then the expression $f(\sigma(t)) \sigma'(t)$ represents the component of the force acting at $\sigma(t)$ in the direction of motion along the path, and therefore the line integral $\int_a^b f(\sigma(t)) \sigma'(t) dt$ is a kind of computation of the work performed by the force field along the path σ .
- ◆ **Example 3:** Compute the line integral of $f(z) = \bar{z}^2$ along the path $\sigma(t) = t + ti$, $0 \leq t \leq 1$.
- 🔑 **Solution:** The path σ is the line segment joining the point $z = 0$ to the point $z = 1 + i$. This is of course a smooth path, so we may use Theorem 5.3.

$$\begin{aligned} \int_{\sigma} f(z) dz &= \int_0^1 (t - it)^2 (1 + i) dt = \int_0^1 (t^2 - 2it^2 - t^2)^2 (1 + i) dt \\ &= -2i(1 + i) \int_0^1 t^2 dt = -2i(1 + i) \frac{t^3}{3} \Big|_0^1 = \frac{2 - 2i}{3} \end{aligned}$$



i Proposition 5.4: Let $f(z)$ be a continuous function along a piecewise smooth path $s: [a, b] \rightarrow \mathbb{C}$. Then

$$\left| \int_{\sigma} f(z) dz \right| \leq \ell(\sigma) \cdot \text{Max}_{a \leq t \leq b} |f(\sigma(t))|$$

✓ Proof: By assumption, the function $|f(\sigma(t))|$ is continuous on the closed interval $[a, b]$, and therefore it has a maximum $M = \text{Max}_{a \leq t \leq b} |f(\sigma(t))|$.

$$\begin{aligned} \left| \int_{\sigma} f(z) dz \right| &= \left| \int_a^b f(\sigma(t)) \sigma'(t) dt \right| \leq \int_a^b |f(\sigma(t))| \cdot |\sigma'(t)| dt \\ &\leq \int_a^b M \cdot |\sigma'(t)| dt = M \int_a^b |\sigma'(t)| dt \\ &= \ell(\sigma) \cdot \text{Max}_{a \leq t \leq b} |f(\sigma(t))| \end{aligned}$$

We used Theorem 5.2, the path-length formula, and the properties of the definite integral. ■

i Proposition 5.5: If $\sigma: [a, b] \rightarrow \mathbb{C}$ is a piecewise smooth path and $f(z)$ is analytic on σ , then:

$$\int_{\sigma} f'(z) dz = f(\sigma(b)) - f(\sigma(a))$$

✓ Proof: By the chain rule (see Theorem 3.25),

$$\frac{d}{dt} [f(\sigma(t))] = f'(\sigma(t)) \cdot \sigma'(t)$$

Therefore, by the definition of the integral along a path,

$$\begin{aligned} \int_{\sigma} f'(z) dz &= \int_a^b f'(\sigma(t)) \cdot \sigma'(t) dt = \int_a^b \frac{d}{dt} [f(\sigma(t))] dt \\ &= f(\sigma(t)) \Big|_{t=a}^{t=b} = f(\sigma(b)) - f(\sigma(a)) \end{aligned}$$

■

i Corollary 5.6: If $\sigma: [a, b] \rightarrow \mathbb{C}$ is a piecewise smooth closed path, and if the function $f(z)$ has an analytic antiderivative on σ , then:

$$\oint_{\sigma} f(z) dz = 0$$

✓ Proof: Let $F(z)$ be an analytic antiderivative of $f(z)$. That is, $F'(z) = f(z)$ at every point



z on the path σ . By the preceding assertion,

$$\int_{\sigma} f(z) dz = F(\sigma(b)) - F(\sigma(a)) = 0$$

This follows because the path σ is closed, and hence $\sigma(a) = \sigma(b)$. ■

Remarks

1. The meaning of assertion 5.5 is that if f has an analytic antiderivative on σ , then the integral along the path σ depends only on the endpoints of σ . That is, for any two paths σ_1, σ_2 , if $\sigma_1(a) = \sigma_2(a)$, $\sigma_1(b) = \sigma_2(b)$, then $\int_{\sigma_1} f(z) dz = \int_{\sigma_2} f(z) dz$.
2. The assumption that $F(z)$ is analytic on the path σ of course includes the assumption that the path σ is contained in an open set D on which $F(z)$ is analytic. But the interior of the path σ need not also be contained in D (when σ is a closed path).
3. Also, note that the path σ is not required to be simple! All that is required is that it be closed and piecewise smooth. It may certainly intersect itself several times!
4. The fundamental theorem of Calculus 1 states that every continuous real-valued function $f(x)$ has a real-valued antiderivative $F(x)$ such that $F'(x) = f(x)$. At present we have no analogous theorem for complex functions, and we will have to work hard in order to prove the Cauchy-Goursat theorem, from which a result analogous to the fundamental theorem of calculus can be deduced.

◆ **Example 4:** Compute $\int_{\sigma} \frac{dz}{z}$ where $\sigma(t) = te^{it}$, $\pi \leq t \leq 2\pi$.

🌸 **Solution:** An antiderivative of $\frac{1}{z}$ is $\text{Log}(z)$. Therefore, by assertion 5.5,

$$\begin{aligned} \int_{\sigma} \frac{dz}{z} &= \text{Log}(\sigma(2\pi)) - \text{Log}(\sigma(\pi)) = \text{Log}(2\pi e^{2\pi i}) - \text{Log}(\pi e^{\pi i}) \\ &= \text{Log}(2\pi) - \text{Log}(-\pi) = \ln 2 - \pi i \end{aligned}$$

📌 **Proposition 5.7:** For every point $z_0 \in \mathbb{C}$ and every real number $r > 0$,

$$\oint_{|z-z_0|=r} \frac{dz}{z-z_0} = 2\pi i$$

where the expression $|z-z_0|=r$ in the integral will always denote the standard parametrization of the circle of radius r centered at z_0 : $\sigma(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$.

✔ **Proof:** We use the definition of the integral:

$$\oint_{|z-z_0|=r} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{\frac{d}{dt}[re^{it}]dt}{(z_0 + re^{it}) - z_0} = \int_0^{2\pi} \frac{ire^{it}dt}{re^{it}} = \int_0^{2\pi} i dt = 2\pi i$$





5.3 The Cauchy-Goursat Theorem (Cauchy-Goursat)

- The Cauchy-Goursat theorem is one of the most important and useful theorems in the theory of complex functions. It expresses one of the most important and useful properties of analytic functions: The integral of an analytic function along a closed path vanishes (under suitable conditions, of course).
- In most books, the theorem is stated for a triangular or rectangular path, or at most for a simple piecewise smooth path, but it is true for every closed path². We shall prove the theorem for a piecewise smooth closed path, despite the additional effort required, because this greatly simplifies the proof of the important consequences that follow from this theorem (in particular, Cauchy's integral formula).

i Theorem 5.8: (Cauchy-Goursat): Let $f(z)$ be analytic in an open set D , and let σ be a piecewise smooth closed path contained, together with its interior, in D . Then

$$\oint_{\sigma} f(z) dz = 0$$

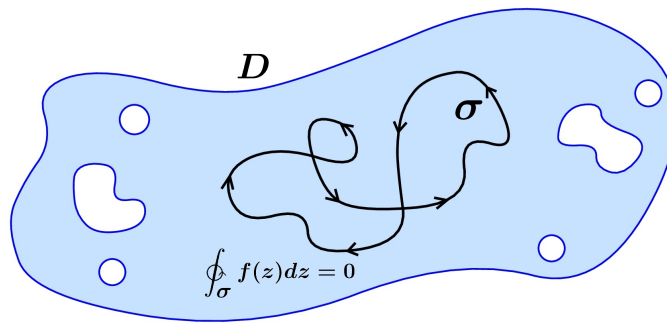


Fig. 5.2: The path σ and its interior are contained in an open set D

i Corollary 5.9: (Cauchy-Goursat 2): Let $f(z)$ be analytic in an open set D , and let σ_1, σ_2 be two paths that start at the same point and end at the same point, such that the domain bounded between them is contained in D . Then

$$\oint_{\sigma_1} f(z) dz = \oint_{\sigma_2} f(z) dz$$

✓ Proof: This assertion is equivalent to the Cauchy-Goursat theorem. If we reverse the direction

² However, this requires a more general definition of an integral along a general path that is not necessarily differentiable. Our definition, which is the customary definition in most textbooks, is based on the smoothness property of the path and therefore does not apply to a path that is merely continuous.



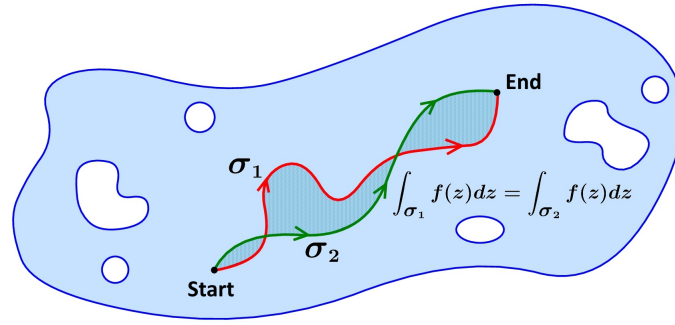


Fig. 5.3: An immediate consequence of the Cauchy-Goursat theorem: the value of the integral depends only on the initial and terminal points, not on the path

of the path σ_2 and concatenate it with the path σ_1 , we obtain a closed path $\sigma_1 + (-\sigma_2)$ that satisfies the hypotheses of the Cauchy-Goursat theorem, and therefore

$$\oint_{\sigma_1 + (-\sigma_2)} f(z) dz = 0$$

But

$$\oint_{\sigma_1 + (-\sigma_2)} f(z) dz = \oint_{\sigma_1} f(z) dz - \oint_{\sigma_2} f(z) dz = 0$$

Therefore, $\oint_{\sigma_1} f(z) dz = \oint_{\sigma_2} f(z) dz$. ■

➤ The proof of the Cauchy-Goursat theorem will be carried out in four stages:

- A. We present an easy proof for the case in which the domain D is a disk or a rectangle.
- B. We prove the theorem for a triangular path (and a general domain D).
- C. We prove the theorem for a general polygonal path (a closed polygon).
- D. We prove the theorem for a piecewise smooth closed path.

A. Proof of the Cauchy-Goursat Theorem for a Disk or Rectangle D

i Proposition 5.10: Let $u(x, y)$ and $v(x, y)$ be two continuously differentiable real-valued functions inside a rectangle or open disk D centered at the point (x_0, y_0) . If $u_y = v_x$ on D , then there exists a twice continuously differentiable real-valued function $P(x, y)$ such that $P_x = u$, $P_y = v$ throughout D .

✓ **Proof:** The function $P(x, y)$ is sometimes called a “potential function”, which is why it is denoted by P . For every point $(x, y) \in D$, define

$$P(x, y) = \int_{x_0}^x u(t, y_0) dt + \int_{y_0}^y v(x, t) dt \quad (5.10)$$

Both integrals are well defined because our domain D is a rectangle or a disk, and therefore the line segment joining the center point (x_0, y_0) to the point (x, y_0) , which is needed for the first integral, is contained entirely in D . Similarly, the line segment joining the point (x, y_0) to the

point (x, y) , which is needed for the second integral, is also contained entirely in D . By the fundamental theorem of differential calculus,

$$P_y(x, y) = v(x, y)$$

The first integral yields a function of x alone, and therefore its derivative with respect to y is 0. This proves the first part of the assertion. For P_x , we again use the fundamental theorem of differential calculus on the first integral in (5.10)

$$\frac{d}{dx} \int_{x_0}^x u(t, y_0) dt = u(x, y_0)$$

and the Leibniz rule on the second integral (see Theorem 5.33 on page 208), using the assumption $u_y = v_x$:

$$\begin{aligned} P_x(x, y) &= \frac{d}{dx} \int_{x_0}^x u(t, y_0) dt + \frac{d}{dx} \int_{y_0}^y v(x, t) dt \\ &= u(x, y_0) + \int_{y_0}^y v_x(x, t) dt \\ &= u(x, y_0) + \int_{y_0}^y u_y(x, t) dt \\ &= u(x, y_0) + u(x, y) - u(x, y_0) \\ &= u(x, y) \end{aligned}$$

Clearly, P is twice continuously differentiable, because its first-order partial derivatives are $P_x = u$, $P_y = v$, and both functions u , v are continuously differentiable. ■

i Theorem 5.11: Let $f(z)$ be analytic on a disk or rectangle D . Then f has a primitive on D .

✓ Proof: Write f in Cartesian form: $f(x + iy) = u(x, y) + iv(x, y)$. The pair of functions u , $-v$ satisfies the Cauchy-Riemann equation $u_y = -v_x$, and therefore the pair u , $-v$ satisfies the hypotheses of assertion 5.10. Thus there exists a twice continuously differentiable potential function $P(x, y)$ satisfying $P_x = u$, $P_y = -v$.

By the Cauchy-Riemann equations, we also have $u_x = v_y$, so there exists a potential function $Q(x, y)$ satisfying $Q_x = v$, $Q_y = u$. Define the function

$$F(x + iy) = P(x, y) + iQ(x, y)$$

It is not difficult to show that F satisfies the Cauchy-Riemann equations, and in addition the partial derivatives of P , Q are continuous on the domain D . Therefore, by Corollary 4.4 (page 119), F is analytic. It is also easy to see that $F'(z) = f(z)$ on the domain D . Thus F is a primitive of f in the domain D . ■

Corollary 5.12: (The Cauchy-Goursat theorem for a disk and a rectangle) Let $f(z)$ be analytic on a disk (or rectangle) D . Then, for every piecewise smooth closed path σ in D ,

$$\oint_{\sigma} f(z) dz = 0$$

Proof: By the preceding theorem, the function $f(z)$ has an antiderivative $F(z)$ on the domain D , and therefore, by Corollary 5.6, $\oint_{\sigma} f(z) dz = 0$. ■

Note that Corollary 5.12 does not concern a simple path! The only requirements are that the path σ be piecewise smooth and closed, but it may intersect itself at intermediate points.

B. Proof of the Cauchy-Goursat Theorem for a Triangular Path

In stage B we prove the Cauchy-Goursat theorem 5.8 for a triangular path in an open set D . The only assumption we need, of course, is that the interior of the triangle σ is contained in the set D .

Set $I = \oint_{\sigma} f(z) dz$. Our goal is to prove that $I = 0$. Assume, to the contrary, that $I > 0$, and derive a contradiction.

Set $\sigma_0 = \sigma$, and denote by Δ_0 the triangle bounded by σ_0 . Divide the triangle Δ_0 into four triangles, as shown in Figure 5.4. It is easy to see from the figure that

$$\begin{aligned} I &= \left| \oint_{\gamma_1} f(z) dz + \oint_{\gamma_2} f(z) dz + \oint_{\gamma_3} f(z) dz + \oint_{\gamma_4} f(z) dz \right| \\ &\leq \left| \oint_{\gamma_1} f(z) dz \right| + \left| \oint_{\gamma_2} f(z) dz \right| + \left| \oint_{\gamma_3} f(z) dz \right| + \left| \oint_{\gamma_4} f(z) dz \right| \end{aligned}$$

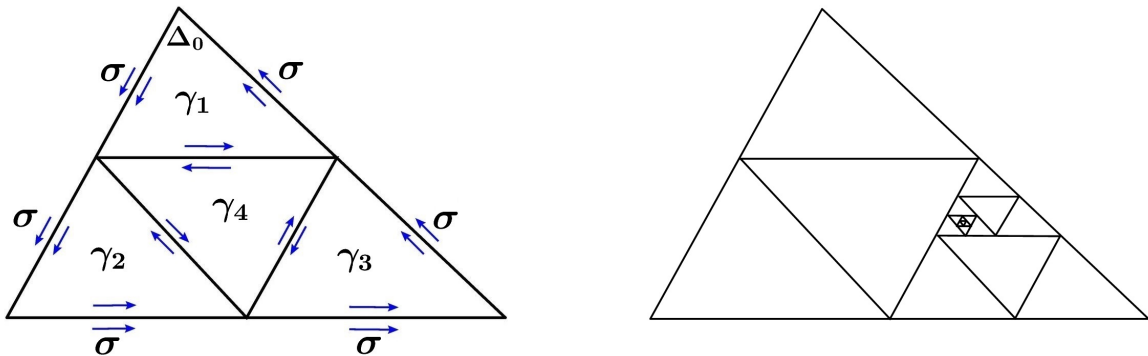


Fig. 5.4: Dividing a triangular path σ into four triangular paths $\gamma_1, \gamma_2, \gamma_3, \gamma_4$

Clearly, at least one of the integrals on the right-hand side must be greater than or equal to $\frac{I}{4}$. Denote the corresponding path by σ_1 and the corresponding triangle by Δ_1 . Repeating this operation over and over yields a sequence of triangles $\{\Delta_n\}_{n=0}^{\infty}$ and a sequence of paths

$\{\sigma_n\}_{n=0}^\infty$, $\sigma_n = \partial\Delta_n$, with the following properties:

$$(5.11) \quad \ell(\sigma_n) = \frac{\ell(\sigma)}{2^n}$$

$$(5.12) \quad \Delta_{n+1} \subseteq \Delta_n$$

$$(5.13) \quad \text{diam}(\Delta_n) \leq \frac{\ell(\sigma)}{2^n} \xrightarrow{n \rightarrow \infty} 0$$

$$(5.14) \quad \left| \oint_{\sigma_n} f(z) dz \right| \geq \frac{I}{4^n}$$

The notion of the diameter $\text{diam}(\Delta_n)$ of a set appears in Definition 2.3. Cantor's lemma (see assertion 2.20 on page 68) guarantees that the intersection of the sequence of triangles consists of one and only one point z_0 , which of course lies in D (and is therefore an interior point of D , since D is open).

$$\bigcap_{n=1}^{\infty} \Delta_n = \{z_0\}$$

Let $r > 0$ be such that $D(z_0, r) \subseteq D$. By Cantor's lemma, there exists a natural number N such that for every $n \geq N$, $\Delta_n \subseteq D(z_0, r)$. Therefore the path σ_n , together with its interior, is contained in the disk $D(z_0, r)$. By Corollary 5.12, $\oint_{\sigma_n} f(z) dz = 0$ contrary to inequality (5.14). $\left| \oint_{\sigma_n} f(z) dz \right| \geq \frac{I}{4^n}$. Therefore $I = 0$ necessarily. That is, we obtain $\oint_{\sigma} f(z) dz = 0$. ■

- Note that without the assumption that the interior of the triangle σ is contained in the open set D , the above argument would not work, because there is no guarantee that the limit point z_0 belongs to the set D ! This is the critical point at which the assumption that the interior of the path σ must be contained in D is needed.

C. Proof of the Cauchy-Goursat Theorem for a Polygon

- We begin with a visual illustration of the idea of the proof using two simple examples: a quadrilateral and a pentagon. The quadrilateral $\sigma = [z_0, z_1, z_2, z_3]$ in Figure 5.5 is decomposed into two triangles

$$\sigma_1 = [z_0, z_1, z_2, z_0], \quad \sigma_2 = [z_2, z_3, z_0, z_2]$$

with a common side $[z_0, z_2]$ (in opposite directions).

Attention must also be paid to the direction of integration (counterclockwise rotation, which is

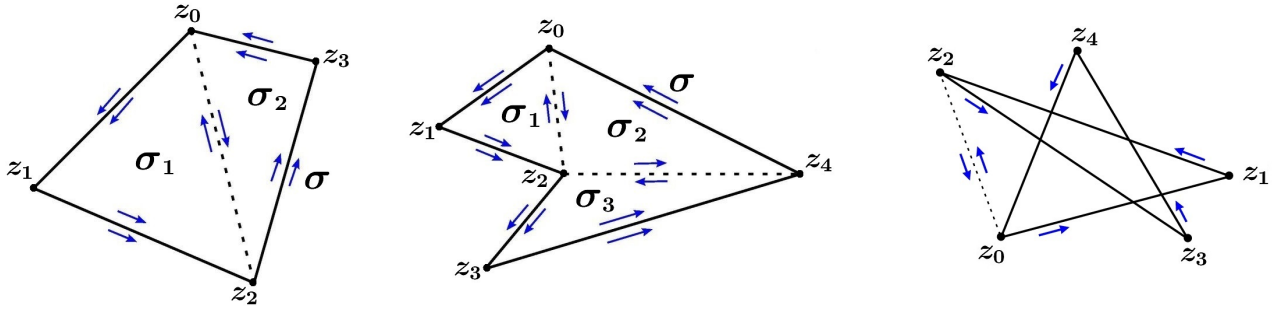


Fig. 5.5: Decomposing polygonal paths into sums of triangular paths

the positive direction).

$$\begin{aligned}
 \oint_{\sigma} f(z)dz &= \oint_{[z_0, z_1]} f(z)dz + \oint_{[z_1, z_2]} f(z)dz + \oint_{[z_2, z_3]} f(z)dz + \oint_{[z_3, z_0]} f(z)dz \\
 &= \left[\oint_{[z_0, z_1]} f(z)dz + \oint_{[z_1, z_2]} f(z)dz + \oint_{[z_2, z_0]} f(z)dz \right] + \left[\oint_{[z_0, z_2]} f(z)dz + \oint_{[z_2, z_3]} f(z)dz + \oint_{[z_3, z_0]} f(z)dz \right] \\
 &= \oint_{\sigma_1} f(z)dz + \oint_{\sigma_2} f(z)dz \\
 &= 0 + 0 = 0
 \end{aligned}$$

In the second line we added the pair of integrals $\oint_{[z_2, z_0]} f(z)dz + \oint_{[z_0, z_2]} f(z)dz$ whose sum is of course zero because the segments have opposite directions. $[z_2, z_0]$, $[z_0, z_2]$. A similar expansion can be constructed for the pentagon, but this is of course not the way to give a formal proof. The formal proof will be carried out by mathematical induction on the number of sides n .

The base case is the preceding result for $n = 3$. Assume that $\oint_{\sigma} f(z)dz = 0$ for every polygon σ with n sides, and prove the result for a simple polygon σ with $n + 1$ sides.

$$\sigma = [z_0, z_1, z_2, \dots, z_{n-1}, z_n, z_0]$$

Assuming that the polygon σ is nondegenerate³, without loss of generality we may assume that every three consecutive vertices $[z_k, z_{k+1}, z_{k+2}]$ form a nondegenerate triangle. Decompose the polygon σ into a triangle

$$\sigma_1 = [z_0, z_1, z_2, z_0]$$

and a polygon with n sides:

$$\sigma_2 = [z_0, z_2, z_3, \dots, z_n, z_0]$$

We see that the polygon σ_2 is obtained from the original polygon σ by removing the vertex z_1 . Since the assertion concerns general polygons (which may intersect themselves at intermediate points), it is advisable to examine the star on the right in Figure 5.5 in order to follow the idea.

³ If σ is a polygon all of whose vertices lie on the same line, the result is trivial

For simplicity and brevity, we omit the expression $f(z)dz$ from the integrals.

$$\begin{aligned}
 \oint_{\sigma} &= \int_{[z_0, z_1]} + \int_{[z_1, z_2]} + \cdots + \int_{[z_{n-1}, z_n]} + \int_{[z_n, z_0]} \\
 &= \left[\int_{[z_0, z_1]} + \int_{[z_1, z_2]} + \int_{[z_2, z_0]} \right] + \left[\int_{[z_0, z_2]} + \int_{[z_2, z_3]} + \cdots + \int_{[z_{n-1}, z_n]} + \int_{[z_n, z_0]} \right] \\
 &= \oint_{\sigma_1} + \oint_{\sigma_2} = 0 + 0 = 0
 \end{aligned}$$

The passage from line 1 to line 2 is made by adding the expression $\oint_{[z_2, z_0]} + \oint_{[z_0, z_2]}$ which is of course equal to zero, and hence the equality is justified. The integral $\oint_{\sigma_1} f(z)dz$ vanishes because the path σ_1 is a triangle, and the integral $\oint_{\sigma_2} f(z)dz$ vanishes by the induction hypothesis (σ_2 is a polygon with n sides). ■

D. Proof of the Cauchy-Goursat Theorem for a Piecewise Smooth Closed Path

➤ At long last, we have reached the proof of the general Cauchy-Goursat theorem 5.8.

The path σ is of course a closed and bounded subset of D , and therefore, by Exercise 28 at the end of Chapter 2, there exists $r > 0$ such that for every point z on the path σ , the disk $\mathbb{D}(z, r)$ is contained entirely in the set D (it is advisable to solve Exercise 28 of Chapter 2 again before proceeding).

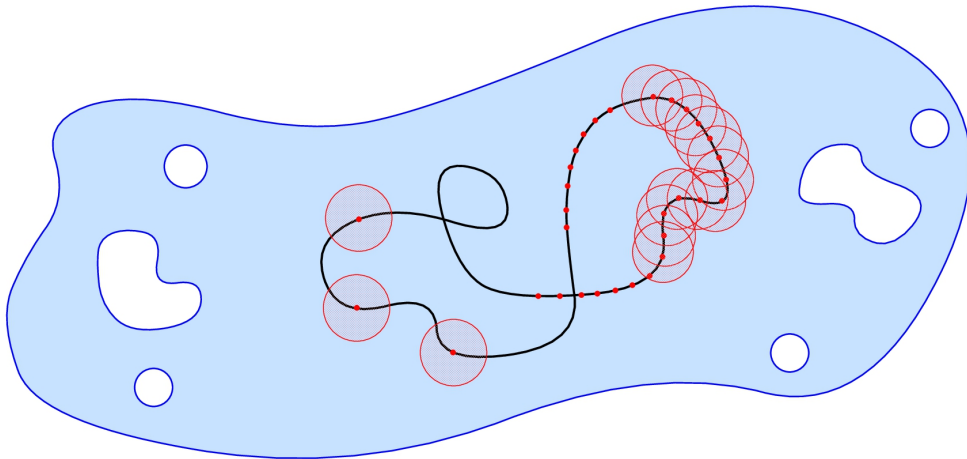


Fig. 5.6: Covering the path σ by open disks of radius r

Without loss of generality, we may assume that the length of the path σ is finite⁴ and therefore we can divide it into finitely many subpaths $\{\sigma_k\}_{k=0}^n$, by choosing finitely many partition points $z_0, z_1, z_2, \dots, z_n$, on the path, in accordance with the direction of traversal of σ (see Figures 5.6, 5.7), such that the length of each subpath σ_k is less than $r/2$. For every $0 \leq k < n$, the path σ_k begins at z_k and ends at z_{k+1} . The path σ_n begins at z_n but ends at z_0 .

Figure 5.7 shows a small portion of the larger picture. Several partition points can be seen there,

⁴ The extreme case in which σ is a path of infinite length can be handled by representing it as a limit of finite paths, but we will not bother to do so here because paths of this kind are not relevant to the present course.

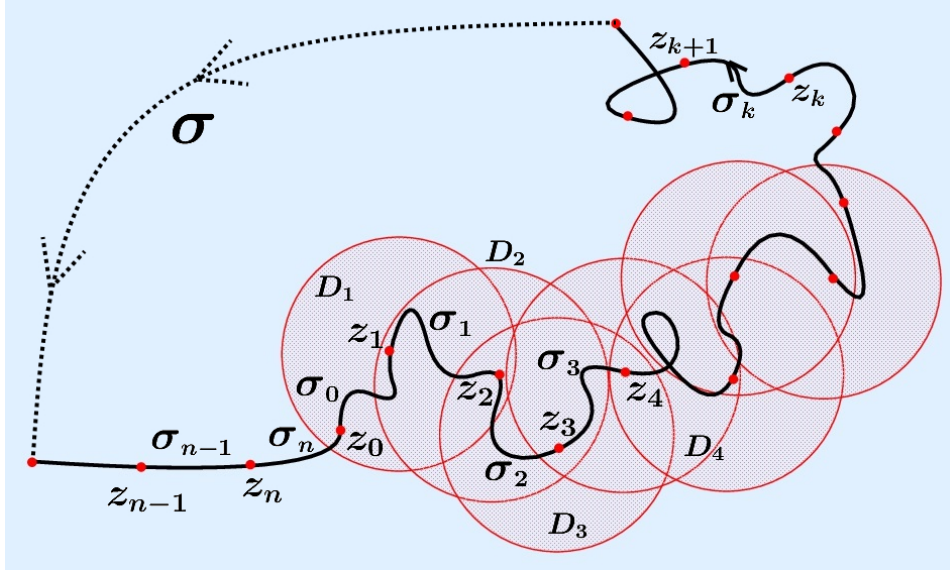


Fig. 5.7: Dividing the path σ into subpaths $\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_n$. Each subpath belongs to a corresponding disk.

together with the disks covering the path (denoted by $D_k = \mathbb{D}(z_k, r)$), and several of the subpaths σ_k joining the partition points z_k, z_{k+1} .

The requirement that the length of each subpath σ_k be less than $r/2$ ensures that the disk $\mathbb{D}(z_k, r)$ contains σ_k in its entirety, and of course also contains the next partition point z_{k+1} , as well as the line segment $[z_k, z_{k+1}]$.

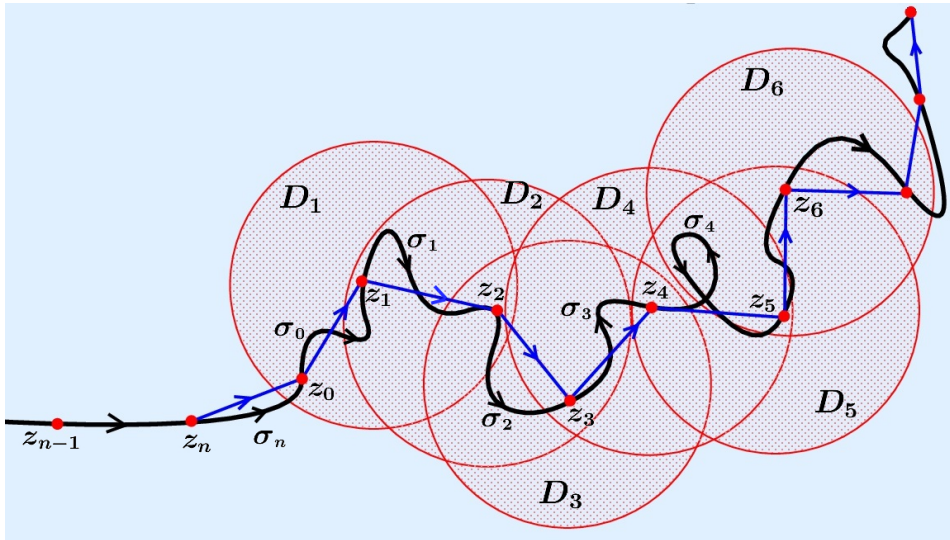


Fig. 5.8: Constructing a polygon equivalent to the path σ by drawing line segments between the centers of the covering disks

If we adjoin the line segment $[z_{k+1}, z_k]$ (in the opposite direction!) to the subpath σ_k , we obtain a closed path inside the disk $\mathbb{D}(z_k, r)$. See Figure 5.8. By Corollary 5.12 (the Cauchy-Goursat theorem for a circular domain),

$$\oint_{\sigma_k + [z_{k+1}, z_k]} f(z) dz = 0$$

and therefore

$$\int_{\sigma_k} f(z) dz = \int_{[z_k, z_{k+1}]} f(z) dz$$

Denote by the following symbol the polygon obtained by joining the partition vertices:

$$\gamma = [z_0, z_1, z_2, \dots, z_n, z_0]$$

By the preceding stage of the proof (the Cauchy-Goursat theorem for a polygonal path),

$$\oint_{\gamma} f(z) dz = 0$$

But

$$\begin{aligned} \oint_{\sigma} f(z) dz &= \int_{\sigma_0} f(z) dz + \int_{\sigma_1} f(z) dz + \dots + \int_{\sigma_n} f(z) dz \\ &= \int_{[z_0, z_1]} f(z) dz + \int_{[z_1, z_2]} f(z) dz + \dots + \int_{[z_{n-1}, z_n]} f(z) dz + \int_{[z_n, z_0]} f(z) dz \\ &= \oint_{\gamma} f(z) dz = 0 \end{aligned}$$

This completes the proof of the full Cauchy-Goursat theorem. ■

➤ We leave the proof of the following corollary to the student as an exercise.

i Corollary 5.13: Let $f(z)$ be analytic on a simply connected domain D . Then f has an antiderivative on D , equal up to a constant to the function

$$F(z) = \int_{z_0}^z f(w) dw$$

where z_0 is a fixed point in the domain D .

Hint: Note that by the Cauchy-Goursat theorem, the integral along any piecewise smooth path from the point z_0 to the point z yields the same result, and therefore the expression $\int_{z_0}^z f(w) dw$ has an unambiguous meaning.

5.4 Cauchy's Integral Formula and Its Consequences

➤ The most important consequence of the Cauchy-Goursat theorem is Cauchy's integral formula, which has several important consequences and many diverse applications, such as computing integrals of real functions that cannot be solved by the methods of Calculus 1 or 2.

The “Bridge” Method for Joining Paths

➤ For this purpose we shall need several topological tools for handling paths. We begin with a



simple technique by which several distinct closed paths can be joined into a single closed path. We note again that we are not restricted to simple paths (which do not intersect at intermediate points).

i Proposition 5.14: Let $\sigma: [a, b] \rightarrow \mathbb{C}$ be a closed path, and let z_0 be any point on the path. Then there exists an equivalent parametrization $\gamma: [a, b] \rightarrow \mathbb{C}$ for which $\gamma(a) = \gamma(b) = z_0$.

✓ Proof: In other words, if σ is a closed path, then every point z_0 on the path has a parametrization in which z_0 is the initial and terminal point of the path. If $\sigma(a) = \sigma(b) = z_0$, we are done. Otherwise, there exists a point $a < t_0 < b$ such that $\sigma(t_0) = z_0$. Define a “rotation” function $\varphi: [a, b] \rightarrow [a, b]$ by

$$\varphi(t) = \begin{cases} t + t_0 - a, & a \leq t \leq a + b - t_0 \\ t + t_0 - b, & a + b - t_0 < t \leq b \end{cases}$$

The term “rotation” refers to a geometric interpretation in which the interval is imagined as a circle on which the points a, b merge into a single point. Figure 5.9 gives a geometric illustration of what happens under the rotation map φ .

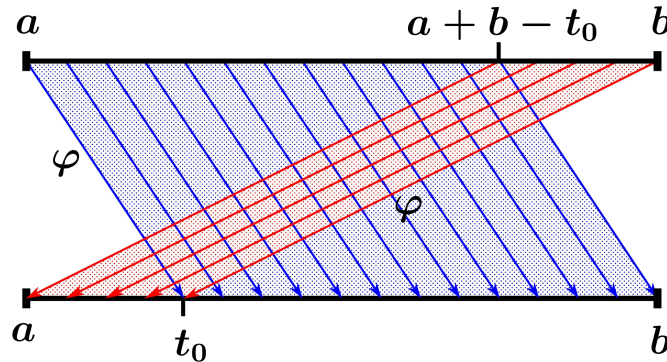


Fig. 5.9: The “rotation” function $\varphi: [a, b] \rightarrow [a, b]$

Define a parametrization $\gamma: [a, b] \rightarrow \mathbb{C}$ by $\gamma(t) = \sigma(\varphi(t))$.

It is not difficult to verify that $\gamma(a) = \gamma(b) = z_0$. ■

➤ The “bridge” method for joining closed paths:

Let $\sigma_1: [a_1, b_1] \rightarrow \mathbb{C}$, $\sigma_2: [a_2, b_2] \rightarrow \mathbb{C}$, be two closed paths. Let z_1 be a point on the path σ_1 , and let z_2 be a point on the path σ_2 . Any path joining z_1 to z_2 can be used, but in most cases the line segment $[z_1, z_2]$ is quite sufficient.

The idea is very simple: By assertion 5.14, we may assume that

$$\sigma_1(a_1) = \sigma_1(b_1) = z_1, \quad \sigma_2(a_2) = \sigma_2(b_2) = z_2$$

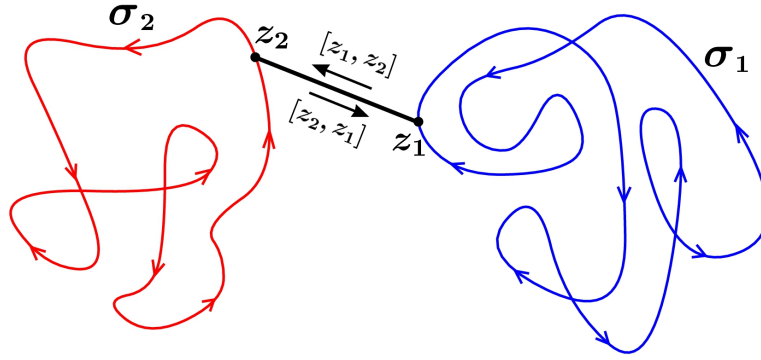


Fig. 5.10: Joining two paths σ_1 , σ_2 by a straight bridge

Define the joined path σ by concatenating the four paths (see Figure 5.10):

$$\sigma = \sigma_1 + [z_1, z_2] + \sigma_2 + [z_2, z_1]$$

The path σ begins at the point z_1 , traverses the path σ_1 and returns to the point z_1 , traverses the line segment $[z_1, z_2]$ and reaches the point z_2 (which, as stated, is the initial point of σ_2), traverses the path σ_2 and returns to the point z_2 , and from there returns through the line segment $[z_2, z_1]$ to the initial point z_1 .

- We begin with the following consequence of the Cauchy-Goursat theorem, which will help us later. The corollary handles cases in which the interior of the path is not entirely contained in the domain of the function. We use the term “Jordan path” to mean a simple closed path.

i Proposition 5.15: Let $f(z)$ be analytic on a domain D . Let σ_1 , σ_2 be two Jordan paths contained in the domain D , and suppose that σ_1 surrounds σ_2 . If f is analytic in the region between σ_1 and σ_2 (but is not necessarily analytic or defined in the interior of σ_2), then

$$\oint_{\sigma_1} f(z)dz = \oint_{\sigma_2} f(z)dz$$

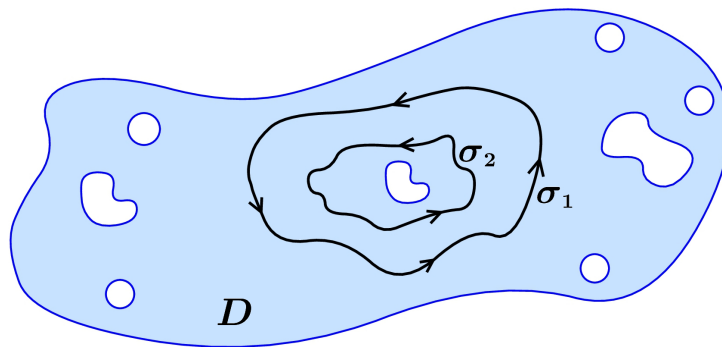


Fig. 5.11: A Jordan path σ_1 surrounds a Jordan path σ_2 , which surrounds a “hole” of D

- ✓ **Proof:** Choose a point z_1 on the path σ_1 and a point z_2 on the path σ_2 such that the distance between z_1 and z_2 is minimal (see Exercise 23 in Chapter 2). Minimality ensures that

the segment $[z_1, z_2]$ does not intersect either path at intermediate points. By assertion 5.14, we may assume that z_1 is the initial and terminal point of σ_1 , and that z_2 is the initial and terminal point of σ_2 . Construct a bridge $[z_1, z_2]$ between the path σ_1 and the path $-\sigma_2$ (this is the path s_2 with its direction reversed), and use it to define a new path σ , as described by the bridge method in the preceding subsection. See Figure 5.12.

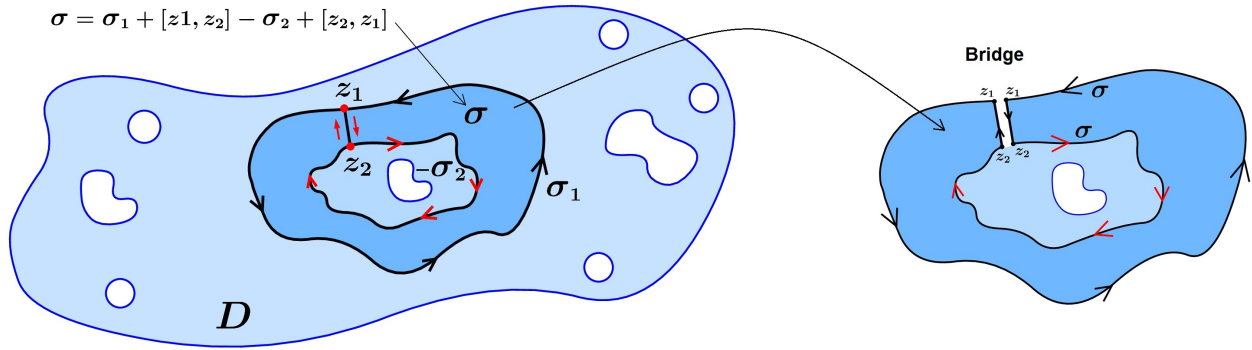


Fig. 5.12: The bridge method for joining two paths into a single path. The bridge is actually a single segment joining the two paths.

This is of course not a simple path, because it traverses the bridge $[z_1, z_2]$ twice, but our version of the Cauchy-Goursat theorem also supports such paths. On the right-hand side of Figure 5.12, we doubled the bridge $[z_1, z_2]$ in order to illustrate the path σ . Of course, this is only an illustrative device; these are not two different segments, but a single segment. We can express the path σ by the following expression:

$$\sigma = \sigma_1 + [z_1, z_2] + (-\sigma_2) + [z_2, z_1]$$

The figure shows that the path σ bypasses the hole surrounded by the paths σ_1 , σ_2 , and therefore σ and its interior are contained entirely in the domain D . Hence, by the Cauchy-Goursat theorem (Theorem 5.8),

$$\oint_{\sigma} f(z) dz = 0$$

Therefore,

$$\begin{aligned} \oint_{\sigma} f(z) dz &= \oint_{\sigma_1 + [z_1, z_2] + (-\sigma_2) + [z_2, z_1]} f(z) dz \\ &= \oint_{\sigma_1} f(z) dz + \int_{[z_1, z_2]} f(z) dz + \oint_{-\sigma_2} f(z) dz + \int_{[z_2, z_1]} f(z) dz \\ &= \oint_{\sigma_1} f(z) dz + \int_{[z_1, z_2]} f(z) dz - \oint_{\sigma_2} f(z) dz - \int_{[z_1, z_2]} f(z) dz \\ &= \oint_{\sigma_1} f(z) dz - \oint_{\sigma_2} f(z) dz = 0 \end{aligned}$$

Therefore,

$$\oint_{\sigma_1} f(z)dz = \oint_{\sigma_2} f(z)dz \quad \blacksquare$$

i Corollary 5.16: Let σ be a Jordan path contained in the domain D , let z_0 be a point in the interior of σ , and let σ_r be a circle centered at z_0 , with radius r , whose interior is contained entirely in the interior of σ . Let $f(z)$ be analytic in the domain D , except possibly at the point z_0 . Then

$$\oint_{\sigma} f(z)dz = \oint_{\sigma_r} f(z)dz \quad (5.15)$$

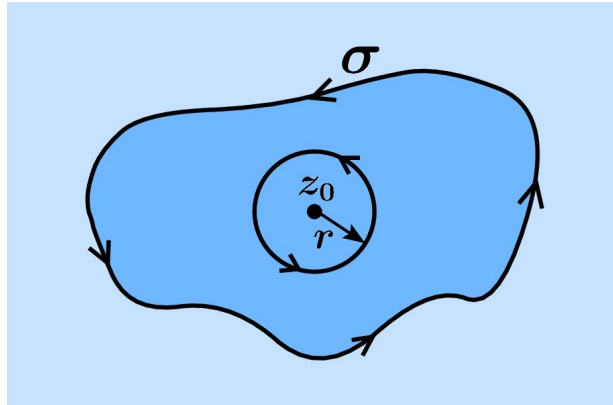


Fig. 5.13: If a piecewise smooth closed Jordan path σ surrounds the point z_0 , then $\oint_{\sigma} \frac{dz}{z-z_0} = 2\pi i$

i Corollary 5.17: Let σ be a piecewise smooth Jordan path, and let z_0 be a point in the interior of σ . Then

$$\frac{1}{2\pi i} \oint_{\sigma} \frac{dz}{z-z_0} = 1 \quad (5.16)$$

✓ Proof: Since the point z_0 lies in the interior of the path σ , there exists $r > 0$ such that the disk $|z - z_0| < r$ is contained in the interior of σ . By the preceding corollary,

$$\oint_{\sigma} \frac{dz}{z-z_0} = \oint_{|z-z_0|=r} \frac{dz}{z-z_0}$$

The integral around the circle can be computed easily using the standard parametrization of the circle: $\sigma_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$

$$\oint_{|z-z_0|=r} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{\frac{d}{dt}[z_0 + re^{it}]dt}{(z_0 + re^{it}) - z_0} = \int_0^{2\pi} \frac{ire^{it} dt}{re^{it}} = i \int_0^{2\pi} dt = 2\pi i$$

and we obtain formula (5.16). ■

➤ By repeated use of the bridge method, assertion 5.15 can be generalized as follows. The following corollary will be very important in the proof of Cauchy's residue theorem (Chapter 7).

i Corollary 5.18: Let $\sigma, \sigma_1, \sigma_2, \dots, \sigma_n$ be $n + 1$ piecewise smooth Jordan paths in a domain D with n holes, as shown in Figure 5.14. Then

$$\oint_{\sigma} f(z)dz = \oint_{\sigma_1} f(z)dz + \oint_{\sigma_2} f(z)dz + \dots + \oint_{\sigma_n} f(z)dz \quad (5.17)$$

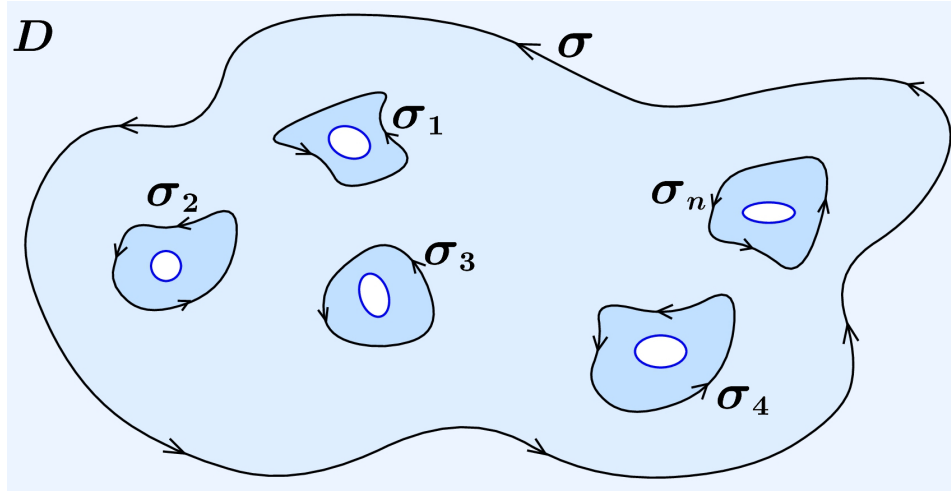


Fig. 5.14: The integral along the surrounding path equals the sum of the integrals along the paths contained within it

✓ Proof: We give only a sketch of the idea of the proof, without entering into all the fine details. As in the proof of assertion 5.15, draw a bridge $[z_k, w_k]$ between each path σ_k and the large path σ . Figure 5.15 provides a visual illustration in which all the bridges are line segments, but in practice there may be situations in which polygonal bridges are required, because a line segment between σ and a path σ_k does not always exist. Divide the path σ into k simple paths $\tau_1, \tau_2, \dots, \tau_n$, according to the partition points $z_1, z_2, \dots, z_n$ (for every $k < n$, $\tau_k = [z_k, z_{k+1}]$, $\tau_n = [z_n, z_1]$).

$$\begin{aligned} \gamma &= [z_1, w_1] + (-\sigma_1) + [w_1, z_1] \\ &\quad + \tau_1 + [z_2, w_2] + (-\sigma_2) + [w_2, z_2] + \\ &\quad + \tau_2 + [z_3, w_3] + (-\sigma_3) + [w_3, z_3] + \\ &\quad \vdots \\ &\quad + \tau_{n-1} + [z_n, w_n] + (-\sigma_n) + [w_n, z_n] + \\ &\quad + \tau_n \end{aligned} \quad (5.18)$$

The figure shows that the interior of the path γ (the region highlighted in dark cyan and surrounded by a dashed line) is contained in the domain D , because it bypasses all the holes (shown in white in the figure) and leaves them outside. Therefore, by the Cauchy-Goursat theorem 5.8, $\oint_{\gamma} f(z)dz = 0$.

Also note that the directions of the paths σ_k have been reversed, and what we see in Figure 5.15

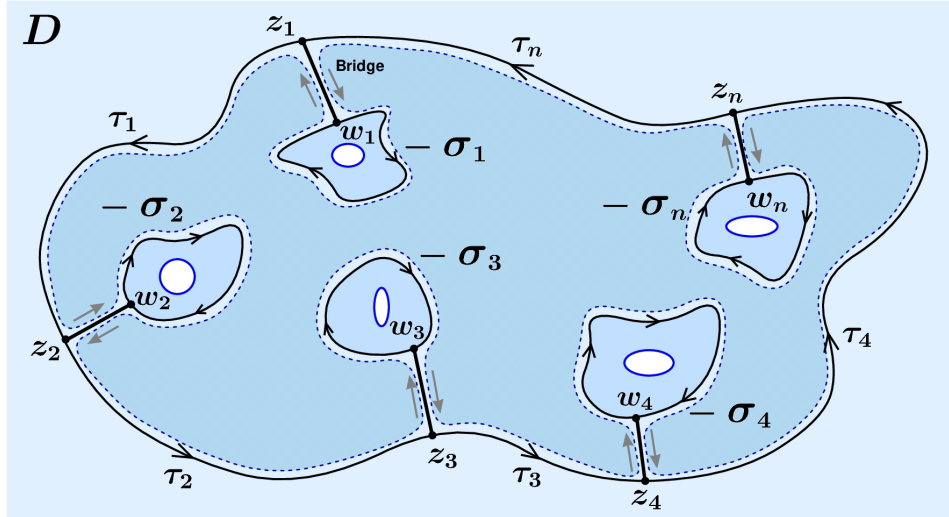


Fig. 5.15: The integral along the large path equals the sum of the integrals along the paths contained within it

are the reversed paths $-\sigma_k$. Define a new path γ by concatenating these paths, as written on the right-hand side of (5.18). From simple path arithmetic applied to the paths on the right-hand side of equality (5.18), we see that all the bridges cancel, and the sum of the paths τ_k gives σ ,

$$\begin{aligned} 0 &= \oint_{\gamma} f(z) dz = \sum_{k=1}^n \int_{\tau_k} f(z) dz + \sum_{k=1}^n \oint_{-\sigma_k} f(z) dz \\ &= \oint_{\sigma} f(z) dz - \sum_{k=1}^n \oint_{\sigma_k} f(z) dz \end{aligned}$$

and after transferring terms from one side to the other, we obtain equality (5.17). ■

Cauchy's Integral Formula (Cauchy)

- The first important consequence we obtain from the Cauchy-Goursat theorem is **Cauchy's integral formula**.
- The integral of a complex function $f(z)$ along a path σ can be interpreted as a kind of “infinite sum” of the values of the function $f(z)$ along the path. If we divide this sum by the length of the path, we obtain the average of $f(z)$ on the path σ . If we multiply each value of $f(z)$ by a weight function $W(z)$, we obtain a weighted average ($W(z)$ is the weight function).
- In short: an expression of the form $\frac{1}{\ell(\sigma)} \oint_{\sigma} f(z) W(z) dz$ represents a weighted average of f over σ . In Cauchy's integral formula, the weight function is $W(z) = \frac{1}{z - z_0}$, for a given point z_0 (that is, the weight of a point decreases inversely with its distance from the center z_0). If σ is a piecewise smooth Jordan path, and if f is analytic on and inside σ , then Cauchy's integral formula states that the weighted average of f along σ with respect to the weight function $\frac{1}{z - z_0}$ gives $f(z_0)$. We note only that the length of σ is represented symbolically by $2\pi i$ (the angular length of a circle).

$$f(z_0) = \frac{1}{2\pi i} \oint_{\sigma} \frac{f(z)}{z - z_0} dz$$

Thus we have obtained an intuitive justification for Cauchy's integral formula. We now turn to the precise statement of the theorem.

- In the two preceding proofs we used the terms “interior” and “exterior” of a closed path without giving them a precise mathematical definition. For a simple path, the Jordan theorem 2.13 (page 63) gives an unambiguous characterization of these two terms, but does not address the general case. For many years it was believed that the Jordan theorem was essential for a course in complex functions. Recently, a new approach has been developing, based on the notion of the index (or winding number) of a path with respect to a point, which can be used to give a precise and more general definition of **interior** and **exterior** for every closed path (even if it intersects itself at intermediate points). In order not to add to the burden of the present chapter, this topic is treated in Exercises 63–59 at the end of the chapter.

i Theorem 5.19: (Cauchy's Integral Formula Cauchy) Let $f(z)$ be a continuous function in a domain D , and let σ be a Jordan path that is piecewise smooth and positively oriented, such that it and its interior are contained in D , and let z_0 be a point in the interior of σ . Then

$$f(z_0) = \frac{1}{2\pi i} \oint_{\sigma} \frac{f(z)}{z - z_0} dz \quad (5.19)$$

✓ **Proof:** Throughout the proof we refer to Figure 5.16.

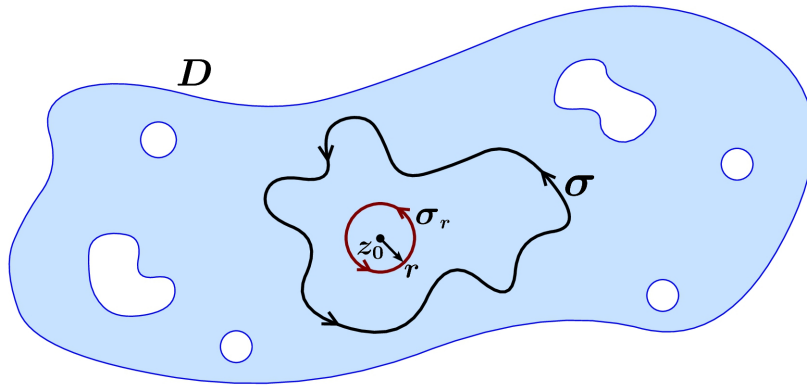


Fig. 5.16: Integrating a function $f(z)$ along a piecewise smooth Jordan path σ surrounding the point z_0

Since z_0 is an interior point, there exists $r > 0$ such that the circle σ_r centered at z_0 is contained entirely in the interior of the path σ .

By Corollary 5.16,

$$\oint_{\sigma} \frac{f(z)}{z - z_0} dz = \oint_{\sigma_r} \frac{f(z)}{z - z_0} dz \quad (1)$$

By the linear approximation formula (assertion 3.20 on page 105), there exists a function $\rho(z)$ in a neighborhood of z_0 such that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)$$

and such that $\lim_{z \rightarrow z_0} \rho(z) = 0$, and therefore

$$(2) \quad \oint_{\sigma_r} \frac{f(z)}{z - z_0} dz = \oint_{\sigma_r} \frac{f(z_0) + f'(z_0)(z - z_0) + \rho(z)(z - z_0)}{z - z_0} dz$$

$$(3) \quad = \oint_{\sigma_r} \frac{f(z_0)}{z - z_0} dz + \oint_{\sigma_r} f'(z_0) dz + \oint_{\sigma_r} \rho(z) dz$$

$$(4) \quad = \oint_{\sigma_r} \frac{f(z_0)}{z - z_0} dz + \oint_{\sigma_r} \rho(z) dz$$

$$(5) \quad = f(z_0) \oint_{\sigma_r} \frac{1}{z - z_0} dz + \oint_{\sigma_r} \rho(z) dz$$

$$(6) \quad = 2\pi i f(z_0) + \oint_{\sigma_r} \rho(z) dz \xrightarrow{r \rightarrow 0} 0$$

A. In equation (3), the second integral $\oint_{\sigma_r} f'(z_0) dz = f'(z_0) \oint_{\sigma_r} 1 \cdot dz$ vanishes because the constant function $g(x) \equiv 1$ has an antiderivative on all of $\mathbb{C}$.

B. It is easy to see that the integral $\oint_{\sigma_r} \rho(z) dz$ tends to zero as $r \rightarrow 0$.

$$\left| \oint_{\sigma_r} \rho(z) dz \right| \leq \ell(\sigma_r) \cdot \sup_{|z - z_0| = r} |\rho(z)| = 2\pi r \cdot \sup_{|z - z_0| = r} |\rho(z)| \xrightarrow{r \rightarrow 0} 0 \cdot 0 = 0$$

But in fact, $\oint_{\sigma_r} \rho(z) dz = 0$ follows directly from Corollary 5.12 (because $\rho(z)$ is analytic and σ_r can be surrounded by a disk contained in the domain).

C. In equation (5), the passage from the integral $\oint_{\sigma_r} \frac{1}{z - z_0} dz$ to the value $2\pi i$ follows from Corollary 5.17.

D. By equation (1), the value of the integral $\oint_{\sigma_r} \frac{f(z)}{z - z_0} dz$ does not depend on the radius r , and this is possible only if $\oint_{\sigma_r} \rho(z) dz = 0$ for every $r > 0$ in the domain.

We are therefore left with the equality

$$\oint_{\sigma_r} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

Therefore, by equality (1),

$$\oint_{\sigma} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

and this is precisely Cauchy's integral formula. ■



❖ **Example 5:** Compute the value of the integral $\oint_{\sigma} \frac{e^z}{z^2 - 2z} dz$ around the canonical circle $|z| = 3$.

🔗 **Solution:** We may choose the parametrization $\sigma(t) = 3e^{it}$, $0 \leq t \leq 2\pi$, but any other parametrization will do, provided it has positive orientation.

$$\oint_{\sigma} \frac{e^z}{z^2 - 2z} dz = \frac{1}{2} \oint_{\sigma} \frac{e^z}{z - 2} dz - \frac{1}{2} \oint_{\sigma} \frac{e^z}{z} dz = \frac{1}{2} \cdot 2\pi i e^2 - \frac{1}{2} \cdot 2\pi i e^0 = \pi i (e^2 - 1)$$

The function $f(z) = e^z$ is analytic on the entire complex plane, and the points $z = 0$, $z = 2$ lie inside the circle $|z| = 3$. Therefore, by Cauchy's integral formula,

$$2\pi i f(z_0) = 2\pi i e^{z_0} = \oint_{\sigma} \frac{f(z)}{z - z_0} dz = \oint_{\sigma} \frac{e^z}{z - z_0} dz$$

Substituting $z_0 = 0$, $z_0 = 2$ into this formula yields the result obtained at the end of our computation.

📌 **Corollary 5.20: (Mean-value formula)** Let $f(z)$ be analytic in the closed disk $\overline{D}(z_0, r)$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt$$

🔗 **Proof:** The assumption of analyticity on the closed disk implies that there exists an open set D containing the closed disk $\overline{D}(z_0, r)$ on which f is analytic. By Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i} \oint_{|z - z_0| = r} \frac{f(z)}{z - z_0} dz$$

Use the parametrization $\sigma(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$, for the circle $|z - z_0| = r$.

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} \cdot ire^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt \quad \blacksquare$$

🔗 The name “mean-value formula” refers to one of the most prominent properties of an analytic function: The value of the function $f(z)$ at the center of a circle is the average of its values around the circle.

🔗 In the next stage we prove that if a function is analytic at a point, then it is necessarily differentiable there to every order! Moreover, we shall also obtain a formula for the n th derivative (Cauchy's derivative formula). Before the main theorem, we need the following auxiliary assertion.

i Proposition 5.21: Let $g(w)$ be a continuous function on a piecewise smooth Jordan path σ . For every point z not on σ , define the function

$$f(z) = \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{w - z} dw$$

Then $f(z)$ is analytic in the set $\mathbb{C} - \sigma$.

✓ Proof: Let z_0 be a point not on the path σ . In Figure 5.18 we chose a point in the interior of σ , but an exterior point could equally well be chosen! Choose $r > 0$ such that the disk $\mathbb{D}(z_0, r)$ does not intersect the path σ . It is clear from the construction that for every point $w \in \sigma$ and every point z in the disk $\mathbb{D}(z_0, \frac{r}{2})$, we have⁵ $|w - z| \geq \frac{r}{2}$.

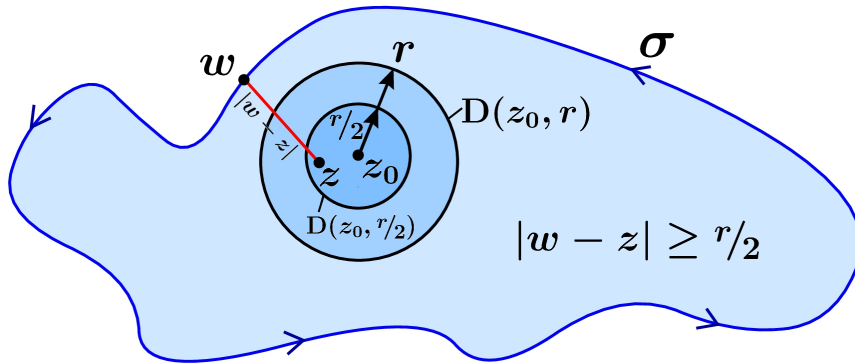


Fig. 5.18: Auxiliary diagram for assertion 5.21

To prove that f is differentiable at the point z_0 , we try to estimate the magnitude of the expression $\frac{f(z) - f(z_0)}{z - z_0}$ and prove that its limit as $z \rightarrow z_0$ exists. We restrict ourselves to values of z lying in the disk $\mathbb{D}(z_0, r)$.

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \frac{1}{z - z_0} \cdot [f(z) - f(z_0)] \\ &= \frac{1}{z - z_0} \left[\frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{w - z} - \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{w - z_0} \right] \\ &= \frac{1}{2\pi i(z - z_0)} \oint_{\sigma} \frac{g(w)[(w - z_0) - (w - z)]}{(w - z)(w - z_0)} \\ &= \frac{1}{2\pi i(z - z_0)} \oint_{\sigma} \frac{g(w)(z - z_0)}{(w - z)(w - z_0)} \\ &= \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w - z)(w - z_0)} \end{aligned}$$

⁵ Indeed: $|w - z| + |z - z_0| \geq |w - z_0|$, and hence $|w - z| \geq |w - z_0| - |z - z_0| \geq r - \frac{r}{2} = \frac{r}{2}$.

From the last line we may conjecture that

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w - z_0)^2} dw$$

To verify the conjecture, we estimate the magnitude of the difference

$$\begin{aligned} \left| \frac{f(z) - f(z_0)}{z - z_0} - \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w - z_0)^2} \right| &= \left| \frac{1}{2\pi i} \oint_{\sigma} \left[\frac{g(w)}{(w - z)(w - z_0)} - \frac{g(w)}{(w - z_0)^2} \right] dw \right| \\ &= \left| \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{w - z_0} \left[\frac{1}{w - z} - \frac{1}{w - z_0} \right] dw \right| \\ &= \left| \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{w - z_0} \cdot \frac{z - z_0}{(w - z)(w - z_0)} dw \right| \\ &\leq \frac{|z - z_0|}{2\pi i} \oint_{\sigma} \frac{|g(w)|}{|w - z| \cdot |w - z_0|^2} dw \\ &\leq \frac{|z - z_0|}{2\pi i} \cdot \ell(\sigma) \cdot \frac{M}{\frac{r}{2} \cdot r^2} \xrightarrow{z \rightarrow z_0} 0 \end{aligned}$$

where $M = \max_{w \in \sigma} |g(w)|$, and $\ell(\sigma)$ is the length of the path σ . All the quantities M , $\ell(\sigma)$, r are constant, but the quantity $|z - z_0|$ tends to zero as $z \rightarrow z_0$! We have therefore obtained

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w - z_0)^2} dw$$

■

i Theorem 5.22: Let $g(w)$ be a continuous function on a piecewise smooth Jordan path σ . For every point z not on σ , define the function

$$f(z) = \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{w - z} dw$$

Then $f(z)$ is differentiable to every order n on the domain $\mathbb{C} - \sigma$, and in addition, for every natural number n ,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w - z)^{n+1}} dw \quad (5.20)$$

✓ Proof: We use mathematical induction. The proof for $n = 1$ is given by the preceding theorem. Assume that the theorem is true for a given natural number n :

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w - z)^{n+1}} dw$$



Then

$$\begin{aligned}
 f^{(n+1)}(z) &= \frac{d}{dz} \left[\frac{n!}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w-z)^{n+1}} dw \right] \\
 &= \frac{n!}{2\pi i} \oint_{\sigma} \frac{d}{dz} \left[\frac{g(w)}{(w-z)^{n+1}} \right] dw \\
 &= \frac{n!}{2\pi i} \oint_{\sigma} \frac{(n+1)g(w)}{(w-z)^{n+2}} dw \\
 &= \frac{(n+1)!}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w-z)^{n+2}} dw
 \end{aligned}$$

The interchange of differentiation and integration in the second line follows from the Leibniz rule for complex functions (it is not difficult to verify that the conditions hold). The precise statement and proof of the Leibniz rule appear in the appendix at the end of the present chapter. ■

Remarks

1. The induction step can also be carried out by estimating the magnitude of the difference

$$\left| \frac{f^{(n)}(z) - f^{(n)}(z_0)}{z - z_0} - \frac{g(w)}{(w - z_0)^{n+2}} \right|$$

but this involves a nontrivial algebraic complication (using the factorization formula for $a^{n+1} - b^{n+1}$), and therefore, in this case, the use of the Leibniz rule simplifies the proof.

2. The Leibniz rule used in the induction step is intended to be a generalization of the original Leibniz rule for complex functions, which of course requires an effort beyond the scope of the course. For interested students, a complete statement and proof appear as an appendix at the end of the present chapter.
3. In fact, the induction can be started at the stage $n = 1$. By definition,

$$f(z) = \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{w - z} dw$$

and therefore, by the Leibniz rule,

$$f'(z) = \frac{1}{2\pi i} \cdot \frac{d}{dz} \oint_{\sigma} \frac{g(w)}{w - z} dw = \frac{1}{2\pi i} \oint_{\sigma} \frac{d}{dz} \left[\frac{g(w)}{w - z} \right] dw = \frac{1}{2\pi i} \oint_{\sigma} \frac{g(w)}{(w - z)^2} dw$$

This of course avoids the calculation at the beginning of the proof of assertion [5.21](#).

4. The immediate consequence of assertion [5.21](#) is that if $f(z)$ is analytic in a domain D , then it has derivatives of every order in D ! In addition, we obtain a generalization of Cauchy's integral



formula for the n th derivative of f .

i Corollary 5.23: (Cauchy's derivative formula) If $f(z)$ is analytic in an open set S , then f is differentiable in S to every order n , and, in addition, for every point $z_0 \in S$,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\sigma} \frac{f(z)}{(z - z_0)^{n+1}} dw \quad (5.21)$$

for every piecewise smooth Jordan path σ surrounding the point z_0 whose interior is contained in S .

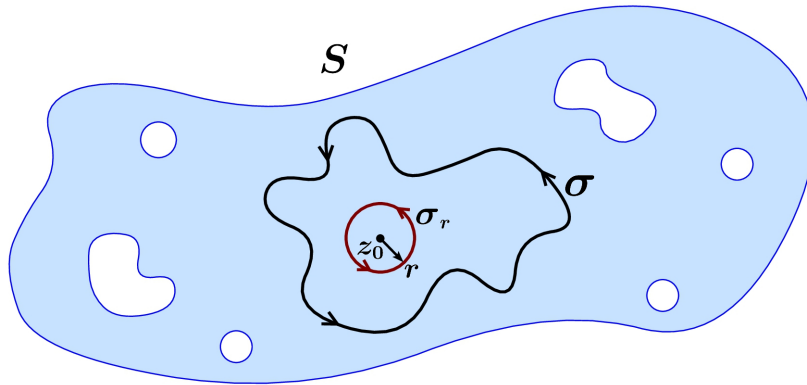


Fig. 5.19: Integration of a function $f(z)$ along a piecewise smooth Jordan path σ surrounding the point z_0

✓ Proof: Let $z_0 \in S$ be a given point in the open set S , and let σ_r be a circular path centered at z_0 with radius r , contained in S together with its interior. By Cauchy's integral formula (Theorem 5.19),

$$f(z_0) = \frac{1}{2\pi i} \oint_{\sigma_r} \frac{f(z)}{z - z_0} dz$$

By Theorem 5.22, f is differentiable to every order, and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\sigma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

If $\sigma: [a, b] \rightarrow \mathbb{C}$ is a piecewise smooth Jordan path surrounding z_0 in the positive direction, then the distance function $h(t) = |\sigma(t) - z_0|$ is continuous on the closed interval $[a, b]$, and therefore attains a minimum $r_0 > 0$ there. Thus we may choose $0 < r < r_0$ so that the circle σ_r is contained in the interior of the path σ , as shown in Figure 5.19. By Corollary 5.16,

$$\oint_{\sigma} \frac{f(z)}{(z - z_0)^{n+1}} dz = \oint_{\sigma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and therefore we also obtain the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\sigma} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

for every piecewise smooth Jordan path σ surrounding z_0 with positive orientation. ■

❖ **Example 6:** Compute the line integral $\oint_{|z|=1} \frac{\cos z}{z^3} dz$.

🔑 **Solution:** The expression z^3 suggests using formula (5.21) (Corollary 5.23). Substitute $f(z) = \cos z$, $z_0 = 0$, $n = 2$.

$$f^{(2)}(0) = -\cos 0 = -1 = \frac{2!}{2\pi i} \oint_{|z|=1} \frac{\cos z}{z^3} dz$$

We have therefore obtained

$$\oint_{|z|=1} \frac{\cos z}{z^3} dz = -\pi i$$

📌 **Corollary 5.24: (Cauchy's inequality)** Let $f(z)$ be analytic on and inside a circular path σ_r centered at z_0 with radius $r > 0$. Then, for every natural number n ,

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}$$

where $M = \max_{z \in \sigma_r} |f(z)|$.

✔ **Proof:** By Cauchy's derivative formula,

$$\begin{aligned} |f^{(n)}(z_0)| &= \left| \frac{n!}{2\pi i} \oint_{\sigma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} \cdot \ell(\sigma_r) \cdot \frac{\max_{z \in \sigma_r} |f(z)|}{r^{n+1}} \\ &= \frac{n!}{2\pi} \cdot 2\pi r \cdot \frac{M}{r^{n+1}} = \frac{n!M}{r^n} \end{aligned}$$

📌 **Corollary 5.25:** Let $f(z)$ and $g(z)$ be two analytic functions defined on and inside a piecewise smooth Jordan path σ . If $f(z) = g(z)$ on the path σ , then the two functions are identical inside the path as well.



Fig. 5.20: Augustin-Louis Cauchy
1789-1857



Fig. 5.21: Édouard Goursat
1858-1936



Fig. 5.22: Joseph Liouville
1809-1882

Liouville's Theorem (Joseph Liouville 1809-1882)

- Another important consequence of Cauchy's integral formula is **לייביל's theorem**, one of the cornerstones of complex analysis. Recall that an entire function is an analytic function defined on the entire complex plane.

i Theorem 5.26: (לייביל's Theorem) Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. If f is bounded, then f is constant.

- ✔ **Proof:** Suppose that $|f(z)| \leq M$ for every $z \in \mathbb{C}$. Let z_0 be a given complex number. For every positive real number $r > 0$, denote by σ_r the circle of radius r centered at the point z_0 (with positive orientation). By Corollary 5.24,

$$|f'(z_0)| \leq \frac{\max_{z \in \sigma_r} |f(z)|}{r} \leq \frac{M}{r} \xrightarrow{r \rightarrow \infty} 0$$

The inequality guaranteed by Corollary 5.24 holds for every radius r . Therefore, as the radius r tends to infinity, the right-hand side of the inequality tends to zero. Thus necessarily $f'(z_0) = 0$ for every point $z_0 \in \mathbb{C}$. By assertion 4.5, f must be constant on all of $\mathbb{C}$. ■

- In other words, Liouville's theorem asserts that every nonconstant entire function is necessarily unbounded. One of the important consequences of Liouville's theorem is the fundamental theorem of algebra.

i Theorem 5.27: (The fundamental theorem of algebra)
Every nonconstant polynomial has at least one root.

- ✔ **Proof:** Let $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a nonconstant polynomial. If $p(z)$ has no roots, then $f(z) = \frac{1}{p(z)}$ is an entire function, and moreover

$$|f(z)| = \left| \frac{1}{p(z)} \right| = \frac{1}{|z|^n} \cdot \frac{1}{\left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n \right|} \xrightarrow{|z| \rightarrow \infty} \frac{1}{\infty} \cdot \frac{1}{|0 + 0 + \cdots + a_n|} = 0$$

Therefore there exists a radius R such that $|f(z)| < 1$ for every $|z| > R$. The modulus $|f(z)|$ is bounded on the disk $|z| \leq R$ by the Weierstrass theorem (a continuous function on a closed and bounded set). Thus $f(z)$ is entire and bounded on $\mathbb{C}$, and hence by Liouville's theorem it must be constant on $\mathbb{C}$, so $p(z)$ is a constant polynomial, contrary to the assumption. Therefore $p(z)$ must have at least one root. ■

i Definition 5.5: A number z_1 is called a **root of multiplicity k** of a polynomial $p(z)$ if there exists a polynomial $q(z)$ such that

$$p(z) = (z - z_1)^k q(z)$$



and in addition $q(z_1) \neq 0$.

i Theorem 5.28: Let $p(z)$ be a polynomial of degree n , and let $z_1, z_2, \dots, z_m$ be all of its distinct roots. Let $k_1, k_2, \dots, k_m$ be the multiplicities of the roots in the corresponding order. Then $k_1 + k_2 + \dots + k_m = n$, and in addition

$$p(z) = (z - z_1)^{k_1} (z - z_2)^{k_2} \dots (z - z_m)^{k_m}$$

✓ Proof: By the fundamental theorem of algebra, there exists a root z_1 . Let k_1 be its multiplicity, and let $q_1(z)$ be a polynomial for which $p(z) = (z - z_1)^{k_1} q_1(z)$. Again, by the fundamental theorem of algebra, there exists a root z_2 of the polynomial $q_1(z)$. Let k_2 be the multiplicity of z_2 , and let $q_2(z)$ be a polynomial for which $q_1(z) = (z - z_2)^{k_2} q_2(z)$. We can continue in this way until we obtain $k_1 + k_2 + \dots + k_m = n$. The second part is clear. ■



Fig. 5.23: Giacinto Morera
1856-1909



Fig. 5.24: Bernhard Riemann
1826-1866

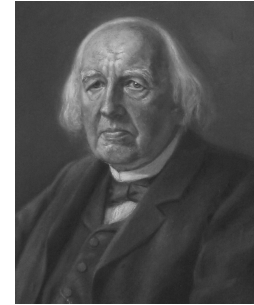


Fig. 5.25: Karl Weierstrass
1815-1897

5.5 Morera's Theorem (Giacinto Morera 1856-1909)

- In the preceding section we proved one of the most important properties of an analytic function: Its line integral along every piecewise smooth closed path contained in its domain of definition always vanishes.
- The Italian mathematician **ג'סינטו מוררה** (1856-1909), who studied complex functions, asked whether this property characterizes analytic functions, or whether nonanalytic functions with this property exist.
- The theorem named after Morera gives an affirmative answer to the question: the vanishing of the line integral over piecewise smooth Jordan paths is indeed a property that characterizes analytic functions alone.

i Theorem 5.29: (Morera's theorem) Let $f(z)$ be a continuous function on a domain D whose line integral vanishes over every triangular path contained in D (together with its interior). Then f is analytic on D .

♥ **Proof:** As we saw in the proof of the Cauchy-Goursat theorem (Theorem 5.8), every polygonal path can be decomposed into a sum of triangular paths, and therefore we obtain $\oint_{\sigma} f(z)dz = 0$ for every polygonal path σ .

By the argument encountered in the proof of Corollary 5.9, the preceding property is equivalent to

$$\oint_{\sigma_1} f(z)dz = \oint_{\sigma_2} f(z)dz$$

for any two polygonal paths σ_1, σ_2 in the domain D that begin and end at the same endpoints (see Figure 5.3). Let $z_0 \in D$ be a given point. By Theorem 2.11 (page 60), for every point $z \in D$, there exists a polygonal path σ in the domain D joining the point z_0 to the point z (see Figure 5.26). Define

$$F(z) = \int_{z_0}^z f(w) dw = \int_{\sigma} f(w) dw$$

where σ is any polygonal path in D joining z_0 to z (since all polygonal paths between z_0 and z yield the same value!).

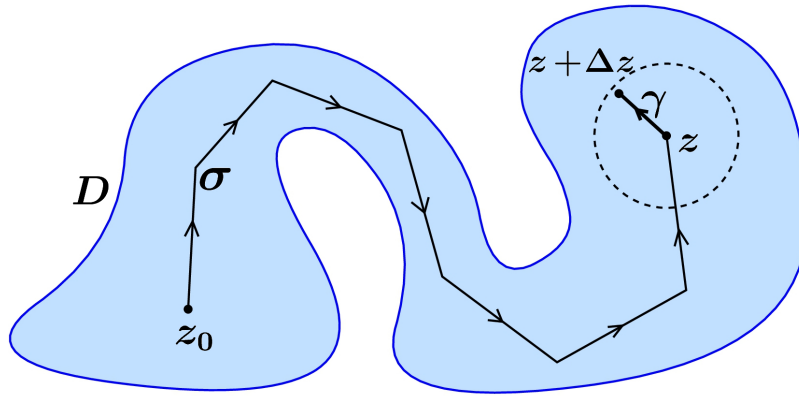


Fig. 5.26: Defining a primitive $F(z) = \int_{z_0}^z$ by an integral along a polygonal path from z_0 to z

We prove that F is analytic and that its derivative is the function $f(z)$. For this purpose, consider a small neighborhood of the point z contained in D , and let $z + \Delta z$ be any point in this neighborhood obtained by adding Δz to the point z . Let $\gamma = [z, z + \Delta z]$ be the segment joining the point z to the point $z + \Delta z$. We must prove that

$$\lim_{\Delta z \rightarrow 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z)$$

For this purpose, observe that

$$F(z + \Delta z) - F(z) = \int_{\sigma + \gamma} f(w)dw - \int_{\sigma} f(w)dw = \int_{\gamma} f(w)dw \quad (5.22)$$

Set $z = x + iy$, $\Delta z = a + bi$. A simple parametrization of the segment γ is

$$\gamma(t) = (x + at) + (y + bt)i, \quad 0 \leq t \leq 1$$

Therefore,

$$\int_{\gamma} f(w)dw = \int_0^1 f(\gamma(t)) \cdot \gamma'(t) dt = \int_0^1 f(\gamma(t)) \cdot (a + bi) dt = \Delta z \cdot \int_0^1 f(\gamma(t)) dt$$

and therefore

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z} \cdot \int_{\gamma} f(w)dw = \int_0^1 f(\gamma(t)) dt$$

Use Cartesian notation $f(z) = u(x, y) + iv(x, y)$. From equality (5.22), we obtain

$$\begin{aligned} \frac{F(z + \Delta z) - F(z)}{\Delta z} &= \int_0^1 f(\gamma(t)) dt \\ &= \int_0^1 u(x + at, y + bt) dt + i \int_0^1 v(x + at, y + bt) dt \end{aligned}$$

For the purposes of the discussion, the functions $u(x + at, y + bt)$ and $v(x + at, y + bt)$ are functions of the single variable t . By the integral mean-value theorem (for a function of the single variable t), there exist t_1, t_2 in the interval $[0, 1]$ for which

$$\begin{aligned} \int_0^1 u(x + at, y + bt) dt &= u(x + at_1, y + bt_1) \\ \int_0^1 v(x + at, y + bt) dt &= v(x + at_2, y + bt_2) \end{aligned}$$

and therefore we obtain

$$\frac{F(z + h) - F(z)}{\Delta z} = u(x + at_1, y + bt_1) + iv(x + at_2, y + bt_2)$$

As stated, $0 \leq t_1 \leq 1$, $0 \leq t_2 \leq 1$. Therefore we obtain

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} &= \lim_{\substack{a \rightarrow 0 \\ b \rightarrow 0}} [u(x + at_1, y + bt_1) + iv(x + at_2, y + bt_2)] \\ &= u(x, y) + iv(x, y) \\ &= f(z) \end{aligned}$$



We have shown that $F(z)$ is differentiable at the point z and satisfies $F'(z) = f(z)$. By Cauchy's derivative formula (Corollary 5.23), $F(z)$ is differentiable to every order in the domain D ! Therefore $f(z)$ is also analytic in the domain D . ■

Remarks

1. In fact, we proved that if f satisfies the hypotheses of the theorem, then it has an antiderivative $F(z)$. This fact may be useful in other contexts.
 2. In most books, Morera's theorem is stated in terms of general closed or rectangular paths. Note that we chose triangular paths! Of course, all these formulations are ultimately equivalent.
 3. Nevertheless, in various exercises there may be an advantage to using rectangular or circular paths specifically! Clearly, we may use them instead of triangular paths if necessary.
 4. Another point worth noting is that in various contexts it is enough to use arbitrarily small triangles, or triangles of a certain kind (such as right triangles), because every triangular path can be decomposed into a sum of arbitrarily small triangular paths (and every triangle can likewise be decomposed into a sum and/or difference of right triangles).
- Morera's theorem has important and interesting applications. For example, we present the following corollary, which would be very difficult to prove without Morera's theorem. The theorem will be understood and presented in detail in Theorem 6.18 in the next chapter on series.
- ◆ **Example 7:** Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on a domain D , converging uniformly to a function $f(z)$. Then the function $f(z)$ is also analytic on the domain D .
- ✓ **Proof:** We prove that $\oint_{\sigma} f(z) dz = 0$ for every triangular path σ in the domain D . Morera's theorem will then imply that f is analytic in the domain D . Uniform convergence on the compact set σ implies that $\text{Max}_{t \in [a,b]} |f(\sigma(t)) - f_n(\sigma(t))| \rightarrow 0$, and therefore

$$\left| \oint_{\sigma} f(z) dz - \oint_{\sigma} f_n(z) dz \right| \leq \oint_{\sigma} |f(z) - f_n(z)| dz \xrightarrow{n \rightarrow \infty} 0$$

We have therefore obtained

$$\oint_{\sigma} f(z) dz = \lim_{n \rightarrow \infty} \oint_{\sigma} f_n(z) dz = 0$$

Thus f is also analytic on D . ■

* The topic of sequences and series of functions is covered in the next chapter, in Theorem 6.18, so readers without the necessary background may skip this example for now.



5.6 The Maximum Principle

- One of the immediate consequences of the mean-value formula proved in the third part of the present chapter (Corollary 5.20) is the **maximum principle**, which states that a nonconstant function analytic on a bounded domain can never attain an absolute maximum or minimum at an interior point of its domain. Likewise, there cannot be a point of local maximum or minimum at an interior point of the domain.

i Definition 5.6: (Absolute minimum and maximum) Let $f(z)$ be a function defined on a set S . A point $z_0 \in S$ is called an **absolute maximum point** of the function f on the set S if, for every $z \in S$, $|f(z)| \leq |f(z_0)|$. The number $M = |f(z_0)|$ is called the **absolute maximum** of f on S . The **absolute minimum point** and **absolute minimum** are defined similarly.

- There are several formulations of the maximum principle. We present two formulations, of which the second will be our official formulation, but the first is important for proving the second. To avoid confusion, the first formulation is customarily called the **maximum modulus principle**, and, in brief, it states: If the modulus $|f(z)|$ of a function analytic on a domain D attains an absolute maximum at an interior point of D , then f is constant on D .

i Theorem 5.30: (Maximum modulus principle) Let f be analytic on a domain D . If there exists an interior point $z_0 \in D$ such that for every $z \in D$, $|f(z)| \leq |f(z_0)|$, then f is constant on D .

- ✓ **Proof:** By assertion 4.7 (page 122), it suffices to prove that $|f(z)|$ is constant on the domain D . Suppose there is a point z_0 in D at which f attains the maximum value M :

$$\forall z \in D : |f(z)| \leq |f(z_0)| = M$$

By the mean-value formula (Corollary 5.20), for every $r > 0$ such that the disk $\overline{D}(z_0, r)$ is contained in the domain D ,

$$|f(z_0)| = \left| \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + re^{it}) dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt$$

From the assumption $M = |f(z_0)| \geq |f(z)|$, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dt = |f(z_0)|$$

and therefore

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt$$

We may also write this as

$$\begin{aligned} 0 &= |f(z_0)| - \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dz - \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} [|f(z_0)| - |f(z_0 + re^{it})|] dt \end{aligned}$$

We have shown that the integral of the continuous nonnegative function $|f(z_0)| - |f(z_0 + re^{it})|$ (recall that $|f(z_0)| = \text{Max } |f(z)|$!) vanishes, and therefore, by Calculus 1, the function must vanish identically⁶. Therefore $f(z) = f(z_0)$ for every z on the circle $|z - z_0| = r$, for every r whose corresponding disk is contained in D . Since this is true for every such circle, it is true throughout the neighborhood.

In summary, we have established the general property: if z is any point at which the maximum $f(z) = M$ is attained, then all the points in a neighborhood of z have the same value M .

Define the following set:

$$E = \{z \in D : |f(z)| = M\}$$

By the property just proved, E is a nonempty open set. The set $D - E$ is also open, because if $z_1 \in D - E$, that is, $|f(z_1)| < M$, then the continuity of f implies that there is a neighborhood of z_1 all of whose points satisfy $|f(z)| < M$. Since the domain D is connected and $E \neq \emptyset$, $D - E = \emptyset$. That is, $E = D$. ■

i Theorem 5.31: (Maximum principle) Let $f(z)$ be analytic on a bounded domain D and continuous on the closed set $\overline{D}$. Then the modulus $|f(z)|$ attains its absolute maximum on the boundary of D .

Proof: Since D is a bounded domain, its closure $\overline{D}$ is compact (closed and bounded). By assumption, $|f(z)|$ is continuous on $\overline{D}$, and therefore, by the Weierstrass theorem (see Theorem 3.14 on page 100), it attains an absolute minimum and maximum on $\overline{D}$.

There are two possibilities: A. The function $|f(z)|$ is constant, in which case the absolute minimum and maximum are attained throughout $\overline{D}$, including on the boundary. B. The function $|f(z)|$ is not constant, and therefore, by the preceding theorem, it cannot attain its absolute maximum inside D , so the maximum must be attained on the boundary of D . ■

⁶ See, for example, the following link: [If integral is zero and function is continuous and non negative then \$f \equiv 0\$](#)



i Corollary 5.32: (Minimum principle) Let $f(z)$ be analytic on a bounded domain D , continuous on the closed set $\overline{D}$, and nonvanishing on $\overline{D}$. Then the modulus $|f(z)|$ attains its absolute minimum on the boundary of D .

✓ **Proof:** Define the function $g(z) = 1/f(z)$. Then g satisfies the hypotheses of the maximum principle and hence attains its absolute maximum on the boundary of D . That is, there exists a point $z_0 \in \partial D$ such that for every $z \in D$, $|g(z)| \leq |g(z_0)|$. Therefore, for every $z \in D$, $|f(z)| \geq |f(z_0)|$, and we have shown that the absolute minimum is attained on the boundary of D . ■

◇ **Example 8:** Let $f(z)$ be a nonconstant function analytic on the disk $|z - 1| \leq 2$ (including the boundary). Suppose that $|f(z)| = 3$ on the boundary of the disk. Prove that f has at least one root inside the disk.

🔗 **Solution:** Assume to the contrary that $f(z) \neq 0$ for every z in the disk. Define $g(z) = \frac{1}{f(z)}$. Then $|g(z)| = \frac{1}{3}$ on the boundary of the disk. By the maximum principle, $|g(z)| \leq \frac{1}{3}$ for every point in the disk $|z - 1| \leq 2$. Therefore $|f(z)| \geq 3$ throughout the disk. We have obtained $|f(z)| = 3$ throughout the disk, and therefore, by assertion 4.7, $f(z)$ is constant on the disk, contrary to the assumption.

◇ **Example 9:** Find the absolute maximum of $f(z) = \sin z$ on the rectangle $[0, \pi] \times [0, 1]$.

🔗 **Solution:** By the maximum principle, the absolute maximum of $|\sin z|$ must be attained on the boundary of the rectangle. It is easy to verify that

$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y} \quad (z = x + iy)$$

The function $\sin^2 x$ attains its maximum value 1 at the point $x = \frac{\pi}{2}$, and the function $\sinh^2 y$ is monotonically increasing, and hence attains its maximum at the point $y = 1$. Therefore the absolute maximum of $\sin z$ must be attained at the point $(x, y) = (\frac{\pi}{2}, 1)$. The absolute maximum of $|\sin z|$ on the rectangle is therefore $\sqrt{1 + (e - e^{-1})/2}$.

➤ Another consequence of the maximum principle is the impossibility of a local maximum for an analytic function. A local maximum point z_0 is defined as a point at which the modulus $|f(z)|$ attains a maximum value in a neighborhood $D(z_0, \varepsilon)$. Such a situation is impossible by the maximum modulus principle, because the function would then be constant in the neighborhood $D(z_0, \varepsilon)$.

➤ A local minimum equal to 0 is possible. For example, $f(z) = z^2(5 - z)$ attains a local minimum at the point $z = 0$. But a strictly positive local minimum is impossible.



5.7 Appendix: The Leibniz Rule for an Integral of a Complex Function

- The Leibniz rule for complex functions is not included in the course syllabus and is generally not included in textbooks on complex functions, but it is very useful for proving several theorems on the subject (whose proofs would otherwise be very complicated). We include this appendix in the notes for students who are motivated to understand the course material more deeply, but all that we need for the course is to understand the theorem itself and be able to use it where required.
- Recall from Calculus or Infinitesimal Calculus 2 that the Leibniz rule allows us to interchange the order of differentiation and integration in various contexts, thereby simplifying computations that might otherwise be very complicated. For example, the Leibniz rule significantly simplifies the proof of Cauchy's derivative formula.
- The proof of the rule requires the use of more advanced tools from analysis and is therefore usually not included in introductory courses. Nevertheless, we shall try to explain it for interested students. We base ourselves on the proof given at the following link (with minor changes in notation):

Rigorous Proof of Leibniz's Rule for Complex Function

i Theorem 5.33: (The Leibniz rule for an integral of a complex function)

Let $g(t, z)$ be a complex function of two variables, $g: [a, b] \times D \rightarrow \mathbb{C}$ where

- (a) $[a, b]$ is a closed real interval, and D is an open set in the complex plane $\mathbb{C}$.
- (b) g is continuous on the set $[a, b] \times D$.
- (c) For every t in the interval $[a, b]$, $g(t, z)$ is analytic in D .
- (d) The partial derivative $\frac{\partial g}{\partial z}$ is continuous on the set $[a, b] \times D$.

Then the function

$$f(z) = \int_a^b g(t, z) dt \quad (5.23)$$

is analytic on D , and in addition

$$f'(z) = \frac{d}{dz} \int_a^b g(t, z) dt = \int_a^b \frac{\partial g}{\partial z}(t, z) dt \quad (5.24)$$

✓ **Proof:** Fix a point $z_0 \in D$. We must prove that

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = \int_a^b \frac{\partial g}{\partial z}(t, z_0) dt$$



By assumption, there exists $r > 0$ such that the function $\frac{\partial g}{\partial z}(t, z)$ is continuous on the closed disk $\overline{\mathbb{D}}(z_0, r)$, and therefore $\frac{\partial g}{\partial z}(t, z)$ is also uniformly continuous on the set $S = [a, b] \times \overline{\mathbb{D}}(z_0, r)$ (a product of two closed and bounded sets).

Let h be a complex number satisfying $0 < |h| < r$. Then

$$g(t, z_0 + h) - g(t, z_0) = \int_{z_0}^{z_0+h} g_z(t, z) dz = h \int_0^1 g_z(t, z_0 + sh) ds$$

The middle integral denotes a line integral along the path joining the point z_0 to the point $z_0 + h$. Define the function

$$\begin{aligned} \rho(t, h) &= \frac{g(t, z_0 + h) - g(t, z_0)}{h} - g_z(t, z_0) \\ &= \int_0^1 [g_z(t, z_0 + sh) - g_z(t, z_0)] ds \end{aligned} \tag{5.25}$$

The second line is easily verified using the Newton-Leibniz formula (or by differentiating the right-hand side with respect to s , as noted below)⁷. The uniform continuity of g_z on S implies that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |h| < \delta \implies |\rho(t, h)| < \varepsilon$$

That is, the family of functions $\{\rho(t, h)\}_{0 < |h| < r}$ (in the variable t) converges uniformly to the zero function on the closed interval $[a, b]$.

$$\lim_{h \rightarrow 0} \rho(t, h) = 0$$

By the definitions of f and ρ in equalities (5.23), (5.25), we obtain

$$\begin{aligned} \int_a^b \rho(t, h) dt &= \int_a^b \frac{g(t, z_0 + h) - g(t, z_0)}{h} dt - \int_a^b g_z(t, z_0) dt \\ &= \frac{\int_a^b g(t, z_0 + h) dt - \int_a^b g(t, z_0) dt}{h} - \int_a^b g_z(t, z_0) dt \\ &= \frac{f(t, z_0 + h) - f(t, z_0)}{h} - \int_a^b g_z(t, z_0) dt \end{aligned}$$

⁷ By Newton-Leibniz, $\int [g_z(t, z_0 + sh) - g_z(t, z_0)] ds = \frac{1}{h} g(t, sh) - sg(t, z_0)$ and after substituting the values $s = 0$, $s = 1$, we obtain the result.



Therefore,

$$\lim_{h \rightarrow 0} \frac{f(t, z_0 + h) - f(t, z_0)}{h} - \int_a^b g_z(t, z_0) dt = f'(z_0) - \int_a^b g_z(t, z_0) dt = \lim_{h \rightarrow 0} \int_a^b \rho(t, h) dt = 0$$

and we have proved the assertion of the theorem. ■

We may also write the Leibniz rule in the form

$$\frac{d}{dz} \int_a^b g(t, z) dt = \int_a^b \frac{\partial g}{\partial z}(t, z) dt \quad (5.26)$$

We can now obtain the Leibniz rule for an integral along a finite path, which we needed in the proof of Cauchy's derivative formula. The finiteness of the path is essential in order to obtain a closed and bounded set.

i Theorem 5.34: (The Leibniz rule for an integral of a complex function along a finite path)

Let $g(w, z)$ be a continuous complex function of two variables, with a continuous partial derivative $\frac{\partial g}{\partial z}$ on a set $\sigma \times D$, where σ is a finite piecewise smooth path in $\mathbb{C}$, and D is an open set in $\mathbb{C}$. Then

$$\frac{d}{dz} \int_{\sigma} g(w, z) dw = \int_{\sigma} \frac{\partial g}{\partial z}(w, z) dw$$

✓ Proof: Let

$$\begin{cases} \sigma(t) = (x(t), y(t)) \\ a \leq t \leq b \end{cases}$$

be a given parametrization of σ . Define the function

$$\begin{cases} G(t, z) = g(\sigma(t), z) \cdot \sigma'(t) \\ t \in [a, b], \quad z \in D \end{cases}$$

It is easy to verify that G satisfies all the hypotheses of the preceding theorem on $[a, b] \times D$. Therefore we may apply to G the Leibniz rule from formula (5.26):

$$\frac{d}{dz} \int_a^b G(t, z) dt = \int_a^b \frac{\partial G}{\partial z}(t, z) dt$$



By the definition of G , we obtain

$$\frac{d}{dz} \int_a^b g(\sigma(t), z) \cdot \sigma'(t) dt = \int_a^b \frac{\partial g}{\partial z}(\sigma(t), z) \cdot \sigma'(t) dt$$

and from the definition of an integral along a path, we obtain

$$\frac{d}{dz} \int_{\sigma} g(w, z) dw = \int_{\sigma} \frac{\partial g}{\partial z}(w, z) dw \quad \blacksquare$$

Remark: If σ is a piecewise smooth path, then it can be decomposed into a sum of finitely many smooth paths, and the theorem can be applied to each of them.

Exercises for chapter 5



1. Compute the value of the integral $\int_{\sigma} e^{iz} dz$ along the path $\sigma(t) = i + e^{it}$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

Answer: $i(1 - e^{-2})$



2. Compute the value of the integral $\int_{\sigma} \frac{3z^2 - 2iz + 1}{z-1} dz$, where $\sigma(t) = 2 \sin t + i \sin 2t$, $0 \leq t \leq 2\pi$.

3. Compute the line integral $\int_{\sigma} \frac{dz}{z^{1/2}}$ where σ is one of the following four paths:
- (a) The upper semicircle of the unit circle, counterclockwise. Here $z^{1/2}$ is the branch for which $1^{1/2} = 1$.
 - (b) The lower semicircle of the unit circle, counterclockwise. Here $z^{1/2}$ is the branch for which $1^{1/2} = -1$.
 - (c) The full unit circle $|z| = 1$, clockwise. Here $z^{1/2}$ is the branch for which $1^{1/2} = 1$.
 - (d) The full unit circle $|z| = 1$, counterclockwise. Here $z^{1/2}$ is the branch for which $(-1)^{1/2} = i$.

Hint: We mean, of course, analytic branches of $z^{1/2}$. There are only two such branches.

The final answers are: (a) $2(i-1)$ (b) $2(i+1)$ (c) 4 (d) -4



4. Prove: (a) $\left| \oint_{|z|=1} \frac{dz}{5z^2 + 3} \right| \leq \pi$ (b) $\left| \oint_{|z|=1} \frac{2z+1}{z^2+5} dz \right| \leq \frac{3\pi}{2}$

5. Let $f(z)$ be a nonconstant function analytic on the disk $|z-1| \leq 2$ (including the boundary). Suppose that $|f(z)| = 3$ on the boundary of the disk. Prove that f has at least one root inside the disk.

Hint: Assume to the contrary that no root exists, and derive a contradiction to the maximum principle for $1/f(z)$.



6. Let $f(z)$ be analytic in the disk $|z| < R$, where R is a positive real number. Define a



real-valued function $g : [0, R] \rightarrow \mathbb{R}$ by

$$g(r) = \sup_{|z|=r} |f(z)|$$

Prove that g is monotonically increasing.

Hint: The maximum principle



7. Prove that $\oint_{\sigma} z e^{z^2} dz = 0$ for every closed path.

8. Compute $\int_{\sigma} \frac{dz}{z^2+z}$ and $\int_{\sigma} \sin^2(z) dz$ where σ is the segment $[4, 4i]$.



9. Compute the following integrals:

$$\begin{aligned} \text{(a)} \quad \oint_{|z+1|=2} \frac{z^2}{4-z^2} dz & \quad \text{(b)} \quad \oint_{|z|=1} \frac{\sin z}{z} dz & \quad \text{(c)} \quad \oint_{|z|=2} \frac{e^z}{z(z-3)} dz & \quad \text{(d)} \quad \oint_{|z|=4} \frac{e^z}{z(z-3)} dz \\ \text{(e)} \quad \oint_{|z|=1} \frac{e^z}{z^n} dz & \quad \text{(f)} \quad \oint_{|z|=2} \frac{z^n}{z-1} dz & \quad \text{(g)} \quad \oint_{|z-1-i|=5/4} \frac{\text{Log}(z)}{(z-1)^2} dz & \quad \text{(h)} \quad \oint_{|z|=1} \frac{dz}{z^2(z^2-4)} \end{aligned}$$

Remark: All paths have positive orientation, of course. In part E, n is an integer. In part F, $n \geq 0$ is a natural number.
Final answers: (a) $2\pi i$ (b) 0 (c) $-\frac{2\pi i}{3}$ (d) $\frac{2\pi i}{3}(e^3 - 1)$ (e) $\frac{2\pi i}{(n-1)!}$ (f) $2\pi i$ (g) $2\pi i$ (h) 0

10. Compute the integral $\oint_{\sigma} |z| dz$ along the rectangle $\sigma = [-1, 1, 1+i, -1+i, -1]$.

Answer: $1 - \sqrt{2} - \ln(\sqrt{2} + 1)$



11. Compute the integral $\oint_{\sigma} \frac{dz}{z}$ along the square $\sigma = [-1-i, 1-i, 1+i, -1+i, -1-i]$.



12. Compute the integral $\oint_{\sigma} z/\bar{z} dz$ where σ is the path shown in Figure 5.27.

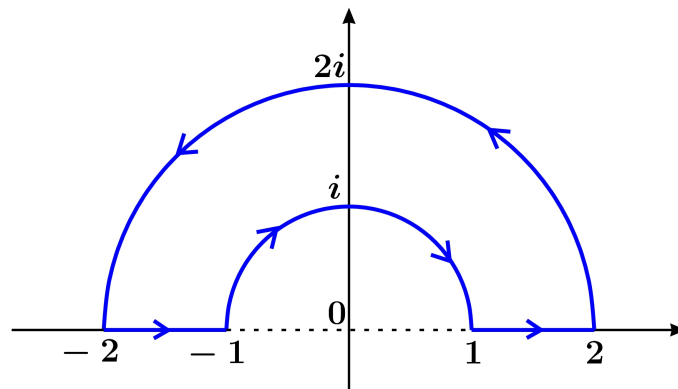


Fig. 5.27: The path σ for Exercise 12



13. Compute the integral $\oint_{|z|=1} \frac{z + \bar{z}}{(z - z_0)^2} dz$ for every point $|z_0| \neq 1$.

Hint: There is a small “trap” here: the numerator is not analytic! Nevertheless, the quotient is analytic (on the path of integration): If we multiply the numerator and denominator by z , we obtain an analytic numerator and denominator. Note that $|z|^2 = 1$ on the path of integration ... Three cases must be considered: z_0 inside, outside, or on the path.

Answer: $\frac{2\pi i}{z_0^2}, 2\pi i, 2\pi i$



14. Prove that for every natural number $m \geq 1$: $\oint_{|z|=1} \left(z + \frac{1}{z}\right)^{2m+1} dz = 2\pi i \binom{2m+1}{m}$.

Hint: By the binomial theorem:

$$\left(z + \frac{1}{z}\right)^{2m+1} = \frac{(z^2 + 1)^{2m+1}}{z^{2m+1}} = \frac{\sum_{k=0}^{2m+1} \binom{2m+1}{k} z^{2k}}{z^{2m+1}} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} z^{2k-2m-1}$$

The integrals of all powers z^n for which $n \neq -1$ vanish because they have an antiderivative (see Corollary 5.6).



15. Compute the integral $\oint_{|z|=2} z^n (1-z)^m dz$ for every pair of integers m, n .



16. Prove: If $f(z)$ is an entire function satisfying $|f(z)| \leq \sqrt{|z|}$ for every $z \in \mathbb{C}$, then necessarily $f(z)$ vanishes identically on the entire complex plane.

Hint: Using Cauchy's derivative formula, show that all the coefficients of the Maclaurin series of f vanish.

17. Find the absolute maximum of each of the following functions on the unit disk $|z| \leq 1$.

(a) $f(z) = \frac{2z+1}{2z-1}$ (b) e^{z^3} (c) $z^2 + iz - 1$ (d) $\frac{z^3}{z-5}$

Answer: (a) 3, (b) e , (c) 3, (d) $\frac{1}{4}$



18. Let $z_1, z_2, \dots, z_n$ be n distinct points on the unit circle $|z| = 1$. Prove that there exists a point z on the unit circle such that

$$(z - z_1)(z - z_2) \cdots (z - z_n) = 1$$

Hint: Let $f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$, Prove that $|f(0)| = 1$, and apply the maximum principle on the unit disk $|z| \leq 1$.



19. The function $f(z) = \frac{1}{z}$ is analytic on the annulus $1 < |z| < 2$. Prove that it has no primitive (antiderivative) on this domain.

Hint: Assume to the contrary that it has a primitive $F(z)$, and show that $F(z)$ differs from $\arg(z)$ by a constant.



20. Prove that the function $f(z) = i \cdot \frac{1-z}{1+z}$ maps the unit disk $D(0, 1)$ onto the upper half-plane $U = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ bijectively.

Hint:

A. Using simple algebra, show that f is one-to-one.

B. Using simple algebra, show that the boundary of the unit disk $|z| = 1$ is mapped to the boundary of U , $\partial U = \mathbb{R}$. Take $z = a + bi$, $a^2 + b^2 = 1$, and show that $f(z) = \frac{-b}{a+1} \in \mathbb{R}$.

C. Show that $f(0) = i$.

D. Apply the maximum principle to conclude that $D(0, 1)$ is mapped onto U .

21. Prove: If $f(z)$ is an entire function satisfying $|f(z)| \geq M > 0$ for every $z \in \mathbb{C}$, then f is necessarily constant.



22. Prove that for every nonconstant polynomial $P(z)$ there exists a point z such that $P(z)$ is



real.



23. Prove: If $P(z)$ is a nonconstant polynomial, then $P(\mathbb{C}) = \mathbb{C}$.

24. Prove: Let $f(z)$ be an entire function. If $|f(z)|$ is bounded on the right half-plane $\operatorname{Re}(z) \geq 0$, and if the derivative $|f'(z)|$ is bounded on the left half-plane $\operatorname{Re}(z) \leq 0$, then f is constant.



25. Construct an example of a function analytic in the unit disk that has exactly three local minimum points there.



26. Prove: If $f(z)$ is an entire function satisfying $|\operatorname{Re}(f(z))| \leq M$ for every $z \in \mathbb{C}$, then f is constant.

Hint: Consider the function $e^{f(z)}$.

27. Let $f(z)$ be analytic in a domain D . Suppose that the function $\operatorname{Re}(f(z))$ has a local maximum point at an interior point of D . Prove that f is constant on D .

28. Using Morera's theorem and the Leibniz rule, prove that the function $f(z) = \int_0^1 \frac{\sin zt}{t} dt$ is entire.



29. Let $f: \mathbb{D}(0, 1) \rightarrow \mathbb{D}(0, 1)$ be an analytic function from the open unit disk into itself. Prove that for every $|z| < 1$, $|f'(z)| \leq \frac{1}{1-|z|}$.

Hint: Take a small circle centered at z , apply Cauchy's derivative formula, and expand the circle until it is tangent to the boundary of $\mathbb{D}(0, 1)$.



30. Prove: If $f(z)$ is analytic on the closed unit disk $|z| \leq 1$ and takes real values on the unit circle $|z| = 1$, then f is constant.

Hint: Define $g(z) = f(z)$ for $|z| \leq 1$, and $g(z) = \overline{f(1/\bar{z})}$ for $|z| > 1$. Prove that g is entire and apply Liouville's theorem.



31. Prove: If $f(z)$ is analytic in a domain D containing the closed unit disk $|z| \leq 1$, and satisfies $|f(z)| = 1$ on the unit circle $|z| = 1$, then f is constant.

Hint: Try an idea similar to that in Exercise 30.



32. It is given that $g(z) = \frac{1}{zf(z) - \sin z}$ is an entire function that maps $\mathbb{C}$ into the plane with the hole $|z| > 2$. Find the function $f(z)$.

33. A set $S \subseteq \mathbb{C}$ is called **convex** if for every $z_1, z_2 \in S$, the segment $[z_1, z_2]$ is contained in S . Let $f(z)$ be analytic on an open convex set S . Suppose that $\operatorname{Re}(f'(z)) > 0$ for every $z \in S$. Prove that f is one-to-one on S .

Hint: Use Cartesian notation $f(x + iy) = u(x, y) + iv(x, y)$. Take two points $z_1 \neq z_2$ in S , and write the line integral $\int_{z_1}^{z_2} f'(z) dz$ along a simple parametrization of the segment $[z_1, z_2]$.



34. Prove that there is no entire function satisfying $|f(z)| \geq |z|$ for every $z \in \mathbb{C}$.





- 35.** Let $f(z)$ be continuous in the unit disk $|z| < 1$ and analytic at every point where $\text{Im}(z) \neq 0$. Prove that in fact f is analytic throughout the unit disk.

Hint: Use Morera's theorem. It suffices to use triangular paths.



- 36.** Based on the insight from Exercise 35, prove the following general assertion: If $f(z)$ is continuous in a domain D and analytic on the set $D - L$, where L is a straight line, then f is analytic throughout the domain D .

Hint: Use Morera's theorem. It suffices to use triangular paths.



- 37.** Let $f(z)$ be an entire function that never vanishes. Let

$$U = \{z \in \mathbb{C} : |f(z)| < 1\}$$

Prove: U is an open set, and all of its connected components are unbounded.

Hint: Show that every connected component is an open set. Assume to the contrary that it is bounded, and derive a contradiction using the maximum principle.



- 38.** Prove **Schwarz's lemma**: If $f: \mathbb{D}(0, 1) \rightarrow \mathbb{D}(0, 1)$ is an analytic function from the unit disk into itself satisfying $f(0) = 0$, then

- A. $|f(z)| \leq |z|$ for every z in the unit disk.
- B. $|f'(0)| \leq 1$
- C. If there exists a point $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$, then there exists $c \in \mathbb{C}$ such that $f(z) = cz$ for every z in the unit disk.

Hint: Try applying the maximum principle to $g(z) = f(z)/z$, $g(0) = f'(0)$ on all the disks $\mathbb{D}(0, r)$.



- 39.** If $f: \mathbb{D}(0, 1) \rightarrow \mathbb{D}(0, 1)$ is an analytic function of the unit disk onto itself (an automorphism), then f must be a Möbius transformation of the form

$$f(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}$$

where θ is a constant angle and $a = f^{-1}(0)$.

Hint: First prove that $g(z) = \frac{a-z}{1-\bar{a}z}$ is an automorphism from the unit disk onto itself satisfying the hypotheses of Schwarz's lemma (Exercise 38). Then prove that $f \circ g^{-1}$ also satisfies the condition in part B of Schwarz's lemma.



- 40.** If $f: \mathbb{D}(0, 1) \rightarrow \mathbb{D}(0, 1)$ is an analytic function from the unit disk into itself with two distinct fixed points, then f is the identity function.

Hint: Recall that a fixed point z of a function f is a point at which $f(z) = z$. Let z_1, z_2 be the two fixed points of f . Consider the map $g(z) = \frac{z_1 - z}{1 - \bar{z}_1 z}$. Verify that $g(0) = z_1$, $g(z_2) = z_3$, $g(z_3) = z_2$, and examine the map $h = g \circ f \circ g$. Verify that $z = 0, z_3$ are fixed points of h . By Schwarz's lemma (Exercise 38), h is a rotation with only one fixed point, and therefore h is the identity function.



- 41.** Every automorphism $f: H \rightarrow H$, where H is the upper half-plane $\text{Im}(z) > 0$, must be a



Möbius transformation.

Hint: Take a Möbius transformation φ mapping H onto $\mathbb{D}(0, 1)$, and apply Exercise 39 to $\varphi \circ f \circ \varphi^{-1}$.

42. Let $f(z)$ be an analytic function from the right half-plane $\operatorname{Re}(z) > 0$ into the unit disk $|z| < 1$. Suppose that $f(1) = 0$. What is the largest possible value of $|f(2)|$?

Hint: Construct a Möbius transformation T from the unit disk $|z| < 1$ onto the right half-plane $\operatorname{Re}(z) > 0$, such that $T(0) = 1$. Show that the composition $f \circ T$ satisfies the hypotheses of Schwarz's lemma (see Exercise 38), and therefore $|f(2)| = |f(T(T^{-1}(2)))|$ is at most $|T^{-1}(2)| = 1/3$.



43. Let $f(z)$ be analytic in a domain D containing the closed disk $\overline{\mathbb{D}}(0, r)$, $r > 0$. Let z_1, z_2 be two distinct points in the disk $\mathbb{D}(0, r)$.

A. Express the integral $\oint_{|z|=r} \frac{f(z)dz}{(z-z_1)(z-z_2)}$ in terms of the two integrals $\oint_{|z|=r} \frac{f(z)dz}{(z-z_1)}$, $\oint_{|z|=r} \frac{f(z)dz}{(z-z_2)}$.

B. Using the preceding part, find another proof of Liouville's theorem.



44. Prove: If f is an entire function taking values only in the upper half-plane $\operatorname{Im}(z) > 0$, then f is constant.

Hint: In Example 14 of Chapter 4, we saw that the Möbius transformation $T(z) = \frac{z-i}{z+i}$ maps the upper half-plane $\operatorname{Im}(z) > 0$ onto the unit disk $|z| < 1$. Therefore $T \circ f$ is an entire function whose image is contained in the unit disk $|z| < 1$. What would Liouville think of that?



45. Let $f(z)$ be analytic on a set S , and let $w \in \mathbb{C}$. The set of preimages of w under f is defined as the set of all points $z \in S$ such that $f(z) = w$. Prove: If f is not constant and S is compact, then the set of preimages is finite.



46. Let $f(z)$ be analytic in a domain D containing the closed unit disk $|z| \leq 1$. Suppose that $|f(z)| = 1$ for every point $|z| = 1$. Prove that the image $f(D)$ also contains the closed unit disk.

Hint: Prove that the equation $f(z) = w_0$ has a root for every $w_0 \in \overline{\mathbb{D}}(0, 1)$. Use the maximum principle.



47. Prove: If $u(x, y)$ is a real-valued harmonic function on a simply connected domain D , then u is infinitely differentiable in both variables on D .

Hint: See Definition 4.3 of a harmonic function on page 129. Use Corollary 4.14 and Theorem 5.23. Note that the definition of harmonicity requires only a continuous second derivative! Thus this is a rare example of a situation in which the existence of two derivatives guarantees the existence of all the infinitely many remaining derivatives!



48. Let $u(x, y)$ be a harmonic function on the entire real plane $\mathbb{R}^2$. Prove: If $u(x, y)$ is bounded on $\mathbb{R}^2$, then u is constant.



49. Let $u(x, y)$ be a harmonic function on the entire real plane $\mathbb{R}^2$. Prove: If $u(x, y)$ is not constant, then there exists at least one point at which it vanishes.

Hint: Suppose not. By the continuity of u , it follows that u is positive at every point or negative at every point. Suppose, for example, that u is positive. Let $v(x, y)$ be the harmonic conjugate of u , so that $f(x + iy) = u(x, y) + iv(x, y)$ is entire (see Corollary 4.14 and Theorem 5.23). Verify that the entire function $g(z) = e^{-f(z)}$ is bounded. Liouville ...





50. Prove that if $f(z)$ is a nonconstant entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Hint: Reminder: A set A is **dense** in a set B if every point $z \in B$ is the limit of a sequence of points $z_n \in A$. Idea: If not, then there exists a point z_0 and an $r > 0$ such that $\mathbb{D}(z_0, r) \cap f(\mathbb{C}) = \emptyset$. Use Liouville's theorem. In fact, by Picard's theorem, which is not included in these notes (at present), the set $f(\mathbb{C})$ is identical to $\mathbb{C}$, with perhaps a single point omitted, but we cannot prove such a result using the tools available to us so far.



51. A function $f(z)$ is called **doubly periodic** if there exist nonzero $\alpha, \beta \in \mathbb{C}$ such that the quotient α/β is not real, and for every $z \in \mathbb{C}$,

$$f(z + \alpha) = f(z + \beta) = f(z)$$

Prove that every doubly periodic entire function is constant.



52. Proof of the Cauchy-Goursat theorem based on Green's theorem: Review Green's theorem, studied in Calculus 2 (or Infinitesimal Calculus 2), and explain how it can be used to prove the Cauchy-Goursat theorem. The principal difference is that, in addition to the analyticity of $f(z)$, one must assume that $f'(z)$ is continuous in the interior and on the path.

Hint: Recall Green's theorem: Let D be a bounded domain, and let $u(x, y)$ and $v(x, y)$ be continuously differentiable functions on a domain containing $D + \partial D$. Then

$$\oint_{\partial D} u(x, y)dx + v(x, y)dy = \iint_D (v_x - u_y) dx dy$$

Let $\sigma(t) = x(t) + iy(t)$ be a parametrization of ∂D . If we use the Cartesian form of a complex function $f(x + iy) = u(x, y) + iv(x, y)$, then we find that the left-hand side of Green's formula is in fact the integral of $f(z)$ along the path σ :

$$\begin{aligned} \oint_{\sigma} f(z) dz &= \oint_{\sigma} [u(x, y) + iv(x, y)](dx + idy) = \oint_{\sigma} (u dx - v dy) + i \oint_{\sigma} (v dx + u dy) \\ &= \iint_D (-v_x - u_y) dx dy + i \iint_D (u_x - v_y) dx dy = 0 \end{aligned}$$

Both double integrals vanish because of the Cauchy-Riemann equations. We leave the details for the student to complete.



53. Poisson's integral formula (Poisson): Let $f(z)$ be analytic in the disk $|z| \leq R$. Prove that for every $0 < r < R$ and every θ ,

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR \cos(\theta - t)} \cdot f(Re^{it}) dt$$

Hint: Use Cauchy's integral formula. Set $z_0 = re^{i\theta}$. Then $f(z_0) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z - z_0} dz$. The point $R^2/\bar{z}_0$ lies outside the disk $|z| \leq R$, and therefore $0 = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z - R^2/\bar{z}_0} dz$. Add the two equations and continue from there.



54. Let $P(z)$ be a polynomial of degree $n > 1$. Recall that for every $w \in \mathbb{C}$, the expression $P^{-1}(w)$ denotes the set of preimages of w , and the expression $|P^{-1}(w)|$ denotes the number of elements in the set $P^{-1}(w)$. Let

$$S = \{ w \in \mathbb{C} : |P^{-1}(w)| = n \}$$

Prove that S is an open set.





- 55.** Let $P(z)$ and $Q(z)$ be two nonconstant polynomials of the same degree, satisfying $|P(z)| = |Q(z)|$ on the unit circle $|z| = 1$. Suppose that all the roots of P , Q lie inside the unit disk $|z| < 1$. Prove that there exists a complex constant $c \in \mathbb{C}$ such that $Q(z) = cP(z)$ for every $z \in \mathbb{C}$.

Hint: Investigate the functions $f(z) = \frac{P(1/z)}{Q(1/z)}$, $g(z) = \frac{Q(1/z)}{P(1/z)}$. Verify that $z = 0$ is a removable singular point of f , g , and that the following limits exist: $\lim_{z \rightarrow \infty} \frac{P(z)}{Q(z)}$, $\lim_{z \rightarrow \infty} \frac{Q(z)}{P(z)}$. Show that f , g are analytic in the disk $|z| < 1 + \varepsilon$, for $\varepsilon > 0$. Use the maximum principle on the disk $|z| \leq 1$.



- 56.** Sketch the plane curve W described by

$$W = \left\{ (x, x \sin \frac{1}{x}) : 0 < x < 1 \right\} \cup \{0\} \times [-1, 0] \cup [0, 1] \times \{-1\} \cup \{1\} \times [-1, \sin(1)]$$

- A.** Give a parametric description $\sigma(t) = x(t) + y(t)i$, $\sigma: [a, b] \rightarrow \mathbb{C}$, of the curve W . It is advisable to decompose it into four parts corresponding to its definition, and then concatenate them.
- B.** Is the curve W bounded? Closed? Simple? Piecewise smooth?
- C.** Try to compute the length of the curve W . An estimate of the length will also suffice. At least try to answer the question: Is its length finite?
- D.** Does the curve W conform to Definition 4.7 of a piecewise smooth path? Does the definition of a piecewise smooth path require its length to be finite? Review the theorems concerning a piecewise smooth path and check whether we have implicitly assumed that the length of a piecewise smooth path is always finite.

Hint: This is another variant of the Warsaw “circle” (Warsaw Circle), which we already encountered in Exercise 33 of Chapter 2. It is advisable to examine that exercise before attempting the present one. The only difference is that instead of the expression $\sin \frac{1}{x}$ in the first part, we have the expression $x \sin \frac{1}{x}$, which makes the curve continuous at the point $x = 0$. A plot of the curve can be viewed by clicking [knit](#) [sint](#). Further discussion of the topic can be found at the following links: [math.stackexchange.1](#), [math.stackexchange.2](#).



- 57.** Let $f(z)$ and $g(z)$ be two entire functions satisfying $f(g(x)) = x$ for every real x . Prove that both functions are one-to-one and onto, and that they are inverses of one another.



- 58.** A difficult problem. For highly motivated students only: Let $f(z)$ be a continuous function on the entire complex plane. Suppose that

$$\oint_{|z-z_0|=r} f(z) dz = 0$$

for every $z_0 \in \mathbb{C}$ and every $r > 0$. Prove that f is entire.

Remark: This is a more refined version of Morera's theorem. In this formulation, the integral of f over every circle vanishes. From this assumption alone, one must prove that f is analytic. To make things a little easier, one may assume that the components of $f = u + iv$ have continuous partial derivatives, and use an appropriate version of Green's theorem. This is a genuinely difficult challenge, intended only for the most dedicated among us.





59. Let $\sigma: [a, b] \rightarrow \mathbb{C}$ be a piecewise smooth closed path, and let z_0 be a point not on the path. Prove that there exists an integer $k \in \mathbb{Z}$ such that

$$\oint_{\sigma} \frac{dz}{z - z_0} = 2k\pi i \quad (5.27)$$

The number k is called the **index of the point z_0 with respect to the path σ** , and is denoted by $\text{Ind}(z_0, \sigma)$.

Hint: In other books, the number k is also called the **winding number** of the path σ with respect to the point z_0 , because it represents the number of times the path σ winds around the point z_0 . This is, of course, a general closed path that is not necessarily a Jordan path (that is, it may intersect itself at intermediate points and wind several times around z_0).

Define the function $h(t) = \int_a^t \frac{\sigma'(s) ds}{\sigma(s) - z_0}$. Show that $h(a) = 0$, $h(b) = \oint_{\sigma} \frac{dz}{z - z_0}$. Define the function $g(t) = [\sigma(t) - z_0] \cdot e^{-h(t)}$. Prove that $g'(t) = 0$ for every t in the interval (a, b) , and therefore g is constant on the interval $[a, b]$. Thus $g(a) = g(b)$. Deduce from this that $e^{-h(b)} = 1 = e^{2k\pi i}$.



60. (Continuation of Exercise 59) Complete the missing equalities in Figure 5.28.

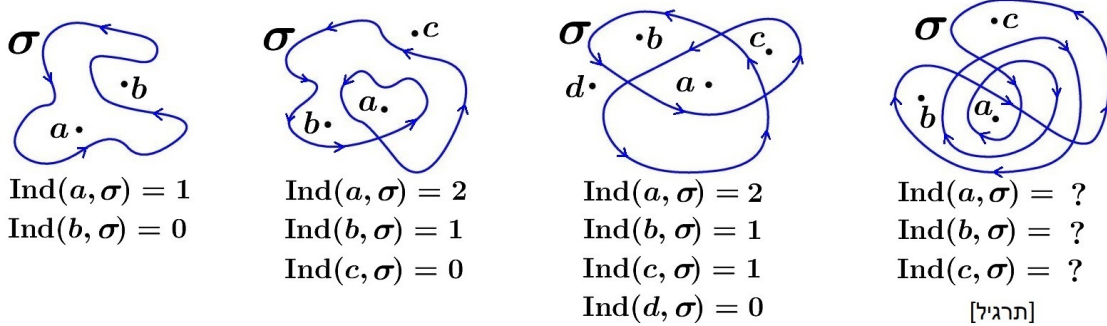


Fig. 5.28: Diagram accompanying Exercise 60: various examples of the index of a point with respect to a path



61. (Continuation of Exercise 59) By examining Figure 5.29, compute the index

$$\text{Ind}(z_k, \sigma) = \frac{1}{2\pi i} \oint_{\sigma} \frac{dz}{z - z_k} \quad (5.28)$$

of each of the points $z_1, z_2, \dots, z_8$ with respect to the closed path σ .



62. Let σ be a piecewise smooth closed path, and let $\mathbb{C} - \sigma$ be the complement of σ (all points not on the path). See Figure 5.30. Prove that the index function $f(z) = \text{Ind}(z, \sigma)$ is constant on each connected component of $\mathbb{C} - \sigma$. In other words, any two points belonging to the same connected component have the same index with respect to the path σ . In addition, prove that all the connected components are bounded, except for one unbounded connected component, the index of all of whose points is zero (in Figure 5.30, all the connected components are bounded except for C_6 , which is colored light cyan).

Hint: Since the index is always an integer, it suffices to prove that f is continuous (and hence cannot take two different values on the same connected component. See Exercise 44 at the end of Chapter 3). Consider the expression

$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \oint_{\sigma} \frac{dz}{z - z_1} - \frac{1}{2\pi i} \oint_{\sigma} \frac{dz}{z - z_2} = \frac{z_1 - z_2}{2\pi i} \oint_{\sigma} \frac{dz}{(z - z_1)(z - z_2)}$$



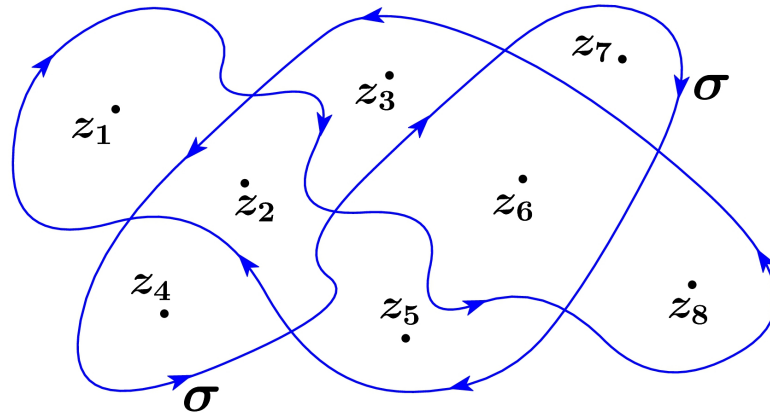
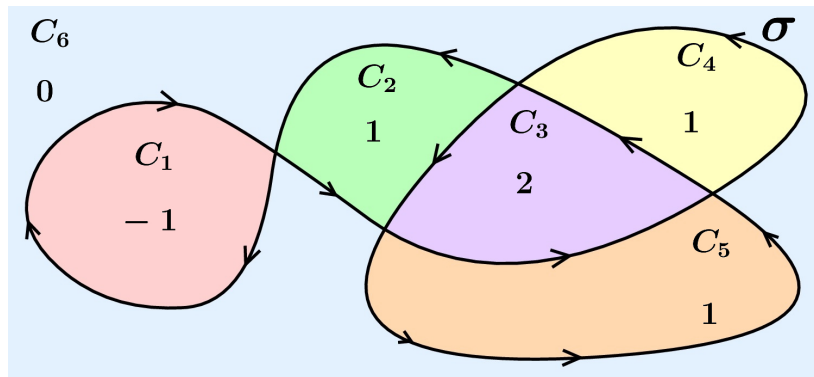


Fig. 5.29: Diagram for Exercise 61

Prove that it is bounded by $|z_1 - z_2| \frac{M \cdot \ell(\sigma)}{2\pi}$, where M is the maximum of $\frac{1}{|z - z_1| |z - z_2|}$ on the path σ .

On the other hand, $f(z) = \text{Ind}(z, \sigma)$ tends to zero as z tends to infinity, and hence necessarily vanishes on the unbounded component (because it must take an integer value, so this value must be zero).

Fig. 5.30: Diagram for Exercise 62: connected components of $\mathbb{C} - \sigma$

63. Let σ be a piecewise smooth closed path. **The interior** of σ is defined as the set of all points $z \in \mathbb{C} - \sigma$ for which $\text{Ind}(z, \sigma) \neq 0$. **The exterior** of σ is defined as the set of all points $z \in \mathbb{C} - \sigma$ for which $\text{Ind}(z, \sigma) = 0$. Find the interior of the path σ in Figure 5.30. Construct an example of a path σ having a bounded connected component that belongs to its exterior.

Hint: For the second part of the question, examine Figure 5.12.



Chapter 6

Series

In the present chapter of the course, we shall generalize concepts and assertions concerning sequences and series from real analysis to complex analysis. In many cases, the definitions and assertions concerning real sequences and series extend automatically to complex numbers, and therefore we shall not bother to provide proofs or detailed explanations for them.

Before proceeding, it is advisable to review the topic of real sequences and series in the Calculus 2 lecture notes (Appendices A-C at the end of the notes). Click the following link to download the notes.



[Calculus 2 Lecture Notes](#)

The topic of series is divided into four subtopics:

- (a) Numerical series
- (b) Sequences of functions
- (c) Series of functions
- (d) Power series (Taylor/Maclaurin)

6.1 Series of Numbers

- This is, of course, a broad subject rich in detail that we cannot cover within the framework of the present course. We assume that the student has encountered most of the concepts and theorems in previous courses, and therefore we shall confine ourselves to a brief and partial review (without proofs).
- Let $\{z_n\}_{n=1}^{\infty}$ be a given sequence of complex numbers. Our aim is to define the total sum of the terms of the sequence and the conditions required for its existence. The infinite sum of the sequence, or the **infinite series**, is denoted by

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \cdots \quad (6.1)$$



- To give meaning to an infinite sum of numbers, we first define the **sequence of partial sums of the series**:

$$\begin{aligned}
 S_1 &= z_1 \\
 S_2 &= z_1 + z_2 \\
 S_3 &= z_1 + z_2 + z_3 \\
 &\vdots \\
 S_n &= z_1 + z_2 + z_3 + \cdots + z_n \\
 &\vdots
 \end{aligned} \tag{6.2}$$

- i Definition 6.1:** Let $\sum_{n=1}^{\infty} z_n$ be a given series. We say that the series $\sum_{n=1}^{\infty} z_n$ **converges** if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges to a **finite limit** S :

$$\lim_{n \rightarrow \infty} S_n = S$$

We say that the series $\sum_{n=1}^{\infty} z_n$ **converges to the sum** S , and write

$$\sum_{n=1}^{\infty} z_n = S$$

If the limit S does not exist, we say that the series $\sum_{n=1}^{\infty} z_n$ **does not converge**, or briefly: it **diverges**.

- i Proposition 6.1:** A complex series $\sum_{n=1}^{\infty} x_n + iy_n$ converges if and only if both real series $\sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y_n$ converge.

- ✓ **Proof:** We use the definition of the sum of a series as the limit of its partial sums. Let $\{A_n\}_{n=1}^{\infty}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} x_n$. That is: $A_n = \sum_{k=1}^n x_k$. Let $\{B_n\}_{n=1}^{\infty}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} y_n$. By assertion 2.16, proved in the section on sequences (page 66), the sequence $\{A_n + iB_n\}_{n=1}^{\infty}$ converges if and only if the two sequences $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ converge. The rest is clear. ■

- The proofs of the following two theorems are identical to their proofs in the real case.

- i Theorem 6.2:** If c is a fixed nonzero complex number, then the series $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} cz_k$ converge or diverge together. In the case of convergence, the following equality holds:

$$\sum_{k=1}^{\infty} cz_k = c \sum_{k=1}^{\infty} z_k$$



i Theorem 6.3: If the series $\sum_{k=1}^{\infty} z_k$ and $\sum_{k=1}^{\infty} w_k$ converge, then the series $\sum_{k=1}^{\infty} z_k \pm w_k$ converge, and in addition the following equality holds:

$$\sum_{k=1}^{\infty} z_k \pm w_k = \sum_{k=1}^{\infty} z_k \pm \sum_{k=1}^{\infty} w_k$$

Elementary Divergence Test

i Theorem 6.4: If the series $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n \rightarrow \infty} z_n = 0$.

v Proof: Let $\{S_n\}_{n=1}^{\infty}$ be the sequence of partial sums of the series. The convergence of the series implies that the limit $\lim_{n \rightarrow \infty} S_n = S$ exists. By the definition of S_n , clearly

$$a_n = S_n - S_{n-1}$$

and therefore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = S - S = 0 \quad \blacksquare$$

i Corollary 6.5: (Elementary divergence test) If the limit $\lim_{n \rightarrow \infty} z_n$ does not exist, or exists but is nonzero, then the series $\sum_{n=1}^{\infty} z_n$ diverges.

Cauchy's Criterion for the Convergence of a Series of Complex Numbers

In various contexts, it is customary to divide an infinite series $\sum_{n=1}^{\infty} z_n$ into three parts:

$$\sum_{k=1}^{\infty} z_k = \underbrace{\sum_{k=1}^m z_k}_{\text{seires eht fo daeh}} + \underbrace{\sum_{k=m+1}^n z_k}_{\text{seires eht fo elddim}} + \underbrace{\sum_{k=n+1}^{\infty} z_k}_{\text{seires eht fo liat}}$$

Of course, the names depend on a suitable choice of the indices m and n . We can characterize convergence of the series in any of the following three equivalent ways:

(a) The series $\sum_{n=1}^{\infty} z_n$ converges if and only if the sequence of heads $S_m = \sum_{k=1}^m z_k$ converges.

Another name for the sequence of heads is the **sequence of partial sums** $\{S_m\}_{m=1}^{\infty}$.

(b) The series $\sum_{n=1}^{\infty} z_n$ converges if and only if the sequence of tails $r_n = \sum_{k=n+1}^{\infty} z_k$ tends to zero.

The number r_n is also called the **remainder of the series** (the remainder begins with the term z_{n+1}).

(c) **Cauchy's criterion** states that a series converges if and only if its middle $\sum_{k=m+1}^n z_k$ tends to



zero as m tends to infinity. The logical formulation of this assertion appears in the following theorem.

Theorem 6.6:



Cauchy's convergence test

The series $\sum_{n=1}^{\infty} z_n$ converges if and only if $\lim_{n,p \rightarrow \infty} \sum_{k=n+1}^{n+p} z_k = 0$. That is, for every $\varepsilon > 0$, there exists N such that for all $n > N$ and p , $\left| \sum_{k=n+1}^{n+p} z_k \right| < \varepsilon$.

$$(6.3) \quad \left| \sum_{k=n+1}^{n+p} z_k \right| = |z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \varepsilon$$

Absolute and Conditional Convergence

Definition 6.2: We say that the series $\sum_{k=1}^{\infty} z_k$ **converges absolutely** if the series of the absolute values of its terms $\sum_{n=1}^{\infty} |z_n|$ converges.

Definition 6.3: We say that a series $\sum_{n=1}^{\infty} z_n$ **converges conditionally** if it converges, but the series of absolute values $\sum_{n=1}^{\infty} |z_n|$ diverges.

Example 1: The canonical example of a conditionally convergent series is the **Leibniz series**, defined by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

Its series of absolute values is the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

Theorem 6.7: If a series converges absolutely, then it converges.

Proof: By Cauchy's convergence test,

$$\forall \varepsilon > 0, \exists N, \forall n > N, \forall p \in \mathbb{N} :$$

$$|z_{n+1}| + |z_{n+2}| + \cdots + |z_{n+p}| < \varepsilon$$

Since, by the triangle inequality,

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| \leq |z_{n+1}| + |z_{n+2}| + \cdots + |z_{n+p}| < \varepsilon$$



we obtain

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \varepsilon$$

Therefore,

$$\left| \sum_{k=n+1}^{n+p} z_k \right| < \varepsilon$$

Thus, by Cauchy's convergence test, the series $\sum_{k=1}^{\infty} z_k$ also converges. ■

◆ **Example 2: The infinite geometric series:** We examine the convergence of the geometric series $\sum_{n=0}^{\infty} z^n$, for a fixed complex number z . We begin by finding a formula for the partial sum

$$\begin{aligned} S_n &= 1 + z + z^2 + z^3 + \cdots + z^n \\ zS_n &= z + z^2 + z^3 + \cdots + z^n + z^{n+1} \end{aligned}$$

Subtracting the last two equations, we obtain

$$S_n - zS_n = S_n(1 - z) = 1 - z^{n+1}$$

This immediately gives the familiar formula for the sum of a finite geometric series, known from high school:

$$S_n = \sum_{k=0}^n z^k = \frac{1 - z^{n+1}}{1 - z}$$

If $|z| < 1$, then $|z|^n$ tends to zero as $n \rightarrow \infty$, and therefore

$$\sum_{n=0}^{\infty} z^n = \lim_{n \rightarrow \infty} S_n = \frac{1}{1 - z}$$

If $|z| \geq 1$, then the limit $\lim_{n \rightarrow \infty} |z|^n$ does not exist, or exists but is nonzero; therefore, by the divergence test (if the general term does not tend to zero, the series diverges), the geometric series diverges. We have therefore shown that the geometric series converges if and only if $|z| < 1$, and we have also obtained its sum formula. Clearly, the convergence is also absolute.

The p -Series Test

◆ For real p , this test states that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

When $p = x + iy$ is a complex number,

$$\left| \frac{1}{n^p} \right| = \left| \frac{1}{n^x \cdot n^{iy}} \right| = \left| \frac{1}{n^x \cdot e^{iy \ln n}} \right| = \frac{1}{n^x} = \frac{1}{n^{\operatorname{Re}(p)}}$$

Thus we obtain the following theorem.



i Theorem 6.8: (The p -series convergence test)

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in \mathbb{C}$, converges absolutely if and only if $\operatorname{Re}(p) > 1$.

The Comparison Test (Comparison Test)

➤ The comparison test for convergence or divergence of a series is characterized by comparing the series under consideration with a second series whose convergence (or divergence) is already known.

➤ Let $\sum_{n=1}^{\infty} z_n$, $\sum_{n=1}^{\infty} w_n$ be two given complex series. The two series may be compared by testing conditions such as $|z_n| \leq |w_n|$, $z_n \sim cw_n$, that is, the existence of the ordinary limit $\lim_{n \rightarrow \infty} \frac{z_n}{w_n}$, or the limit superior $\limsup_{n \rightarrow \infty} \frac{z_n}{w_n}$.

i Theorem 6.9: (Comparison test) Let $\sum_{n=1}^{\infty} z_n$, $\sum_{n=1}^{\infty} w_n$ be two given complex series, and let r be a positive real constant. Suppose that $|z_n| \leq r|w_n|$ for every natural number, with perhaps finitely many exceptions. Then If the series $\sum_{n=1}^{\infty} w_n$ converges absolutely, then the series $\sum_{n=1}^{\infty} z_n$ also converges absolutely.

➤ A more general version of the comparison test is given in the following theorem. It is sometimes called the limit comparison test.

i Theorem 6.10:



timil eht fi .seires xelpmoc nevig owt eb $\sum_{n=1}^{\infty} w_n$, $\sum_{n=1}^{\infty} z_n$ teL

(6.4)

$$\lim_{n \rightarrow \infty} \frac{z_n}{w_n} = c$$

.rehtegot egrevid ro rehtegot egrevnoc $\sum_{n=1}^{\infty} w_n$ dna $\sum_{n=1}^{\infty} z_n$ seires eht neht, $|c| > 0$ fi **a)**

osla $\sum_{n=1}^{\infty} z_n$ seires eht ,yletulosba segrevnoc $\sum_{n=1}^{\infty} w_n$ seires eht fi neht, $c = 0$ fi **b)**
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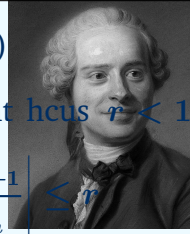
osla $\sum_{n=1}^{\infty} w_n$ seires eht ,yletulosba segrevnoc $\sum_{n=1}^{\infty} z_n$ seires eht fi neht, $c = \infty$ fi **c)**
.yletulosba segrevnoc

D'Alembert's Ratio Test (Jean Le Rond d'Alembert 1717-1783)

➤ The geometric series is a fundamental and simple example of a series whose convergence or divergence is easy to determine. The Cauchy and D'Alembert convergence tests use the geometric series as a basis of comparison to test the convergence of new series.

➤ For convenience, we have included in the following theorem the two different versions of the ratio test (or D'Alembert's test, as it is sometimes called).



i Theorem 6.11: (Ratio test D’alembert)

.seires xelpmoc a eb $\sum_{n=1}^{\infty} z_n$ teL

,drawno mret emos morf ,taht $r < 1$ tnatsnoc laer a stsixe ereht fl)a(

$$\left| \frac{z_{n+1}}{z_n} \right| \leq r$$

.yletulosba segrevnoc $\sum_{n=1}^{\infty} z_n$ seires eht neht

neht ,stsixe $\lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right| = r$ timil eht fl)b(

.yletulosba segrevnoc $\sum_{n=1}^{\infty} z_n$ seires eht neht , $r < 1$ fl ♦

.segrevid $\sum_{n=1}^{\infty} z_n$ seires eht neht , $r > 1$ fl ♦

denimreted eb tonnac $\sum_{n=1}^{\infty} z_n$ seires eht fo ecnegrevid ro ecnegrevnoc eht , $r = 1$ fl ♦

.tset siht yb

Cauchy's Root Test (Cauchy)

- Here too, for convenience, we have included in the following theorem the two different versions of Cauchy's root test. In the first version a comparison is made, while in the second a limit must be computed.

i Theorem 6.12: (Cauchy)

.seires xelpmoc nevig a eb $\sum_{n=1}^{\infty} z_n$ teL

yfstas seires eht fo smret eht ,drawno tniop emos morf ,fl)a(

$$\sqrt[n]{|z_n|} \leq r < 1$$

.yletulosba segrevnoc $\sum_{n=1}^{\infty} z_n$ seires eht neht

timil eht fl)b(

$$\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = r$$

neht ,stsixe

.yletulosba segrevnoc $\sum_{n=1}^{\infty} z_n$ seires eht neht , $r < 1$ fl ♦

.segrevid $\sum_{n=1}^{\infty} z_n$ seires eht neht , $r > 1$ fl ♦

.tset siht yb denimreted eb tonnac ecnegrevid ro ecnegrevnoc , $r = 1$ fl ♦

- The proofs of all the preceding convergence tests differ only slightly from their proofs for real series. Recommended exercise: Review the proofs in the real case and try to understand how they can be generalized to the complex case.

6.2 Sequences of Functions

i **Definition 6.4:** We say that a sequence of complex functions $\{f_n(z)\}_{n=1}^{\infty}$ **converges pointwise on a set S** to a function $f(z)$ if, for every point $z \in S$,

$$\lim_{n \rightarrow \infty} f_n(z) = f(z)$$

The function $f(z)$ is called the **limit function** of the sequence.

i **Definition 6.5:** We say that a sequence of complex functions $\{f_n(z)\}_{n=1}^{\infty}$ **converges uniformly on a set S** to a function $f(z)$ if for every $\varepsilon > 0$, there exists a natural number N such that for every $n \geq N$ and every $z \in S$, $|f(z) - f_n(z)| < \varepsilon$. The notation for uniform convergence is $f_n \Rightarrow f$.

➤ In logical notation, this is written as follows:

$$f_n \Rightarrow f \iff \forall \varepsilon > 0, \exists N, \forall n \geq N, \forall z \in S: |f(z) - f_n(z)| < \varepsilon$$

The phrase “uniformly” indicates that the same N works for all points $z \in S$, unlike the situation in pointwise convergence.

➤ A more precise formulation of the second definition of **uniform convergence** (Uniform Convergence) of a sequence of functions is based on the notion of the **uniform norm** (Uniform norm), defined by

$$\|f\|_S = \sup_{z \in S} |f(z)|$$

That is, the uniform norm of a function $f(z)$ on a set S is the supremum (supremum) of $|f(z)|$ over the set S .

➤ When this notion is applied to the difference of two functions $f(z)$, $g(z)$ on a set S , we obtain the notion of the **uniform distance** between two functions on a set S :

$$\|f - g\|_S = \sup_{z \in S} |f(z) - g(z)|$$



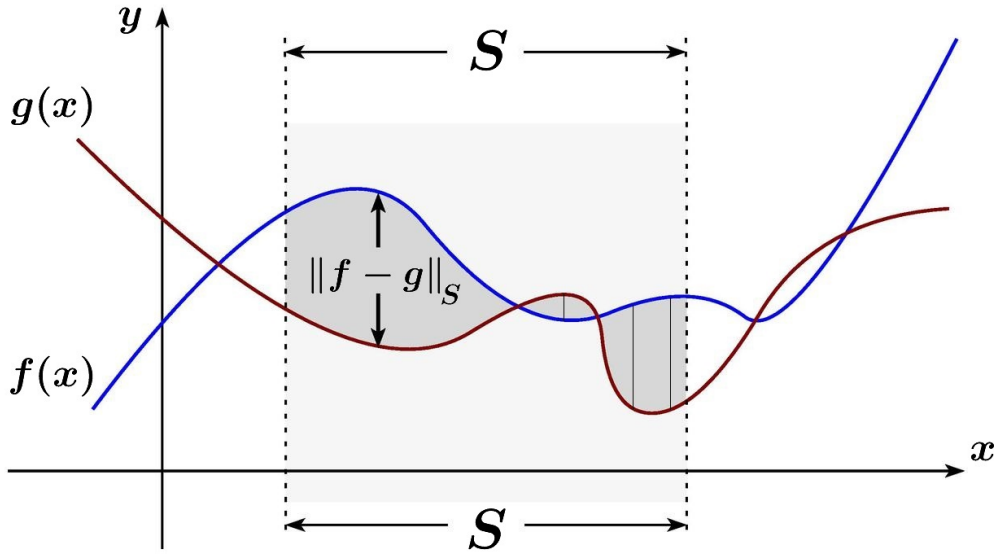


Fig. 6.1: The uniform distance between the function $f(x)$ and the function $g(x)$ equals $\sup_{x \in D} |f(x) - g(x)|$

As can be seen in Figure 6.1, the uniform distance between two functions on a domain D is the maximum distance between $f(x)$ and $g(x)$ for $x \in D$. In cases where the maximum does not exist, this is the supremum, which may sometimes also be ∞ .

i Proposition 6.13: (Properties of the uniform norm)

- (a) $\|f + g\|_D < \|f\|_D + \|g\|_D$
- (b) If $D_1 \subseteq D_2$, then $\|f\|_{D_1} \leq \|f\|_{D_2}$.
- (c) $\|f\|_{D_1 \cup D_2} = \max\{\|f\|_{D_1}, \|f\|_{D_2}\}$

✓ Proof:

- (a) The assertion follows easily from the triangle inequality for real numbers.

$$\begin{aligned}
 \|f + g\|_D &= \sup_{x \in D} |f(x) + g(x)| \\
 &\leq \sup_{x \in D} |f(x)| + |g(x)| \\
 &\leq \sup_{x \in D} |f(x)| + \sup_{x \in D} |g(x)| \\
 &= \|f\|_D + \|g\|_D
 \end{aligned}$$

- (b) Trivial.
- (c) A simple exercise. ■

i Proposition 6.14: A sequence of functions $\{f_n(z)\}_{n=1}^{\infty}$ *converges uniformly* to a function $f(z)$ on a set S if and only if

$$\lim_{n \rightarrow \infty} \|f - f_n\|_S = 0$$

➤ As stated above, a thorough treatment of series of functions appears in Appendix B of the

following lecture notes:



Calculus 2 Lecture Notes

- Uniform convergence of a sequence of functions is a stronger type of convergence than pointwise convergence, because it ensures that the limit function retains properties of the functions in the sequence.
- One of the fundamental tests for uniform convergence is Cauchy's convergence test. It also serves as a theoretical basis for proving more advanced tests.

i Theorem 6.15: (Cauchy's test for convergence of a sequence of functions)

A sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly on a set S if and only if for every $\varepsilon > 0$, there exists a natural number N such that for every $m, n > N$,



$$\|f_m - f_n\|_S < \varepsilon$$

✓ Proof:

$\Leftarrow$: Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions converging uniformly on a set S to a limit function $f(x)$. That is,

$$\lim_{n \rightarrow \infty} \|f - f_n\|_S = 0$$

Let $\varepsilon > 0$. There exists a natural number N such that for every $n > N$, we have $\|f - f_n\|_S < \frac{\varepsilon}{2}$, and therefore, for any two natural numbers $m, n > N$,

$$\begin{aligned} \|f_m - f_n\|_S &= \|f_m - f + f - f_n\|_S \\ &\leq \|f_m - f\|_S + \|f - f_n\|_S \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus we have obtained $\|f_m - f_n\|_S < \varepsilon$.

$\Rightarrow$: Let $\varepsilon > 0$ be given. Suppose there exists N such that for every $m, n > N$, we have $\|f_m - f_n\|_S < \varepsilon$. Let $x \in S$. Then, for every $m, n > N$,

$$|f_m(x) - f_n(x)| < \varepsilon$$

because

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_S < \varepsilon$$

Therefore, by Cauchy's criterion for sequences of real numbers, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ con-



verges to a limit, which we denote by $f(x)$. Moreover,

$$|f(x) - f_n(x)| = \lim_{m \rightarrow \infty} |f_m(x) - f_n(x)| \leq \varepsilon$$

The last inequality holds for every $x \in S$, and hence we have obtained

$$\|f - f_n\|_S \leq \varepsilon$$

Therefore the sequence of functions f_n converges uniformly to the function f . ■

- Unlike pointwise convergence, under suitable conditions uniform convergence preserves properties such as continuity, analyticity, and integrability. For example: If a sequence of continuous functions $\{f_n(z)\}_{n=1}^{\infty}$ converges uniformly to a function $f(z)$, then $f(z)$ is also continuous.

i Theorem 6.16: If a sequence of continuous functions $\{f_n(z)\}_{n=1}^{\infty}$ converges uniformly to a function $f(z)$ on a domain D , then $f(z)$ is continuous on the domain D .

Proof: Let z_0 be any point in the domain D . Then, for every natural number n ,

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)| \\ &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| \\ &\leq 2\|f - f_n\|_D + |f_n(z) - f_n(z_0)| \end{aligned}$$

Let $\varepsilon > 0$. By uniform convergence, there exists n such that

$$\|f - f_n\|_D < \frac{\varepsilon}{3}$$

By the definition of continuity, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f_n(z) - f_n(z_0)| < \frac{\varepsilon}{3}$$

Therefore, for every z satisfying $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| \leq 2\|f - f_n\|_D + |f_n(z) - f_n(z_0)| < 2 \cdot \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

By the definition of continuity, f is continuous at the point z_0 . ■

- ◆ **Example 3:** The sequence of functions $f_n(z) = \sqrt[n]{z}$ does not converge uniformly on the disk $|z| \leq 1$. If it did, then the sequence of real functions $f_n(x) = \sqrt[n]{x}$ on the closed interval $[0, 1]$



would also converge uniformly. It is not difficult to see that the limit function

$$f(x) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

is not continuous, contrary to part 1 of Theorem 6.16. Figure 6.2 illustrates that convergence to the function $f(x)$ is pointwise but not uniform.

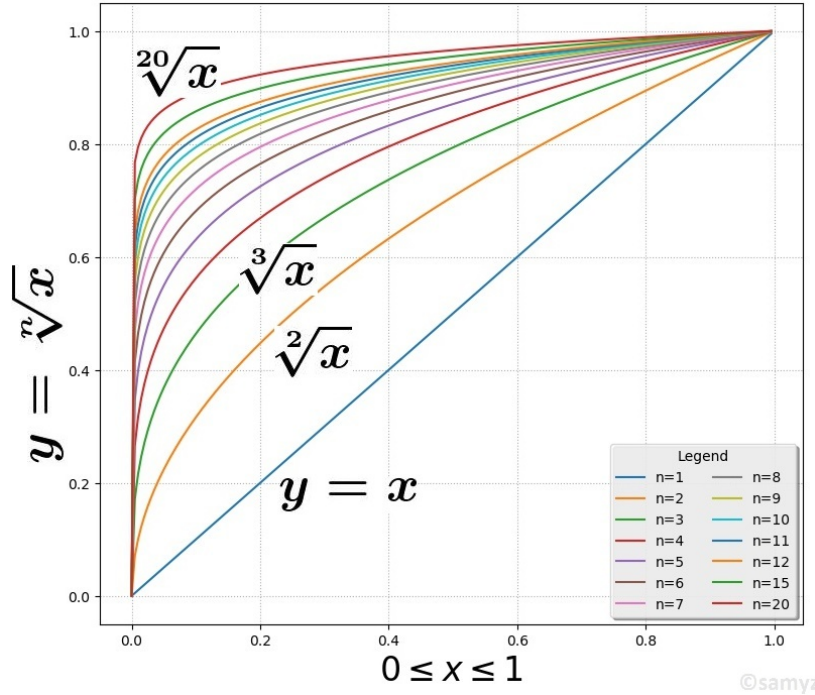


Fig. 6.2: The sequence of functions $f_n(x) = \sqrt[n]{x}$ on the interval $[0, 1]$

Theorem 6.17: Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of continuous functions converging uniformly to a function $f(z)$ on a domain D , and let $\sigma: [a, b] \rightarrow D$ be a piecewise smooth path in D . Then

$$\int_{\sigma} f(z) dz = \int_{\sigma} \left[\lim_{n \rightarrow \infty} \{f_n(z)\}_{n=1}^{\infty} \right] dz = \lim_{n \rightarrow \infty} \int_{\sigma} f_n(z) dz$$

Proof: It suffices to prove the following limit:

$$\left| \int_{\sigma} f(z) dz - \int_{\sigma} f_n(z) dz \right| = \left| \int_{\sigma} [f(z) - f_n(z)] dz \right| \leq \int_{\sigma} |f(z) - f_n(z)| dz \xrightarrow{n \rightarrow \infty} 0$$

By the definition of a smooth path, the function $\sigma'(t)$ is continuous on the interval $[a, b]$, and is therefore bounded by a real number M . Uniform convergence implies that $\lim_{n \rightarrow \infty} \|f - f_n\|_{\sigma} =$

0. Therefore,

$$\begin{aligned} \int_{\sigma} |f(z) - f_n(z)| dz &= \int_{\sigma} |f(\sigma(t)) - f_n(\sigma(t))| \sigma'(t) dt \\ &\leq \int_{\sigma} \|f - f_n\|_{\sigma} \cdot |\sigma'(t)| dt \\ &\leq M \cdot \ell(\sigma) \cdot \|f - f_n\|_{\sigma} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

We have obtained

$$\lim_{n \rightarrow \infty} \left| \int_{\sigma} f(z) dz - \int_{\sigma} f_n(z) dz \right| = 0$$

and therefore $\lim_{n \rightarrow \infty} \int_{\sigma} f_n(z) dz = \int_{\sigma} f(z) dz$. ■

i Theorem 6.18: Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on a domain D , converging uniformly to a function $f(z)$. Then the function $f(z)$ is also analytic on the domain D .

✓ Proof: The continuity of f is guaranteed by Theorem 6.16 (this is the first condition required for applying Morera's theorem). In the second stage, we prove that $\oint_{\sigma} f(z) dz = 0$ for every piecewise smooth Jordan path σ in the domain D . Morera's theorem will then imply that f is analytic in the domain D . Uniform convergence on the path σ implies that

$$\|f - f_n\|_{\sigma} = \sup_{t \in [a, b]} |f(\sigma(t)) - f_n(\sigma(t))| \xrightarrow{n \rightarrow \infty} 0$$

The piecewise continuous function $\sigma'(t)$ is bounded on the closed interval $[a, b]$, and is therefore bounded there by the Weierstrass theorem. Let $M = \max_{a \leq t \leq b} |\sigma'(t)|$. Therefore,

$$\begin{aligned} \left| \oint_{\sigma} f(z) dz - \oint_{\sigma} f_n(z) dz \right| &\leq \oint_{\sigma} |f(z) - f_n(z)| dz \\ &\leq \int_{\sigma} \|f - f_n\|_{\sigma} \cdot |\sigma'(t)| dt \\ &\leq M \cdot \ell(\sigma) \cdot \|f - f_n\|_{\sigma} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

We have therefore obtained

$$\oint_{\sigma} f(z) dz = \lim_{n \rightarrow \infty} \oint_{\sigma} f_n(z) dz = 0$$

Thus, by Morera's theorem (Theorem 5.29), f is analytic on D . ■



6.3 Series of Functions

i Definition 6.6: Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of functions defined on a set S . The expression

$$\sum_{n=1}^{\infty} f_n(z) = f_1(z) + f_2(z) + f_3(z) + \cdots + f_n(z) + \cdots \quad (6.5)$$

is called a **series of functions**.

➤ We wish to define the **sum of a series of functions**. For this purpose, note that if we substitute a specific point z_0 in expression (6.5), we obtain an **ordinary numerical series**, for which we already have many convergence tests.

i Definition 6.7: We say that the series of functions $\sum_{n=1}^{\infty} f_n(z)$ **converges pointwise** to a sum $f(z)$ on a set S if, for every $z_0 \in S$, the numerical series $\sum_{n=1}^{\infty} f_n(z_0)$ converges to the sum $f(z_0)$. The function $f(z)$ is also called the **sum function** of the series.

➤ The basic fact that, for every point z in the domain of convergence, the series $\sum_{n=1}^{\infty} f_n(z)$ is actually a numerical series allows us to use all the known convergence tests for an ordinary numerical series.

◆ **Example 4:** Find the domain of convergence of each of the following series:

(a) $\sum_{n=1}^{\infty} \frac{z^n}{n+1}$

(b) $\sum_{n=1}^{\infty} z^n \sin \frac{z}{n+1}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{z}{1+2z} \right)^n$

(d) $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

Solution:

(a) We test absolute convergence using the ratio test (Theorem 6.11).

$$\lim_{n \rightarrow \infty} \frac{|z_{n+1}|}{|z_n|} = \lim_{n \rightarrow \infty} \left| \frac{z^{n+1}}{n+2} \right| : \left| \frac{z^n}{n+1} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} |z| = |z|$$

By the ratio test, the series converges for $|z| < 1$ and diverges for $|z| > 1$. The ratio test gives no answer for $|z| = 1$ (the unit circle). Substituting $z = 1$ gives a divergent harmonic series, while substituting $z = -1$ gives a convergent Leibniz series. At present it is not clear what happens at other points on the unit circle.

(b) We test absolute convergence using the limit comparison test (Theorem 6.10). Compare with the preceding series:

$$z_n = z^n \sin \frac{z}{n+1}$$

$$w_n = \frac{z^n}{n+1}$$



and therefore

$$\lim_{n \rightarrow \infty} \frac{|z_n|}{|w_n|} = \lim_{n \rightarrow \infty} \frac{|z^n \sin \frac{z}{n+1}|}{\left| \frac{z^n}{n+1} \right|} = \lim_{n \rightarrow \infty} \frac{\left| \sin \frac{z}{n+1} \right|}{\frac{1}{n+1}} = |z|$$

The last equality follows from L'Hôpital's rule, by substituting $t = \frac{1}{n+1}$ and computing $\lim_{t \rightarrow 0^+} \frac{\sin zt}{t}$.

Therefore, by the limit comparison test, the series $\sum_{n=1}^{\infty} z^n \sin \frac{z}{n+1}$ converges in the open unit disk $|z| < 1$. The series diverges for $|z| > 1$. The situation on the unit circle $|z| = 1$ is unclear.

(c) We use the root test (Theorem 6.12). The absolute value of the general term is

$$|z_n| = \left| \frac{(-1)^n}{n} \left(\frac{z}{1+2z} \right)^n \right| = \frac{1}{n} \left| \frac{z}{1+2z} \right|^n$$

and therefore

$$\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} \left| \frac{z}{1+2z} \right| = \left| \frac{z}{1+2z} \right|$$

By the root test, the series converges when $\left| \frac{z}{1+2z} \right| < 1$. We must find the inverse image of the unit disk under the Möbius map $T(z) = \frac{z}{1+2z}$. Using the methods learned in Chapter 3, it is not difficult to find that the inverse image is the disk $|z - \frac{2}{3}| < \frac{1}{3}$. The root test gives no answer for points on the circle $|z - \frac{2}{3}| = \frac{1}{3}$. But for points outside the circle, the root test states that the series diverges.

(d) It is easy to verify from the ratio test that the series converges for every complex z .

❖ The sequence of partial sums $\{S_n(z)\}_{n=1}^{\infty}$ of a series of functions $\sum_{n=1}^{\infty} f_n(z)$ is defined by

$$S_n(z) = \sum_{k=1}^n f_k(z)$$

i Definition 6.8: We say that a series of functions $\sum_{n=1}^{\infty} f_n(z)$ **converges uniformly** on a set S if its sequence of partial sums $\{S_n(z)\}_{n=1}^{\infty}$ converges uniformly on the set S .

i Theorem 6.19: (Cauchy's test for convergence of a series of functions)

The series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on a set S if and only if for every $\varepsilon > 0$ there exists a natural number N (depending only on ε) such that for every $m, n > N$, we have

$$\|S_m - S_n\|_S < \varepsilon$$

✔ **Proof:** follows immediately from Cauchy's test for a sequence of functions (Theorem 6.15). ■



The Weierstrass M -Test (Weierstrass M -Test)

- A fundamental and very useful convergence test for determining uniform convergence of a series of functions is the Weierstrass M -test. Although this test is not the most general, it will suffice to establish most of the important results of the chapter.

i Theorem 6.20: (The Weierstrass M -test, Weierstrass M -Test)

Let $\sum_{n=1}^{\infty} f_n(z)$ be a given series of functions on a set S . If there exists a convergent positive numerical series $\sum_{n=1}^{\infty} r_n$, and a natural number N such that for every $n > N$,

$$\|f_n\|_S \leq r_n$$

then the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly and absolutely on the set S .



- ✔ **Proof:** Let $\varepsilon > 0$. Then, by Cauchy's convergence test for the positive series $\sum_{n=1}^{\infty} r_n$, there exists N such that for every $n > m > N$, we have $\sum_{k=m+1}^n r_k < \varepsilon$. Therefore,

$$\begin{aligned} \|S_n - S_m\|_S &= \|f_{m+1} + f_{m+2} + \cdots + f_n\|_S \\ &\leq \|f_{m+1}\|_S + \|f_{m+2}\|_S + \cdots + \|f_n\|_S \\ &\leq r_{m+1} + r_{m+2} + \cdots + r_n < \varepsilon \end{aligned}$$

In passing from the first line to the second, we used assertion 6.13. ■

- ◆ **Example 5:** The series of functions $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges uniformly on the closed unit disk $|z| \leq 1$. This follows immediately from the Weierstrass M -test. The series is bounded by the numerical series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is known to converge by the p -test (Theorem 6.8). Note that the Weierstrass M -test only determines whether the series converges uniformly, but does not provide a formula for its sum. At this stage, the sum function of the series is not known to us.

- ◆ **Exercise 6.1:** Test the series $\sum_{n=1}^{\infty} \frac{1}{9^n} \cos(nz)$ for uniform convergence on the entire complex plane $\mathbb{C}$.

- ◆ **Exercise 6.2:** Prove that if $\sum_{n=1}^{\infty} f_n(z)$ and $\sum_{n=1}^{\infty} g_n(z)$ are two series converging uniformly on a set S , then the series $\sum_{n=1}^{\infty} c f_n(z)$, $\sum_{n=1}^{\infty} [f_n(z) \pm g_n(z)]$ also converge uniformly on the set S .

- i** **Theorem 6.21:** Let $\sum_{n=1}^{\infty} f_n(z)$ be a series of continuous functions on a domain D that converges uniformly to a function $f(z)$ on the domain D . Then

(a) The sum function $f(z)$ is continuous on the domain D .



(b) If, in addition, the functions $f_n(z)$ are analytic on the domain D , then the sum function $f(z)$ is also analytic on the domain D .

(c) For every piecewise smooth path $\sigma: [a, b] \rightarrow D$ in the domain D ,

$$\int_{\sigma} f(z) dz = \int_{\sigma} \sum_{n=1}^{\infty} f_n(z) dz = \sum_{n=1}^{\infty} \int_{\sigma} f_n(z) dz$$

✓ **Proof:** Pass to the sequence of partial sums

$$S_n(z) = \sum_{k=0}^n f_k(z)$$

The theorem follows easily from the corresponding theorems for a sequence of functions. [6.16](#), [6.17](#), [6.18](#). As an example, we prove part C: By assumption, the partial sums $\{S_n(z)\}_{n=1}^{\infty}$ are continuous functions on the domain D , and therefore, by Theorem [6.17](#),

$$\begin{aligned} \int_{\sigma} f(z) dz &= \int_{\sigma} \left[\lim_{n \rightarrow \infty} \{S_n(z)\}_{n=1}^{\infty} \right] dz = \lim_{n \rightarrow \infty} \int_{\sigma} S_n(z) dz \\ &= \lim_{n \rightarrow \infty} \int_{\sigma} \left[\sum_{k=0}^n f_k(z) \right] dz = \lim_{n \rightarrow \infty} \sum_{k=0}^n \int_{\sigma} f_k(z) dz = \sum_{n=0}^{\infty} \int_{\sigma} f_n(z) dz \end{aligned}$$

The conclusion is simply a translation from the language of sequences to the language of series.

■

◆ **Example 6:** We return to the p -series encountered earlier. If z is a free complex variable, then the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is a series of functions $f_n(z) = \frac{1}{n^z}$, defined on all of $\mathbb{C}$. We prove that this series converges absolutely and uniformly on every closed and bounded subset of the domain

$$D = \{z \in \mathbb{C} : \operatorname{Re}(z) > 1\}$$

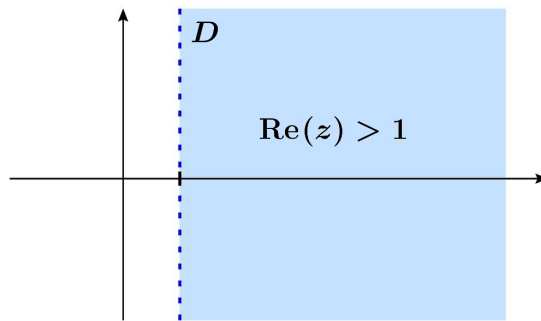


Fig. 6.3: The domain of absolute convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$

Let K be a closed and bounded subset of D . By the Weierstrass theorem, it can be proved

that there exists a real number $a > 1$ such that for every $z \in K$, $z \geq a^1$. In the preceding example we saw that

$$\left| \frac{1}{n^z} \right| = \frac{1}{n^{\operatorname{Re}(z)}} \leq \frac{1}{n^a}$$

Set $M_n = \frac{1}{n^a}$. Then the series $\sum_{n=1}^{\infty} M_n$ satisfies the hypotheses of the Weierstrass M -test with respect to the series of functions $\sum_{n=1}^{\infty} \frac{1}{n^z}$, and therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on the set K . The function $f_n(z) = \frac{1}{n^z}$ is analytic on the domain D , and therefore, by Theorem 6.21, the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges to an analytic function (throughout D). The sum function is known as the **Riemann zeta function**.

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

The famous Riemann hypothesis states:



The Riemann Hypothesis

which still awaits a solution (since 1859). Riemann discovered an analytic continuation of the function $\zeta(z)$ to the entire complex plane, and his hypothesis concerns the extended function.

❖ **Exercise 6.3:** Prove that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges to an analytic function in the open unit disk $\mathbb{D}(0, 1)$.

6.4 Power Series

➤ Power series are a particular kind of series of functions in which the general term has the form

$$f_n(z) = a_n(z - z_0)^n$$

❗ **Definition 6.9:** A series of functions of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + \cdots + a_n(z - z_0)^n + \cdots \quad (6.6)$$

is called a **power series**. The point z_0 is called the **center of the series**, and the constants a_n are called the **coefficients of the series**.

➤ In many examples, the center of the series is $z_0 = 0$. In this case, the power series has the

¹ The function $g: K \rightarrow \mathbb{R}$, $g(z) = \operatorname{Re}(z)$ is continuous and therefore attains a minimum a at a point $z \in K$. a must satisfy $a > 1$, for otherwise $\operatorname{Re}(z) = 1$, contrary to the assumption.



simpler form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots \quad (6.7)$$

Such a series is called a **Maclaurin series**, after the Scottish mathematician Colin Maclaurin, who studied it. In fact, every power series of the general form (6.6) can be converted to a simple power series of the form (6.7) by the substitution $t = z - z_0$:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n t^n$$

It would therefore have been enough to study only power series of the form (6.7).

❖ **Example 7:** Every polynomial is in fact a power series all but finitely many of whose coefficients vanish.

$$-7z^2 + 14iz - (3 + 5i) = -(3 + 5i) + 14iz - 7z^2 + 0z^3 + 0z^4 + 0z^5 + \cdots$$

Thus a power series is an infinite-dimensional generalization of the notion of a polynomial, and indeed, in some contexts, a power series is sometimes called an "infinite polynomial".

❖ **Example 8:** The best-known power series is the geometric series

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$$

The center of the series is $z_0 = 0$, and all its coefficients equal 1 (that is, for every n , $a_n = 1$). Recall that its domain of convergence is the open unit disk $|z| < 1$.

❖ The Scottish mathematician Colin Maclaurin (Colin Maclaurin 1698-1746) used this term in his search for practical methods of computing real transcendental functions such as $\sin x$, $\cos x$, $\log x$, a^x . His basic idea was to find some way to "convert" these functions into polynomials, thereby making them easier to compute (a polynomial requires only multiplication and addition, operations with which every child is thoroughly familiar from the first years of elementary school). But of course he understood very well that there is no finite-degree polynomial $P(x)$ satisfying $\sin x = P(x)$ for every real x . The reason is very simple: every polynomial of degree n has at most n roots, whereas the function $\sin x$, for example, has infinitely many roots.

❖ **Exercise 6.4:** Prove that there is no finite-degree polynomial $P(z)$ identical to the function $\sin z$ on any domain $D \subseteq \mathbb{C}$ (that is nonempty).

Hint: Assume that $\sin z = P(z)$ for every z in the domain D , differentiate both sides of the equality $n + 1$ times (where n is the degree of the polynomial), and derive a contradiction.

❖ The research efforts of Colin Maclaurin (and Brook Taylor) eventually led to the discovery of power



series, which in a certain sense are polynomials of infinite degree. Colin Maclaurin eventually succeeded in proving that all the transcendental functions known in his time can be expressed by means of power series. Figure 6.4 presents the known Maclaurin series for elementary real functions familiar from Calculus 1 courses.

e^x	$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
$\sin x$	$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
$\cos x$	$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
$\arctan x$	$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$
$\ln(1+x)$	$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$
$(1+x)^\alpha$	$= \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$	$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots, (\alpha \in \mathbb{R})$
$\frac{1}{1-x}$	$= \sum_{n=0}^{\infty} x^n$	$= 1 + x + x^2 + x^3 + \dots$
$\frac{1}{1+x}$	$= \sum_{n=0}^{\infty} (-1)^n x^n$	$= 1 - x + x^2 - x^3 + \dots$

Fig. 6.4: Table of Maclaurin series for elementary (real) functions

- Later in the chapter we shall prove that the formulas in Table 6.4 remain valid for the complex versions of the functions in the left-hand column of the table.

Domain of Convergence of a Power Series

- Clearly, every power series $\sum_{n=0}^{\infty} a_n z^n$ converges at the point $z = 0$.
- The faster the growth rate of the coefficient sequence a_n , the smaller the domain of convergence of the series.
- For example, the power series $\sum_{n=0}^{\infty} n^n z^n$ converges only at the point $z = 0$! (because $n^n z^n = (nz)^n$ diverges if $z \neq 0$).
- In contrast, the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for every complex z .

i Theorem 6.22: (Elementary auxiliary theorem)

If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at a point $z_1 \neq z_0$, then it converges absolutely in the open disk $\mathbb{D}(z_0, r)$, where $r = |z_1 - z_0|$, and it converges uniformly and absolutely on every closed disk $\overline{\mathbb{D}}(z_0, q)$, for $q < r$.

Remarks:

1. Note that the assertion does not require the series to converge at additional points on the circle $|z - z_0| = r$ (apart from the point z_1 , at which the series converges by assumption).
2. The assertion requires absolute convergence at every point z whose distance from the center z_0 is less than r , and uniform convergence on every closed subdisk $\mathbb{D}(z_0, q)$, for every $q < r$.
3. This assertion will be used at several important points.

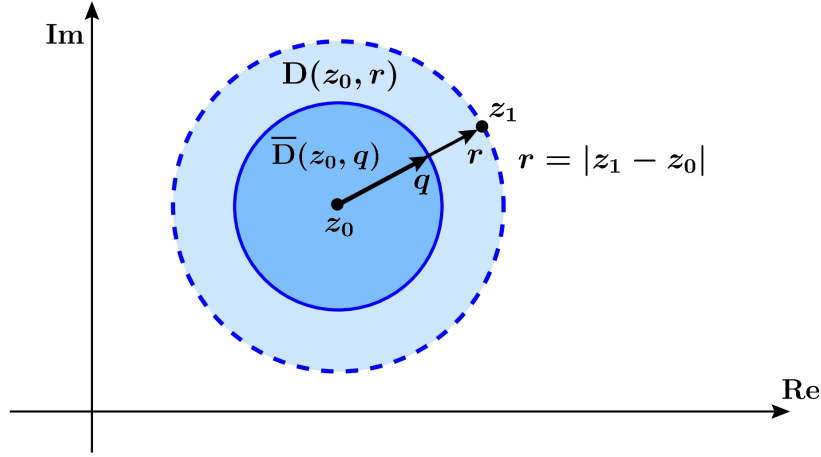


Fig. 6.5: If a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges at a point z , then it converges absolutely

in the open disk $D(z_0, r)$, $r = |z - z_0|$, and converges absolutely and uniformly on every closed disk $\overline{D}(z_0, q)$, $q < r$

✓ **Proof:** Let z be an interior point of the open disk $D(z_0, r)$, and set $q = |z - z_0|$. By assumption, the series $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges. Therefore the sequence $|a_n(z_1 - z_0)^n| = |a_n|r^n$ tends to zero, and hence the sequence $\{|a_n|r^n\}_{n=1}^{\infty}$ is bounded by a real number K . Now

$$|a_n(z - z_0)^n| = |a_n q^n| = \left| a_n r^n \cdot \frac{q^n}{r^n} \right| = |a_n r^n| \cdot \left(\frac{q}{r} \right)^n \leq K \cdot \left(\frac{q}{r} \right)^n$$

The series $\sum_{n=0}^{\infty} K \left(\frac{q}{r} \right)^n$ is a convergent geometric series ($\frac{q}{r} < 1$) that bounds the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. Substitute $M_n = K \cdot \left(\frac{q}{r} \right)^n$ in the Weierstrass M -test to obtain that the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely and uniformly on the closed disk $\overline{D}(z_0, r)$. ■

❗ **Corollary 6.23:** If the power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ diverges at a point z_1 , then it diverges at every point outside the disk $D(z_0, r)$, where $r = |z_1 - z_0|$.

➤ Theorem 6.22 and the last corollary form the basis for defining the **radius of convergence** of a power series: $\sum_{n=0}^{\infty} a_n(z - z_0)^n$

$$R = \sup \left\{ |z - z_0| : \text{converges } \sum_{n=0}^{\infty} a_n(z - z_0)^n \right\}$$

The number R exists because the series converges for $z = z_0$, and it may also be ∞ (if the series converges on the entire complex plane). The following theorem summarizes the situation so far.

❗ **Theorem 6.24:** Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a given power series, and let R be its radius of convergence. The following are all the possible cases:

A. $R = 0$:

The series converges only at the point z_0 .

B. $0 < R < \infty$:

- ◆ The series converges absolutely in the open disk $D(z_0, R)$.
- ◆ The series converges absolutely and uniformly on every closed disk $\overline{D}(z_0, r)$, $r < R$.
- ◆ The series diverges in the plane with the closed disk removed, $\mathbb{C} - \overline{D}(z_0, R)$.
- ◆ At points on the circle $|z - z_0| = R$, the series may converge at some points and diverge at others. Nothing can be determined in advance; it depends on the particular series.

C. $R = \infty$:

The series converges absolutely on the entire complex plane $\mathbb{C}$, and converges uniformly on every bounded set.

- In view of the last theorem, methods for computing the radius of convergence of a given series are of great importance. We present two such methods, which will suffice for most of our needs in the course.

i Theorem 6.25: (The Cauchy-Hadamard formula Cauchy-Hadamard)

Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ be a given power series whose coefficient sequence $\{a_n\}_{n=1}^{\infty}$ has a limit (in the extended sense):

$$Q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

A. If $Q = 0$, then the radius of convergence of the series is $R = \infty$.

B. If $Q = \infty$, then the radius of convergence of the series is $R = 0$.

C. If $0 < Q < \infty$, then the radius of convergence is $R = \frac{1}{Q}$.



- ✓ **Proof:** A detailed proof can be found at the following link (Appendix C at the end of the notes):



Calculus 2 Lecture Notes

- The theorem can be summarized briefly by the following formula (**the Cauchy-Hadamard formula**):

$$R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}} \quad (6.8)$$

In most of the cases we encounter, the limit $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ exists, and therefore there is no need to compute a limit superior.



i Theorem 6.26: (The ratio formula for computing the radius of convergence)

Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ be a given power series whose coefficient sequence $\{a_n\}_{n=1}^{\infty}$ has a limit (in the extended sense):

$$Q = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$$

A. If $Q = 0$, then the radius of convergence of the series is $R = \infty$.

B. If $Q = \infty$, then the radius of convergence of the series is $R = 0$.

C. If $0 < Q < \infty$, then the radius of convergence is $R = \frac{1}{Q}$.

➤ The theorem can be summarized briefly by the following formula (the ratio formula for computing the radius of convergence):

$$R = \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} \quad (6.9)$$

➤ We note that Hadamard's formula has an advantage over the ratio formula in that it is intended to give the exact radius of convergence for every possible series, whereas the ratio formula depends on the existence of the limit. The only problem with Hadamard's formula is that the limit it requires is not always easy to compute.

◆ **Example 9:** The radius of convergence of the alternating geometric series

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

is $R = 1$. The sum-function formula is obtained easily by substituting $-z$ in the ordinary geometric series (see Example 8). The radius of convergence follows immediately from Hadamard's formula (6.8), and also by using the ratio formula (6.9). By Theorem 6.24, the series converges absolutely in the open disk $\overline{D}(0, 1)$, and converges absolutely and uniformly on every closed disk $\overline{D}(0, r)$, $r < 1$. We prove that the series does not converge uniformly in the open disk $\overline{D}(0, 1)$. As in Example 8, it is easy to obtain the partial-sum formula

$$S_n(z) = \sum_{k=0}^n (-1)^k z^k = \frac{1 - (-1)^{n+1} z^{n+1}}{1 + z}$$

and therefore

$$\left| \frac{1}{1+z} - \frac{1 - (-1)^{n+1} z^{n+1}}{1+z} \right| = \left| \frac{(-1)^{n+1} z^{n+1}}{1+z} \right| = \left| \frac{z^{n+1}}{1+z} \right|$$

It is easy to verify that

$$\sup_{|z|<1} \left| \frac{z^n}{1+z} \right| = \sup_{|z|<1} \frac{|z|^n}{|1+z|} = \infty$$

For example, as z approaches the point -1 from the right, the denominator $|1+z|$ tends to zero, while the numerator $|z|^n$ tends to 1 . Therefore the quotient on the right-hand side tends to the form $\frac{1}{0} = \infty$. Thus the series does not converge uniformly in the open disk $D(0, 1)$.

- If we set $f(z) = \sum_{n=1}^{\infty} (-1)^n z^n$, then we may write $f(z) = \frac{1}{1+z}$ for every $|z| < 1$. The series diverges for $|z| \geq 1$, so the domain of definition of $f(z)$ is only $|z| < 1$. The domain of definition of the function $\frac{1}{1+z}$, on the other hand, is much larger, $\mathbb{C} - \{-1\}$: the entire complex plane except for the point $z = -1$.
- The gap between the domain of convergence of the series and the domain of definition of the target function it is supposed to compute, $\frac{1}{1+z}$, creates a sense of missed opportunity, as though the series does not fulfill the purpose for which it was intended: The power series of $\frac{1}{1+z}$ does its job only for values $|z| < 1$, which form a very small part of the domain of definition of $\frac{1}{1+z}$. As we shall learn later, this is a recurring pattern: every singular point (in our case $z = -1$) of the target function is a barrier to the radius of convergence of the series.
- Because of the singular point $z = -1$, we cannot develop a power series for $\frac{1}{1+z}$ that converges throughout the domain of definition of the function, but power series with a larger radius of convergence can be developed by choosing a center farther from the singular point $z = -1$. Here is a method for constructing a power series for the function $\frac{1}{1+z}$ centered at $z_0 = 5$ with radius of convergence $R = 6$:

$$\begin{aligned} \frac{1}{1+z} &= \frac{1}{6+(z-5)} = \frac{1}{6} \cdot \frac{1}{1+\frac{1}{6}(z-5)} \\ &= \frac{1}{6} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{6^n} (z-5)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} (z-5)^n \end{aligned}$$

It is easy to verify from Hadamard's formula that the radius of convergence of this series is $R = 6$.

- The function $\frac{1}{1+z}$ is called an **analytic continuation** of the function $\sum_{n=1}^{\infty} (-1)^n z^n$, or of the function $\sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} (z-5)^n$, because it is identical to both of them on their domains of convergence, but is defined on a much larger domain.

❖ **Exercise 6.5:** Find the radius and domain of convergence of the following series:

$$(a) \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (b) \sum_{n=0}^{\infty} (3^n - 2^n) z^n \qquad (c) \sum_{n=0}^{\infty} (-2)^n \frac{(z-5)^n}{n^4}$$

Hint: (a) Directly from the ratio test, we obtain $R = \infty$. (b) Hint: $3^n - 2^n = 2^n \left[\left(\frac{3}{2}\right)^n - 1 \right]$.



❖ **Exercise 6.6:** Find an example of a power series $\sum_{n=0}^{\infty} a_n z^n$ for which the limit

$$R = \lim_{n \rightarrow \infty} \frac{1}{n \sqrt[n]{|a_n|}}$$

exists, but the limit

$$R = \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|}$$

Hint: does not. Try combining 2^n and 3^n .

❖ **Exercise 6.7:** If the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is R , what can you say about the radius of convergence of the series $\sum_{n=0}^{\infty} n a_n z^n$?

Hint: Use Hadamard's formula to verify that the radius of convergence is the same.

Differentiation and Integration of a Power Series

➤ Given a power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converging to a function $f(z)$, we wish to know under what conditions $f(z)$ is differentiable, and whether both sides can be differentiated while retaining an equality between $f'(z)$ and the sum of the derivatives of the terms of the series. Similar questions arise in connection with integration. To answer these questions, we need the following topological auxiliary assertion.

❗ **Proposition 6.27:** Let z_0 be a given point in the complex plane, and let $R > 0$ be a positive real number. If S is a closed subset of the open disk $\mathbb{D}(z_0, R)$, then there exists a positive real number $r < R$ such that $S \subseteq \mathbb{D}(z_0, r)$.

✓ **Proof:** Assume to the contrary that no such r exists. Then, for every natural number n , there exists $z_n \in S$ satisfying $R - \frac{1}{n} \leq |z_n - z_0| < R$. The sequence $\{z_n\}_{n=1}^{\infty}$ is a closed and bounded set, and therefore, by the Bolzano-Weierstrass theorem 2.18, it has a convergent subsequence $\{z_{n_k}\}_{n=1}^{\infty}$ whose limit $w = \lim_{k \rightarrow \infty} z_{n_k}$ satisfies $|w - z_0| \geq R - \frac{1}{n_k}$ for every natural number k . This is of course possible only if $|w - z_0| \geq R$. Since S is closed, $w \in S \subseteq \mathbb{D}(z_0, R)$, and hence $|w - z_0| < R$. We have obtained a logical contradiction, so the assertion of the theorem must be true. ■

❗ **Corollary 6.28:** Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with radius of convergence $R > 0$. Then the series converges absolutely and uniformly on every closed subset $S \subseteq \mathbb{D}(z_0, R)$.

✓ **Proof:** By the preceding assertion, there exists $r < R$ such that $S \subseteq \mathbb{D}(z_0, r)$. By Theorem 6.24, the series converges absolutely and uniformly on $\overline{\mathbb{D}(z_0, r)}$, and hence it also converges



absolutely and uniformly on S . ■

i Theorem 6.29: (Integration of a power series)

Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series with radius of convergence R , and let $\sigma: [a, b] \rightarrow \mathbb{C}$ be a piecewise smooth path in the disk $\mathbb{D}(z_0, R)$. Then

$$\int_{\sigma} f(z) dz = \int_{\sigma} \left[\sum_{n=0}^{\infty} a_n(z - z_0)^n \right] dz = \sum_{n=0}^{\infty} \int_{\sigma} a_n(z - z_0)^n dz$$

✓ Proof: Without loss of generality, we may assume that σ is a smooth path. Since σ is a continuous function from the real interval $[a, b]$ to $\mathbb{D}(z_0, R)$, it follows that the image $\sigma([a, b])$ is a closed subset of $\mathbb{D}(z_0, R)$ (see Theorem 3.13 on page 100). By assertion 6.27, there exists $r < R$ such that $\sigma([a, b]) \subseteq \mathbb{D}(z_0, r)$. By Theorem 6.24, the series converges absolutely and uniformly on $\overline{\mathbb{D}(z_0, r)}$, and therefore, by Theorem 6.29, the operations of integration and summation may be interchanged. ■

i Theorem 6.30: Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series with radius of convergence R that converges to the sum function $f(z)$. Then f is analytic in the open disk $\mathbb{D}(z_0, R)$.

✓ Proof: By Theorem 6.24, the series converges absolutely and uniformly on every disk $\mathbb{D}(z_0, r)$, $r < R$. Therefore, by Theorem 6.21, $f(z)$ is analytic in every disk $\mathbb{D}(z_0, r)$, $r < R$. Thus $f(z)$ is analytic in the open disk $\mathbb{D}(z_0, R)$. ■

i Theorem 6.31: (Differentiation of a power series)

Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series with radius of convergence R . Then, for every point $|z| < R$,

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

Moreover, the two series $\sum_{n=0}^{\infty} a_n z^n$, $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence R .

✓ Proof: It is easy to verify using Hadamard's formula (6.8) that the radius of convergence of the derivative series is also R .

$$\frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|n a_n|}} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{n} \cdot \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}} = \frac{1}{1 \cdot \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}} = R$$

Therefore the series $\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ converges absolutely for $|z - z_0| < R$. Set

$$g(w) = \sum_{n=1}^{\infty} n a_n (w - z_0)^{n-1}$$



and prove that $f'(z) = g(z)$.

Let z be a given point in the domain of convergence $|z - z_0| < R$. Use the path

$$\sigma(t) = z_0 + t(z - z_0), \quad 0 \leq t \leq 1$$

(which is the line segment joining z_0 to z). Clearly $\sigma'(t) = z - z_0$. By Theorem [6.29](#),

$$\begin{aligned} \int_{z_0}^z g(w) dw &= \int_{z_0}^z \left[\sum_{n=1}^{\infty} n a_n (w - z_0)^{n-1} \right] dw = \sum_{n=1}^{\infty} a_n \int_{z_0}^z n (w - z_0)^{n-1} dw \\ &= \sum_{n=1}^{\infty} a_n \int_0^1 n (t(z - z_0))^{n-1} \cdot (z - z_0) dt = \sum_{n=1}^{\infty} a_n (z - z_0)^n \int_0^1 n t^{n-1} dt \\ &= \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} \cdot t^n \Big|_0^1 = \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1} = f(z) - a_0 \end{aligned}$$

By Corollary [5.13](#) (page 185), the function $\int_{z_0}^z g(w)$ is an antiderivative of $g(z)$. Therefore the series $f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$ is an antiderivative of the series $\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$. ■

◆ **Example 10:** Compute the sums of the series $\sum_{n=0}^{\infty} n z^n$, $\sum_{n=0}^{\infty} n^2 z^n$.

🌟 **Solution:** Use the sum formula for the geometric series:

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$$

Differentiate both sides to obtain

$$\frac{1}{(1 - z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \tag{6.10}$$

Multiply both sides by z to obtain

$$\sum_{n=0}^{\infty} n z^n = \frac{z}{(1 - z)^2}$$

Differentiating [\(6.10\)](#) once more gives

$$\frac{2}{(1 - z)^3} = \sum_{n=2}^{\infty} n(n - 1) z^{n-2}$$



Therefore,

$$\begin{aligned}
 \sum_{n=0}^{\infty} n^2 z^n &= \sum_{n=0}^{\infty} n(n-1)z^n + \sum_{n=0}^{\infty} n z^n \\
 &= z^2 \cdot \sum_{n=0}^{\infty} n(n-1)z^{n-2} + \sum_{n=0}^{\infty} n z^n \\
 &= z^2 \cdot \frac{2}{(1-z)^3} + \frac{z}{(1-z)^2} = \frac{2z^2}{(1-z)^3} + \frac{z}{(1-z)^2} \\
 &= \frac{z^2+z}{(1-z)^3}
 \end{aligned}$$

- The last theorem allows us to differentiate a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ as many times as we wish,

$$\begin{aligned}
 f(z) &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + a_4(z-z_0)^4 + \dots \\
 f'(z) &= a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + 4a_4(z-z_0)^3 + \dots + na_n(z-z_0)^{n-1} + \dots \\
 f''(z) &= 2a_2 + 3 \cdot 2a_3(z-z_0) + 4 \cdot 3a_4(z-z_0)^2 + \dots + n(n-1)a_n(z-z_0)^{n-2} + \dots \\
 f'''(z) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(z-z_0) + \dots + n(n-1)(n-2)a_n(z-z_0)^{n-3} + \dots
 \end{aligned}$$

where, in all these series, the radius of convergence remains the same as that of the original series.

- At the point z_0 (the center of the series), we obtain the following equalities:

$$\begin{aligned}
 f(z_0) &= a_0 \\
 f'(z_0) &= a_1 \\
 f''(z_0) &= 2a_2 \\
 f'''(z_0) &= 3 \cdot 2a_3 \\
 \vdots &= \vdots \\
 f^{(k)}(z_0) &= k!a_k
 \end{aligned}$$

- The conclusion from the last calculation is that the coefficients of the series can be expressed in terms of the derivatives $f^{(k)}(z)$ at the point z_0 .

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

i Theorem 6.32: (Higher derivatives of a power series)

Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence R , $R > 0$. Then its sum $f(z)$ has derivatives of every order in the domain $|z| < R$. The derivative of order k of $f(z)$ can be obtained by differentiating the series term by term k times. In addition, the



following relation holds between the derivative of order k and the coefficient a_k of the series:

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

6.5 Expansion of a Function as a Power Series (Taylor)

i Theorem 6.33: (Uniqueness of a power series)

If two power series

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ g(z) &= \sum_{n=0}^{\infty} b_n (z - z_0)^n \end{aligned}$$

satisfy $f(z) = g(z)$ for every z in an open disk $|z - z_0| < r$, for $r > 0$, then, for every natural number n , $a_n = b_n$.

✓ Proof: From the assumption $r > 0$, we conclude that the radius of convergence of both series is positive, and therefore, by the preceding theorem, for every natural number k ,

$$\begin{aligned} a_k &= \frac{f^{(k)}(z_0)}{k!} \\ b_k &= \frac{g^{(k)}(z_0)}{k!} \end{aligned}$$

But for every k , $f^{(k)}(z_0) = g^{(k)}(z_0)$, and we are done. ■

- The meaning of the preceding theorem is that if a function $f(z)$ can be represented at all by a power series centered at a point z_0 , then this can be done in only one way!
- The question that interested mathematicians such as Taylor and Maclaurin was this: Given a function, under what conditions does a power series for the function exist, and what is its domain of convergence? Taylor and Maclaurin dealt with real functions, but most of their work applies to complex functions as well.

i Definition 6.10: Let $f(z)$ be a function defined around a point z_0 . That is, there exists $r > 0$ (including the possibility that $r = \infty$) such that the function $f(z)$ is defined in the neighborhood $\mathbb{D}(z_0, r)$. We say that $f(z)$ can be expanded as a power series in the disk $\mathbb{D}(z_0, r)$ if there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ such that

$$\forall z \in \mathbb{D}(z_0, r) : f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

- ⤵ Unlike the real case, in which a function may be infinitely differentiable on an open domain but have no Taylor series, in the complex case, every analytic function has a Taylor series in an open disk around each point at which it is analytic.

i Theorem 6.34: (Taylor's formula) Let $f(z)$ be analytic in the open disk $\mathbb{D}(z_0, r)$ centered at the point z_0 . Then, for every z in the disk,

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$



- ⤵ That is, $f(z)$ can be expanded as a power series centered at z_0 , whose coefficients are $a_n = \frac{f^{(n)}(z_0)}{n!}$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad (6.11)$$

- ⤵ The series (6.11) is called **the Taylor series of f centered at the point z_0** ! In many problems, choosing the right point about which to expand the series is the key to solving the problem.
- ⤵ That is, the same function has different Taylor series, depending on the point z_0 about which the series is expanded.
- ✓ **Proof:** The proof of Taylor's theorem is based on Cauchy's integral formula. Choose a point $z_1 \in \mathbb{D}(z_0, r)$, and let $r_1 = |z_1 - z_0|$. To apply Cauchy's formula at the point z_1 , choose an intermediate radius r_2 , $r_1 < r_2 < r$. Let $\sigma(t) = z_0 + r_2 e^{it}$, $0 \leq t \leq 2\pi$, be a parametrization of the circle $|z - z_0| = r_2$. By Cauchy's integral formula (Theorem 5.19 on page 192),

$$f(z_1) = \frac{1}{2\pi i} \oint_{\sigma} \frac{f(z)}{z - z_1} dz \quad (6.12)$$

See Figure 6.6. Use the technique encountered in Example 9 to express the quotient $\frac{1}{z - z_1}$ as the sum of a geometric series:

$$\frac{1}{z - z_1} = \frac{1}{(z - z_0) - (z_1 - z_0)} = \frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{z_1 - z_0}{z - z_0}} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z_1 - z_0}{z - z_0} \right)^n$$

In the last step, we used the geometric-series formula $\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$, where

$$q = \frac{z_1 - z_0}{z - z_0}$$



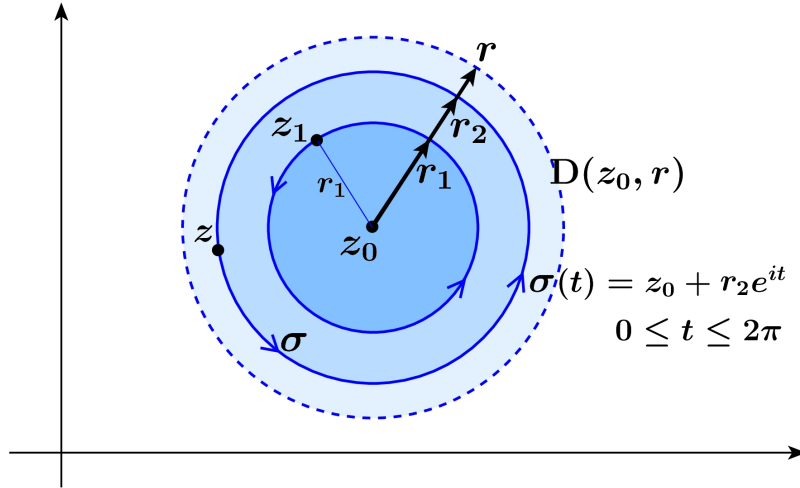


Fig. 6.6: Using Cauchy's integral formula to prove Taylor's formula centered at the point z_0

Note that

$$|q| = \left| \frac{z_1 - z_0}{z - z_0} \right| = \frac{|z_1 - z_0|}{|z - z_0|} = \frac{r_1}{r_2} < 1$$

and therefore the geometric series $\sum_{n=0}^{\infty} \left(\frac{z_1 - z_0}{z - z_0} \right)^n$ converges absolutely and uniformly on the circle $|z - z_0| = r_2$.

Now express the integrand on the right-hand side of (6.12) as a series

$$\frac{f(z)}{z - z_1} = \frac{f(z)}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z_1 - z_0}{z - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{f(z)}{(z - z_0)^{n+1}} (z_1 - z_0)^n$$

The function $|f(z)|$ is continuous on the closed path σ , and therefore has a maximum M . We test the convergence of the above series.

$$\left| \frac{f(z)}{(z - z_0)^{n+1}} (z_1 - z_0)^n \right| = \left| \frac{f(z)}{z - z_0} \right| \cdot \left| \frac{(z - z_0)^n}{(z_1 - z_0)^n} \right| \leq \frac{M}{r_2} \cdot |q|^n$$

The geometric series $\sum_{n=1}^{\infty} \frac{M}{r_2} \cdot |q|^n$ converges because $|q| < 1$, and therefore, by the Weierstrass M -test (Theorem 6.20), our series also converges. By Theorem 6.21, we may integrate term by term along the path σ .

$$\begin{aligned} f(z_1) &= \frac{1}{2\pi i} \oint_{\sigma} \frac{f(z)}{z - z_1} dz = \frac{1}{2\pi i} \oint_{\sigma} \sum_{n=0}^{\infty} \frac{f(z)}{(z - z_0)^{n+1}} (z_1 - z_0)^n dz \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{\sigma} \frac{f(z)}{(z - z_0)^{n+1}} dz \right] (z_1 - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z_1 - z_0)^n \end{aligned}$$

In the last step, we used Cauchy's derivative formula (see formula (5.21) on page (5.21)). Re-

placing the variable z_1 by the variable z , we obtain Taylor's formula in its standard form.

■

- One conclusion that can be drawn from the proof of Taylor's theorem is a method for computing the radius of convergence of a Taylor series.

i Corollary 6.35: (Radius of convergence of a Taylor series) Let $f(z)$ be analytic in a domain D . Then, for every point $z_0 \in D$, the radius of convergence of the Taylor series of $f(z)$ centered at z_0 is the largest value of r for which $\mathbb{D}(z_0, r) \subseteq D$.

- ◆ This is clear, because this disk satisfies all the hypotheses of Taylor's theorem.
- ◆ If the function $f(z)$ can be extended to a larger domain in which it is analytic, then the radius increases accordingly.

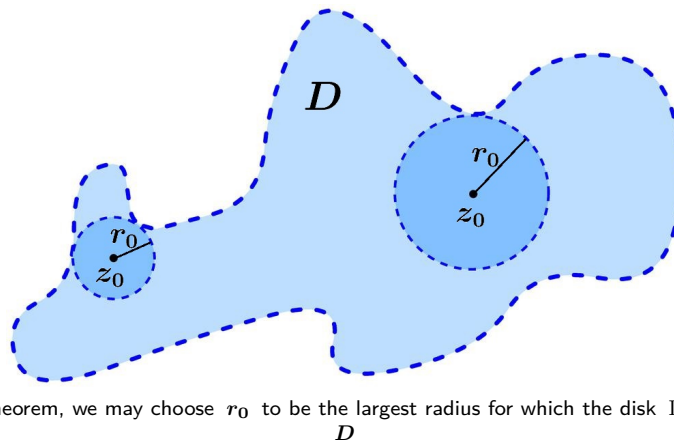


Fig. 6.7: In the proof of Taylor's theorem, we may choose r_0 to be the largest radius for which the disk $\mathbb{D}(z_0, r_0)$ is contained in the domain D .

- ◆ **Example 11:** A *Maclaurin series* is a Taylor series centered at $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad (6.13)$$

The formula is of course valid for functions analytic in a neighborhood of $z_0 = 0$. Table 6.8 presents the Maclaurin series of several elementary functions. Their expansions are identical to those in the real case, and it is advisable to try several of them as a refresher. Exercise: What is the domain of convergence of the series?

- ◆ **Example 12:** Using formula (6.13) to compute a Maclaurin series is not always the shortest method. In many cases, one Maclaurin series can be found by differentiating or integrating another Maclaurin series. For example, compute the Maclaurin series of the principal branch of the logarithm function $f(z) = \text{Log}(1 + z)$ by integrating the alternating geometric series.

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

e^z	$= \sum_{n=0}^{\infty} \frac{z^n}{n!}$	$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$
$\sin z$	$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$	$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$
$\cos z$	$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$	$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$
$\arctan z$	$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}$	$= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \dots$
$\text{Log}(1+z)$	$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$	$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$
$(1+z)^\alpha$	$= \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$	$= 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \dots, (\alpha \in \mathbb{C})$
$\frac{1}{1-z}$	$= \sum_{n=0}^{\infty} z^n$	$= 1 + z + z^2 + z^3 + \dots$
$\frac{1}{1+z}$	$= \sum_{n=0}^{\infty} (-1)^n z^n$	$= 1 - z + z^2 - z^3 + \dots$

Fig. 6.8: Table of Maclaurin series for elementary (complex) functions

By Theorem 6.29, we may integrate $\int_0^z$ term by term.

$$\text{Log}(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$$

The radius of convergence of the series is $R = 1$, because the logarithm function does not exist on the negative real axis, and therefore the largest disk centered at $z = 0$ contained in the domain of analyticity of $\text{Log}(1+z)$ is the unit disk.

◆ **Example 13:** Without computing the Taylor series of the function $f(z) = \frac{1}{e^z + i}$, we can determine its radius of convergence about the point $z_0 = 1 + \pi i$. The domain of f is the entire complex plane except for the two points that make the denominator $e^z + i = 0$ vanish. Substituting $z = x + iy$ in the last equality gives $e^x e^{yi} = -1$, and hence $x = 0$, $y_1 = -\frac{\pi}{2}$, $y_2 = \frac{3\pi}{2}$. As can be seen in the diagram on the left, the radius of the largest disk centered at $z_0 = 1 + \pi i$ that does not contain either of these points is $R = \sqrt{1 + \frac{\pi^2}{4}}$.

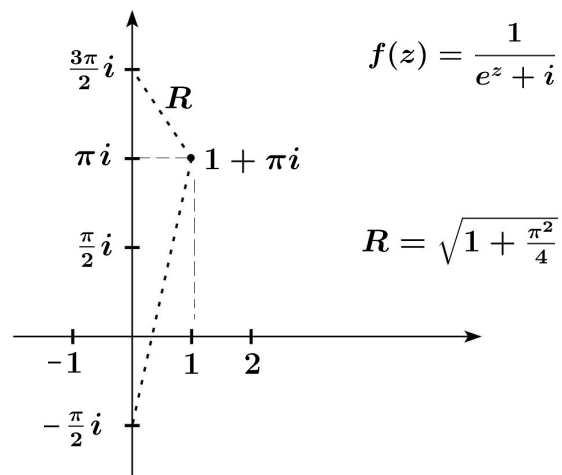


Fig. 6.9: The radius of convergence of the Taylor series centered at $z_0 = 1 + \pi i$ is determined by the shortest distance between z_0 and the points at which f is not analytic

◆ **Exercise 6.8:** Let $f(z)$ be analytic in a domain D containing the open unit disk $|z| < 1$. Suppose that for every point $|z| < 1$, $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^2}$. Prove that at least one point on the circle $|z| = 1$ does not belong to the domain D .

Hint: If $\overline{\mathbb{D}}(0, 1) \subseteq D$, then there exists $r > 1$ such that $\mathbb{D}(0, r) \subseteq D$. By the uniqueness theorem for series 6.33 and Taylor's theorem 6.34, $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$ must be the Taylor series of f in the disk $\mathbb{D}(0, r)$ as well, but this is impossible because the

radius of convergence of the series is 1! Further details about this example can be found at the following link (including a proof of the existence of an analytic continuation of the series to the slit plane $\mathbb{C} - [1, \infty)$).

singularity of analytic continuation of $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$

- If f is entire (analytic on the entire complex plane $D = \mathbb{C}$), then $r = \infty$. That is, the Taylor series centered at any point z_0 converges on the entire complex plane.
- It should be qualified, however, that if the singular point is removable, then it is not a barrier to the radius of convergence of the Taylor series.

✦ **Exercise 6.9:** Prove: If $f(z)$ is entire, and if $|f(z)| \leq K|z|^n$ for a real K and natural number n , then $f(z)$ must be a polynomial.

🔗 **Solution:** For every real $r > 0$, define

$$M(r) = \text{Sup}\{|f(z)| : |z| = r\}$$

We use the inequality

$$|f^{(n+k)}(0)| \leq \frac{M(r) \cdot (n+k)!}{r^{n+k}}$$

By assumption, $M(r) \leq r^n$, and therefore

$$|f^{(n+k)}(0)| \leq \frac{r^n \cdot (n+k)!}{r^{n+k}} = \frac{(n+k)!}{r^k} \xrightarrow{r \rightarrow \infty} 0$$

It follows that all derivatives of order $n+1$ and higher of f vanish at $z=0$, and hence the Taylor series of f is a polynomial of degree at most n .

✦ **Exercise 6.10:** An alternative proof of Liouville's theorem: Using the preceding exercise, show that if $f(z)$ is entire and bounded, $|f(z)| \leq M$, then f is constant.

Hint: Note that in the solution of the preceding exercise, the Taylor-series coefficients that vanish are a_m , $m > n$, where n is the degree of the polynomial!

6.6 Laurent Series (Pierre Alphonse Laurent 1813-1854)



Fig. 6.10: Colin Maclaurin
1698-1746



Fig. 6.11: Brook Taylor
1685-1731



Fig. 6.12: Pierre Alphonse
Laurent 1813-1854

- Laurent series are named after the French mathematician **Pierre Alphonse Laurent**, who lived from 1813-1854, studied them, and proved the principal theorems concerning them.
- Laurent series are a generalization of Taylor series for functions with singular points or a “hole” in their domain of definition. A Laurent series is a two-sided power series of the form $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, where n may also be negative (but integral). We begin with a simple example.

◆ **Example 14:** Consider the geometric series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, whose domain of convergence is the disk $|z| < 1$. If we replace the variable z by the expression $\frac{1}{z}$, we obtain a series with negative powers $\sum_{n=-\infty}^0 z^n$, which converges in the plane with a hole $|z| > 1$ to the sum function $\frac{z}{z-1}$.

Explanation: If $|z| > 1$ is a point in the plane with a hole, then $|\frac{1}{z}| < 1$ is a point in the open unit disk, and therefore the geometric series $\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n$ converges to the sum

$$\sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

Thus we have obtained the Laurent series of the function $f(z) = \frac{z}{z-1}$ which converges on the plane with a hole $|z| > 1$.

$$\frac{z}{z-1} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} z^{-n}$$

In this case, the series has negative powers only, but it is the key to understanding what a Laurent series is, so we begin with its formal definition.

Definition 6.11:

(a) A **negative-power series** is a series of the form $\sum_{n=1}^{\infty} a_n(z - z_0)^{-n}$ which can also be written in the form $\sum_{n=-1}^{-\infty} a_n(z - z_0)^n$, or in the form $\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$. The point z_0 is called the **center of the series**.

(b) An ordinary power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ will also be called a **positive-power series**.

(c) A **Laurent series** $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ is a combination of a negative-power series $\sum_{n=-\infty}^{-1} a_n(z - z_0)^n$ and a positive-power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. We say that a Laurent series **converges** if and only if both series converge.

i Proposition 6.36: The domain of convergence of a negative-power series $\sum_{n=1}^{\infty} a_n(z - z_0)^{-n}$ is a **plane with a hole** $|z - z_0| > \frac{1}{R}$, where R is the radius of convergence of the positive-power series $\sum_{n=1}^{\infty} a_n(z - z_0)^n$ with the same coefficients.

* There is, of course, the possibility of convergence at some points of the circle $|z - z_0| = \frac{1}{R}$, which is not mentioned in the assertion.

✓ Proof: Begin with the ordinary power series $\sum_{n=1}^{\infty} a_n(z - z_0)^n$. Suppose its radius of convergence is R . In place of the variable z , substitute the expression $z_0 + \frac{1}{z - z_0}$.

$$\sum_{n=1}^{\infty} a_n \left(\left(z_0 + \frac{1}{z - z_0} \right) - z_0 \right)^n = \sum_{n=1}^{\infty} \frac{a_n}{(z - z_0)^n} = \sum_{n=1}^{\infty} a_n(z - z_0)^{-n}$$

We have obtained a negative-power series whose domain of convergence is

$$\left| \left(z_0 + \frac{1}{z - z_0} \right) - z_0 \right| < R \iff \left| \frac{1}{z - z_0} \right| < R \iff \boxed{|z - z_0| > \frac{1}{R}}$$

A **plane with a hole** (the complement of a disk) centered at the point z_0 , with hole radius $\frac{1}{R}$. ■

⦿ To clarify, if $R = 0$, then the negative-power series diverges for every $z \in \mathbb{C}$, and if $R = \infty$, then it converges in the domain $|z - z_0| > 0$ (a punctured plane).

Definition: A **negative-power series** is a series of the form $\sum_{n=1}^{\infty} a_n(z - z_0)^{-n}$ which can also be written in the form $\sum_{n=-1}^{-\infty} a_n(z - z_0)^n$, or in the form $\sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}$. By the preceding subsection, the domain of convergence of such a series is a **plane with a hole** $|z - z_0| > R$ (with the possibility of convergence at some points of the circle $|z - z_0| = R$).

i Corollary 6.37: (Radius of convergence of a negative-power series)

The radius of convergence of the negative-power series $\sum_{n=1}^{\infty} a_n(z - z_0)^{-n}$ is given by Hadamard's formula

$$R = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

or by the ratio formula (if the limit exists):

$$R = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$$

Of course, this is a plane with a hole: $|z - z_0| > R$!

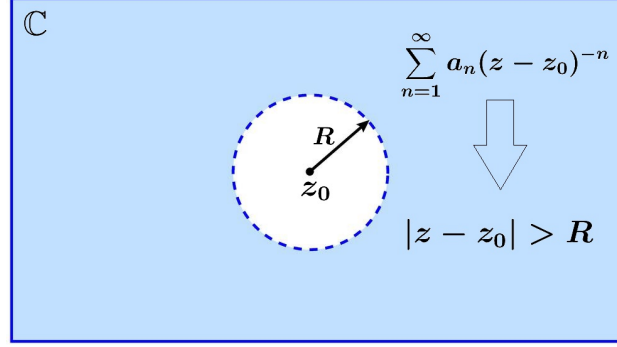


Fig. 6.13: Domain of convergence of a negative-power series $\sum_{n=1}^{\infty} a_n(z - z_0)^{-n}$

- A Laurent series (Laurent) is a combination of a negative-power series and an ordinary Taylor series. A Laurent series can be represented in several different ways. It is easy to verify that all the following forms are equivalent:

$$\begin{aligned}
 (1) \quad & \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \\
 (2) \quad & \sum_{n=1}^{\infty} a_n(z - z_0)^{-n} + \sum_{n=0}^{\infty} b_n(z - z_0)^n \\
 (3) \quad & \sum_{n=1}^{\infty} \frac{a_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} b_n(z - z_0)^n \\
 (4) \quad & \sum_{n=-1}^{-\infty} a_n(z - z_0)^n + \sum_{n=0}^{\infty} a_n(z - z_0)^n
 \end{aligned}$$

In representation (4), the coefficients of the negative powers are a_{-1} , a_{-2} , and so on.

- We say that the Laurent series converges absolutely (or uniformly) on a set S if and only if the two series converge absolutely (or uniformly) on the set S .
- Just as the typical convergence neighborhood of a Taylor series is a disk centered at a point z_0 , the typical convergence neighborhood of a Laurent series is an annulus

$$R_1 < |z - z_0| < R_2$$

obtained by intersecting the disk $|z - z_0| < R_2$ (the domain of convergence of the Taylor series) with the plane with a hole $|z - z_0| > R_1$ (the domain of convergence of the negative-power

series). See Figure 6.14 for a visual illustration.

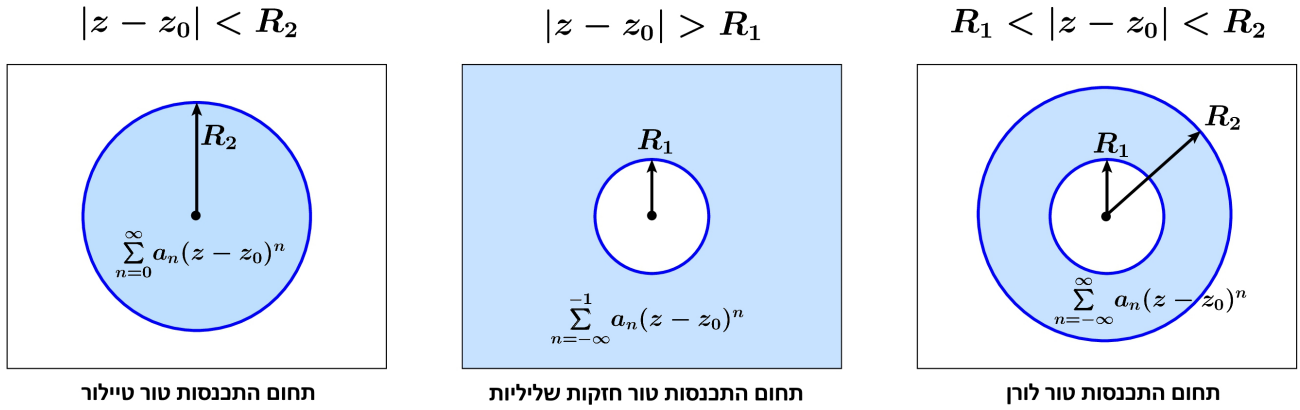


Fig. 6.14: Domains of convergence of a positive-power series, a negative-power series, and a Laurent series

The pair of radii R_1 , R_2 satisfy the following constraint:

$$0 \leq R_1 \leq R_2 \leq \infty$$

- For every Laurent series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, there exist R_1 , R_2 for which the series converges absolutely in the annulus $R_1 < |z - z_0| < R_2$, and diverges for every z satisfying $|z - z_0| < R_1$ or $|z - z_0| > R_2$.
- When $R_2 = \infty$, the domain of convergence is a plane with a hole, and when $R_1 = 0$, we obtain a plane or a punctured circular neighborhood (the point $z = z_0$ is included in the domain only when all the negative coefficients vanish).
- The series may also converge at some points on the boundary of the annulus, $|z - z_0| = R_1$ or $|z - z_0| = R_2$.
- **Uniform convergence:** In addition (as we shall see later), for every $R_1 < r_1 \leq r_2 < R_2$, the series converges uniformly on the annulus $r_1 \leq |z - z_0| \leq r_2$ (see Figure 6.15).

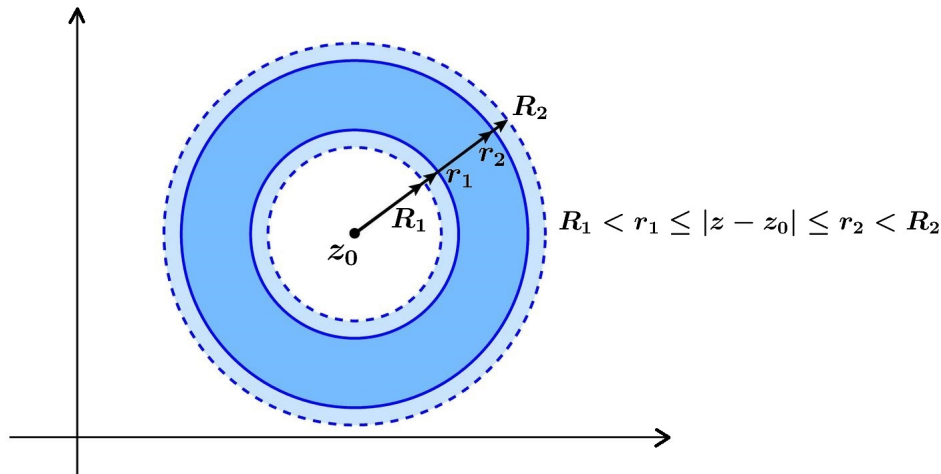


Fig. 6.15: A Laurent series converges uniformly on every closed subannulus $R_1 < r_1 \leq |z - z_0| \leq r_2 < R_2$

◆ **Example 15:** The Laurent expansion of $e^{1/z}$ is obtained easily from the Taylor series of e^z . Recall that the Taylor series of e^z is

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

and its radius of convergence is $R = \infty$. Therefore,

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

and by assertion 6.36, the domain of convergence of the Laurent series is $|z| > \frac{1}{R} = \frac{1}{\infty} = 0$. That is, the Laurent series converges on the punctured plane $\mathbb{C} - \{0\}$.

◆ **Example 16:** Let $a \in \mathbb{C}$, $a \neq 0$. In the domain $|z| < |a|$,

$$\frac{1}{z-a} = -\frac{1}{a} \cdot \frac{1}{1 - \left(\frac{z}{a}\right)} = -\frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n = \sum_{n=0}^{\infty} -\frac{1}{a^{n+1}} z^n$$

In the domain $|z| > |a|$,

$$\frac{1}{z-a} = \frac{1}{z} \cdot \frac{1}{1 - \left(\frac{a}{z}\right)} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty} a^n z^{-n-1} = \sum_{n=1}^{\infty} a^{n-1} z^{-n}$$

◆ **Example 17:** The function $f(z) = \frac{1}{(z-1)(z-2)}$ can be expanded as a Laurent/Taylor series centered at $z_0 = 0$ in each of the following domains:

- | | | |
|-----|---------------|--------------------------|
| (1) | $ z < 1$ | (Taylor) |
| (2) | $1 < z < 2$ | (Laurent) |
| (3) | $2 < z $ | (negative powers series) |

First write $f(z)$ as a sum of simple functions:

$$f(z) = \frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{1}{z-2}$$

By the preceding result, we obtain a Taylor series for $f(z)$ in the disk $|z| < 1$:

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2} = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}} z^n = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n$$

For the annulus $1 < |z| < 2$, compute a negative-power series for $-\frac{1}{z-1}$ in the domain $|z| > 1$,



and a Taylor series for $\frac{1}{z-2}$ in the disk $|z| < 2$. Again, by the preceding result,

$$\begin{aligned} -\frac{1}{z-1} &= -\sum_{n=1}^{\infty} z^{-n} & (1 < |z|) \\ \frac{1}{z-2} &= \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}} z^n & (|z| < 2) \end{aligned}$$

Therefore the Laurent series of $f(z) = \frac{1}{(z-1)(z-2)}$ in the annulus $1 < |z| < 2$ is

$$f(z) = \frac{1}{(z-1)(z-2)} = -\sum_{n=1}^{\infty} z^{-n} - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$$

We leave the task of computing the negative-power series in the domain $|z| > 2$ to the student.

i Theorem 6.38: (Laurent) Let $f(z)$ be analytic in an annulus $R_1 < |z - z_0| < R_2$, where $0 \leq R_1 < R_2 \leq \infty$. Then f has a unique Laurent-series expansion in this annulus:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad (6.14)$$

The coefficients a_n of the series can be computed by

$$a_n = \frac{1}{2\pi i} \int_{\sigma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (6.15)$$

where $R_1 < r < R_2$ and σ_r is a positively oriented parametrization of the circle $|z - z_0| = r$. The series (6.14) converges absolutely and uniformly on every subannulus

$$R_1 < r_1 \leq |z - z_0| \leq r_2 < R_2$$

✓ Proof: Let z be a given point in the annulus $R_1 < |z - z_0| < R_2$. Throughout the proof, we regard z as a constant. Choose radii r_1, r_2, r such that

$$R_1 < r_1 < |z - z_0| < r_2 < R_2$$

Denote by $\sigma_{r_1}, \sigma_{r_2}, \sigma_r$ the circular paths corresponding to these radii around the point z_0 . By Cauchy's integral formula (Theorem 5.19),

$$f(z) = \frac{1}{2\pi i} \int_{\sigma_{r_2}} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_{\sigma_{r_1}} \frac{f(w)}{w - z} dw \quad (6.16)$$

This is, of course, a more general integral formula for a domain with a hole, which is easily proved as an exercise using the bridge method (see the proof of the corollaries following the

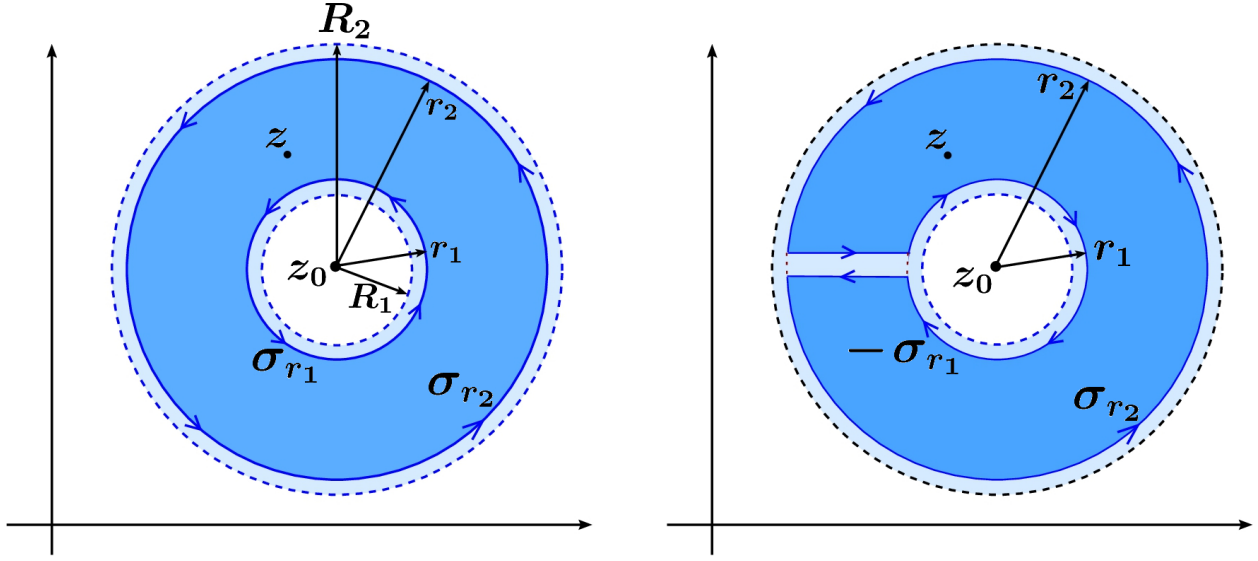


Fig. 6.16: Domain of absolute convergence of a Laurent series in an annulus

Cauchy-Goursat theorem 5.8). Set

$$f_1(z) = \frac{1}{2\pi i} \int_{\sigma_{r_1}} \frac{f(w)}{w-z} dw \quad ; \quad f_2(z) = \frac{1}{2\pi i} \int_{\sigma_{r_2}} \frac{f(w)}{w-z} dw$$

Begin by representing the function f_2 as a Taylor series (in the variable w).

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w-z_0 - (z-z_0)} = \frac{1}{w-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}} \\ &= \frac{1}{w-z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0} \right)^n \end{aligned}$$

The result follows because $w \in \sigma_{r_2}$, and hence $|w-z_0| = r_2$, and because $|z-z_0| < r_2$. Thus we have obtained a convergent geometric series, because its ratio $q = \left| \frac{z-z_0}{w-z_0} \right| < 1$. Therefore, for every $w \in \sigma_{r_2}$,

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0} \right)^n$$

The function $f(w)$ is continuous on the compact set σ_{r_2} and is therefore bounded. Hence the series of functions $\sum_{n=0}^{\infty} \frac{f(w)}{w-z_0} \cdot \left(\frac{z-z_0}{w-z_0} \right)^n$ converges uniformly on the circle σ_{r_2} . Therefore (by Theorem 6.21), we may integrate term by term in the following computation:

$$\begin{aligned} f_2(z) &= \frac{1}{2\pi i} \int_{\sigma_{r_2}} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_{\sigma_{r_2}} \left[\sum_{n=0}^{\infty} \frac{f(w)}{w-z_0} \cdot \left(\frac{z-z_0}{w-z_0} \right)^n \right] dw \end{aligned}$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\sigma_{r_2}} \frac{f(w)}{(w - z_0)^{n+1}} dw \right] \cdot (z - z_0)^n$$

By Corollary 5.23 (the corollary to Cauchy's derivative formula),

$$a_n = \frac{1}{2\pi i} \int_{\sigma_{r_2}} \frac{f(w)}{(w - z_0)^{n+1}} dw = \frac{f^{(n)}(z_0)}{n!}$$

and we have obtained a Taylor series for $f_2(z)$.

$$\begin{aligned} f_2(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ a_n &= \frac{1}{2\pi i} \int_{\sigma_{r_2}} \frac{f(w)}{(w - z_0)^{n+1}} dw \end{aligned}$$

We now turn to the function $f_1(z)$.

$$\begin{aligned} -\frac{1}{w - z} &= \frac{1}{z - w} = \frac{1}{z - z_0 - (w - z_0)} \\ &= \frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{w - z_0}{z - z_0}} \\ &= \frac{1}{z - z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^n \end{aligned}$$

The result follows because $w \in \sigma_{r_1}$, and hence $|w - z_0| = r_1$, and because $|z - z_0| > r_1$. Thus we have obtained a convergent geometric series, because its ratio $q = \left| \frac{w - z_0}{z - z_0} \right| < 1$. Therefore, for every $w \in \sigma_{r_1}$,

$$\frac{f(w)}{w - z} = \frac{f(w)}{z - z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^n$$

The function $f(w)$ is continuous on the compact set σ_{r_1} and is therefore bounded there. Hence the series of functions $\sum_{n=0}^{\infty} \frac{f(w)}{z - z_0} \cdot \left(\frac{w - z_0}{z - z_0} \right)^n$ converges uniformly on the circle σ_{r_1} . Therefore (by Theorem 6.21), we may integrate term by term in the following computation:

$$\begin{aligned} f_1(z) &= \frac{1}{2\pi i} \int_{\sigma_{r_1}} \frac{f(w)}{w - z} dw \\ &= -\frac{1}{2\pi i} \int_{\sigma_{r_1}} \left[\sum_{n=0}^{\infty} \frac{f(w)}{z - z_0} \cdot \left(\frac{w - z_0}{z - z_0} \right)^n \right] dw \\ &= -\sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\sigma_{r_1}} \frac{f(w)}{(w - z_0)^{-n}} dw \right] \cdot (z - z_0)^{-(n+1)} \end{aligned}$$



We have obtained the coefficients of the Laurent series (Laurent) of the function $f_1(z)$.

$$a_{-(n+1)} = \frac{1}{2\pi i} \int_{\sigma_{r_1}} \frac{f(w)}{(w - z_0)^{-n}} dw$$

By equation (6.16),

$$\begin{aligned} f(z) &= f_2(z) - f_1(z) \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\sigma_{r_2}} \frac{f(w)}{(w - z_0)^{n+1}} dw \right] \cdot (z - z_0)^n + \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\sigma_{r_1}} \frac{f(w)}{(w - z_0)^{-n}} dw \right] \cdot (z - z_0)^{-(n+1)} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{\sigma_{r_2}} \frac{f(w)}{(w - z_0)^{n+1}} dw \right] \cdot (z - z_0)^n + \sum_{n=-1}^{-\infty} \left[\frac{1}{2\pi i} \int_{\sigma_{r_1}} \frac{f(w)}{(w - z_0)^{n+1}} dw \right] \cdot (z - z_0)^n \end{aligned}$$

Let r be any radius between r_1 and r_2 : $r_1 < r < r_2$.

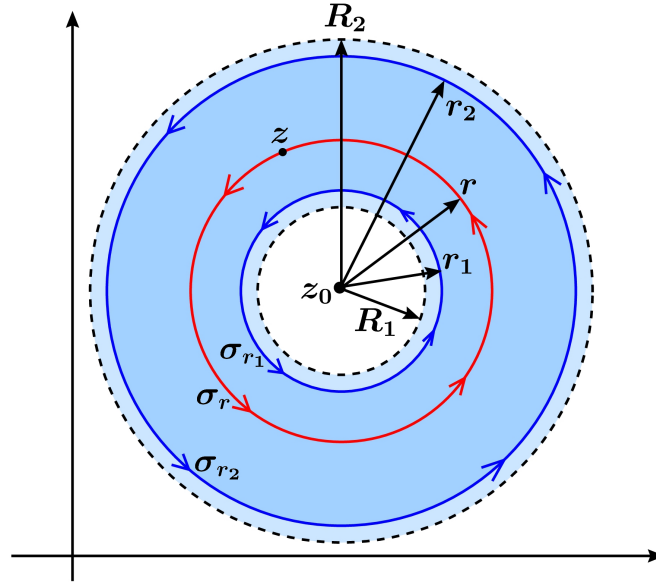


Fig. 6.17: An intermediate path σ_r between σ_{r_1} and σ_{r_2}

The function $\frac{f(w)}{(w - z_0)^n}$ is analytic in the annulus $R_1 < |z - z_0| < R_2$, and therefore, by assertion 5.15,

$$\begin{aligned} \int_{\sigma_{r_2}} \frac{f(w)}{(w - z_0)^n} dw &= \int_{\sigma_r} \frac{f(w)}{(w - z_0)^n} dw \\ \int_{\sigma_{r_1}} \frac{f(w)}{(w - z_0)^n} dw &= \int_{\sigma_r} \frac{f(w)}{(w - z_0)^n} dw \end{aligned}$$

Thus we may write the Laurent-series expansion of $f(z)$ in the following form:

$$f(z) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi i} \int_{\sigma_r} \frac{f(w)}{(w - z_0)^{n+1}} dw \right] \cdot (z - z_0)^n$$

If we set

$$a_n = \frac{1}{2\pi i} \int_{\sigma_r} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

we obtain the conclusion of the theorem. ■

► The uniqueness of the Laurent series follows from the fact that if $f(z)$ has two Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n$$

on the same annulus, then the coefficients of both series must also satisfy formula (6.15).

$$a_n = b_n = \frac{1}{2\pi i} \int_{\sigma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Therefore $f(z)$ has a unique Laurent-series expansion in a given annulus.

◆ **Example 18:** Expand $f(z) = \frac{1}{(z-1)^2}$ as a Laurent series centered at the point $z_0 = 0$ in the domain $|z| > 1$.

🔗 **Solution:** This can be done by a simple algebraic method using the sum formula for a geometric series.

$$\begin{aligned} \frac{1}{(z-1)^2} &= \frac{d}{dz} \left[\frac{-1}{z-1} \right] = -\frac{d}{dz} \left[\frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} \right] \\ &= -\frac{d}{dz} \left[\frac{1}{z} \cdot \sum_{n=0}^{\infty} z^{-n} \right] = -\frac{d}{dz} \sum_{n=0}^{\infty} z^{-(n+1)} \\ &= \sum_{n=0}^{\infty} (n+1) z^{-(n+2)} = \sum_{n=-2}^{-\infty} -(n+1) z^n \end{aligned}$$

We try to verify this result using formula (6.15) from Laurent's theorem 6.38.

$$f(z) = \frac{1}{(z-1)^2} = \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\sigma_r} \frac{f(w)}{(w - z_0)^{n+1}} dw = \frac{1}{2\pi i} \oint_{\sigma_r} \frac{1}{(w-1)^2 \cdot w^{n+1}} dw$$

where σ_r is a circle of radius r (arbitrary) centered at the point $z_0 = 0$. Letting the radius r tend to infinity, for every integer $n \geq -1$,

$$|a_n| \leq \frac{1}{2\pi} \left(\frac{1}{(r-1)^2 \cdot r^{n+1}} \right) \cdot 2\pi r = \frac{r}{(r-1)^2 \cdot r^{n+1}} \xrightarrow{r \rightarrow \infty} 0$$



Therefore necessarily $a_n = 0$ for every $n \geq -1$ ²

For $n < -1$, set $m = -n - 1$. Then $m \geq 1$, and we obtain

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \oint_{\sigma_r} \frac{dw}{w^{n+1} \cdot (w-1)^2} \\ &= \frac{1}{2\pi i} \oint_{\sigma_r} \frac{w^m dw}{(w-1)^2} \\ &= \frac{d}{dz} [z^m] \Big|_{z=1} = m z^{m-1} \Big|_{z=1} = m \\ &= -n - 1 \end{aligned}$$

In the last step, we used Cauchy's derivative formula for the function z^m . Thus we obtain again that

$$\frac{1}{(z-1)^2} = \sum_{n=-2}^{-\infty} (n+1)z^n$$

i Proposition 6.39: (Term-by-term differentiation of a Laurent series) Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be a Laurent series converging in the annulus $R_1 < |z-z_0| < R_2$. Then the derivative of $f(z)$ is obtained by differentiating the series term by term:

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z-z_0)^{n-1}$$

✓ Proof: See Exercise 27 at the end of the chapter.

Exercises for chapter 6

Remark: Most of the exercises focus on power series (Taylor and Maclaurin series), because these notes are oriented toward integral transforms and signal processing in electrical engineering (in the sequel), and therefore we did not find it necessary to add a further burden that deviates from this goal. If there is demand for it, we will add further exercises on general sequences and general series of functions in the future.

1. Let $a > 0$ be a real number. Prove that the series $f(z) = \sum_{n=1}^{\infty} e^{-an^2 z}$ converges to an analytic function on the upper half-plane $\operatorname{Re}(z) > 0$.
2. What is the largest domain on which the series $f(z) = \sum_{n=1}^{\infty} \frac{1}{z^2+n^2}$ converges to an analytic function?

Answer: $\mathbb{C} \setminus \pm in : n = 1, 2, 3, \dots$

3. What is the largest domain on which the series $f(z) = \sum_{n=1}^{\infty} \frac{\cos nz}{n^2}$ converges uniformly to an

² We used the inequality $|w-1| \geq |w| - 1 = r - 1$, which is contained in the triangle inequality in Theorem 1.3.



analytic function?

Answer: $z \in \mathbb{C} : \operatorname{Im} z = 0$

4. Let $a > 0$ be a real number. Prove that the series $f(z) = \sum_{n=0}^{\infty} \frac{e^{-anz}}{(a+n)^2}$ converges to an analytic function on the upper half-plane $\operatorname{Re}(z) > 0$.



5. Prove that the series $f(z) = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{z^2+n}{n^2}$ converges uniformly to an analytic function on every disk $|z| \leq r$, but does not converge uniformly on $\mathbb{C}$.

6. Let $a > 0$ be a real number. Prove that the series $f(z) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^z}$ converges to an analytic function in the domain $\operatorname{Re}(z) > 1$.



7. Find the Maclaurin series of the following functions and determine their domains of convergence.

(a) $\frac{z+1}{z-1}$ (b) $\frac{1}{1+z^2}$ (c) $\frac{z^7}{8+z^3}$ (d) $\cos^2 z$

8. Find the Taylor series of $f(z) = \sin z$ centered at the point $z_0 = \pi$, and compute its radius of convergence.

9. Find the Taylor series of $f(z) = \frac{1}{z}$ centered at the point $z_0 = 1+i$, and compute its radius of convergence.



10. Compute the sums of the series $\sum_{n=0}^{\infty} r^n \sin(n\theta)$, $\sum_{n=0}^{\infty} r^n \cos(n\theta)$, where r is a real number in the interval $(-1, 1)$, and θ is any angle.

Hint: Use the sum formula for a geometric series. Final answer: $\frac{1-r \cos \theta}{1-2r \cos \theta + r^2}$, $\frac{r \sin \theta}{1-2r \cos \theta + r^2}$



11. Let $f(z)$ be an entire function satisfying $|f'(z)| \leq |z|$ for every $z \in \mathbb{C}$. Prove that

$$f(z) = a + bz^2, \quad |b| \leq \frac{1}{2}$$

12. Prove: If an entire function $f(z)$ satisfies $|f(z)| \leq M$ on the circle $|z| = R$, then all the coefficients of its Taylor series centered at $z = 0$ satisfy $|a_n| \leq \frac{M}{R^n}$.

Hint: $a_n = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z) dz}{z^{n+1}}$.

13. Using the result of Exercise 12, prove that if the polynomial $P(z)$ satisfies $|P(z)| \leq 1$ on the unit circle, then all its coefficients are bounded in absolute value by 1.

Hint: A polynomial is a Taylor series with only finitely many nonzero coefficients ...

14. Prove that the power series $\sum_{n=0}^{\infty} n z^n$ converges inside the unit disk $|z| < 1$, but diverges at every point on the unit circle $|z| = 1$.

15. Prove that the power series $\sum_{n=1}^{\infty} \frac{1}{n} z^n$ converges at every point $|z| \leq 1$, except at the point



$$z = 1.$$

16. Prove that for every z in the disk $|z - 2| < 2$: $\frac{1}{4} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{z-2}{2} \right)^n = \frac{1}{z^2}$



17. Prove that the following power series

$$f(z) = \sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + z^8 + z^{16} + \dots$$

converges in the unit disk $|z| < 1$, but diverges at every point $|z| = 1$.

Hint: Note that if real $r \rightarrow 1^-$, then $f(r) \rightarrow \infty$. In addition, $f(z) = z + f(z^2)$. Examine what happens near points of the form $z = e^{2k\pi i/2^n}$, $k = 1, 2, \dots, 2^{n-1}$: What happens to $f(z)$ as its argument moves along the corresponding radius toward such a point?



18. Prove that if an entire function $f(z)$ satisfies $|f(z)| \leq |z|^2$ for every $z \in \mathbb{C}$, then $f(z) = cz$ for a complex constant c , and in addition $|c| \leq 1$.

Hint: For every $z_0 \in \mathbb{C}$ and every $R > 0$, $f'''(z_0) = \frac{3!}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^4} dz$



19. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \sin n$.

Hint: $|a_n| = |\sin(n)| \leq 1$, and therefore, by comparison with a geometric series, $R \geq 1$. Show that the sequence of numbers $\sin(n)$ is dense in the interval $[0, 1]$. After some thought, you may take a look [at this link](#).



20. Find a rational function whose Taylor series in the unit disk is $\sum_{n=0}^{\infty} n^2 z^n$.

Hint: Begin with the geometric series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, and experiment with linear combinations of its derivatives (first and second).



21. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function analytic in the unit disk $|z| < 1$. Suppose that $a_1 \neq 0$, and $\sum_{n=2}^{\infty} n|a_n| < |a_1|$. Prove that f is one-to-one.

Suggestion: Try using an idea similar to the one in Exercise 33, Chapter 5.



22. Find the Maclaurin series of the function

$$f(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{cz}{n}$$

where c is a given constant. In addition, find the radius of convergence of the Maclaurin series.

Hint: Note that f is defined as the sum of a trigonometric series! This is not a power series! Use appropriate theorems to verify that it may be differentiated term by term and to compute the values of $f^{(n)}(0)$. Use the evenness and oddness of $\sin$, $\cos$.



23. Find the Laurent series centered at $z = -1$ of the function $f(z) = \frac{e^z}{(z+1)^5}$ in the shortest possible way.



Hint: $e^z = e^{-1} \cdot e^{z+1}$. Final answer: $\sum_{n=1}^{\infty} \frac{e^{-1}}{n!} (z+1)^n$.



24. Use a Laurent series to compute the integral $\oint_{|z|=1} z^5 e^{\frac{2}{z}}$.

Hint: $e^{\frac{2}{z}} = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{-n}$. Final answer: $\frac{8\pi i}{45}$.

25. Prove: For every polynomial $P(z)$, the function $f(z) = P(z)e^{\frac{1}{z}}$ cannot be extended to an entire function.



26. Let $f(z)$ be analytic in the annulus $R_1 < |z| < R_2$. Prove that there exists a function $f_2(z)$ analytic in the disk $|z| < R_2$, and a function $f_1(z)$ analytic in the plane with a hole $|z| > R_1$, such that $f(z) = f_1(z) + f_2(z)$ on the annulus.



27. Using Exercise 26, prove assertion 6.39 concerning term-by-term differentiation of a Laurent series.

28. Find the coefficient of $(z+i)^{-3}$ in the Laurent-series expansion of $f(z) = \frac{1}{z^2-1}$ centered at the point $z_0 = -i$ in the domain $|z+i| > \sqrt{2}$.

Hint: $\frac{1}{z^2-1} = \frac{1}{2} \left(\frac{1}{z-1} - \frac{1}{z+1} \right)$. Answer: $a_{-3} = 0$.



29. Compute the third term ($n = -3$) in the negative-power series of $f(z) = (z-3) \sin \frac{1}{z+2}$ centered at the point $z = -2$.

Hint: Substitute $w = z+2$ or $w = \frac{1}{z+2}$. Answer: $\frac{5}{6}$.



30. Find the coefficient of the term $(z-1)^{-2}$ in the Laurent series of the function $f(z) = \frac{z^2-2z+3}{z-2}$ in the domain $|z-1| > 1$.

Hint: $\frac{z^2-2z+3}{z-2} = z + \frac{3}{z-2} = z + \frac{3}{z} \cdot \frac{1}{1-\frac{2}{z}}$. Answer: $a_{-2} = 6$.

31. Expand the function $f(z) = \frac{1}{(z+1)(z^2+1)}$ as a Laurent series in each of the following domains:

(a) $1 < |z| < \sqrt{2}$ (b) $|z| > \sqrt{2}$ (c) $|z| < 1$

32. Expand the function $f(z) = \frac{3z-1}{z^2-2z-3}$ as a Laurent series in each of the following domains:

(a) $1 < |z| < 3$ (b) $|z| > 3$ (c) $|z| < 1$

33. Expand the function $f(z) = \frac{z+4}{z^2(z^2+3z+2)}$ as a Laurent series in each of the following domains:

(a) $0 < |z| < 1$ (b) $1 < |z| < 2$ (c) $|z| > 2$ (d) $0 < |z+1| < 1$

34. Find the largest radius $R > 0$ for which the Laurent series of $f(z) = \frac{1}{e^{z^2}-1}$ converges in the domain $0 < |z| < R$.

Answer: $R = \sqrt{2\pi}$





35. Find the Laurent series of $f(z) = z^2 \sin \frac{1}{z^2}$ in the domain $0 < |z| < \infty$.

Answer: $1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}$



36. Prove that the Maclaurin series of an entire function $f(z)$ converges uniformly to f on all of $\mathbb{C}$ if and only if f is a polynomial.



37. Let $f(z)$ and $g(z)$ be two entire functions satisfying $|f(z)| \leq |g(z)|$ for every $z \in \mathbb{C}$. Then there exists $\lambda \in \mathbb{C}$ such that $f(z) = \lambda g(z)$ for every $z \in \mathbb{C}$.



38. Let D be a domain symmetric with respect to the origin (see Definition 2.6 on page 48). A function $f(z)$ on the domain D is called an **even function** if $f(-z) = f(z)$ for every $z \in D$. f is called an **odd function** if $f(-z) = -f(z)$ for every $z \in D$. Prove that if f is even, then all the odd coefficients of the Laurent series of f centered at $z = 0$ vanish. If f is odd, then all the even coefficients of the Laurent series of f centered at $z = 0$ vanish.

39. Let $f(z)$ be analytic on a domain D , and let $z_0 \in D$ be a given point. Prove: If $f'(z_0) \neq 0$, then

$$\frac{2\pi i}{f'(z_0)} = \oint_{|z-z_0|=\varepsilon} \frac{dz}{f(z) - f(z_0)}$$

where $|z - z_0| \leq \varepsilon$ is a closed disk centered at z_0 and contained entirely in the domain D .

Hint: Use the Taylor series of f centered at z_0 : $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. By a simple manipulation, obtain:

$f(z) - f(z_0) = a_1(z - z_0) \left[1 + \frac{a_2}{a_1}(z - z_0) + \frac{a_3}{a_1}(z - z_0)^2 + \dots \right]$. Use Cauchy's integral formula (Theorem 5.19). If this does not work, there is a shorter solution using the residue theorem. See Exercise 30 in the next chapter.



40. The Fibonacci sequence $\{a_n\}_{n=1}^{\infty}$ is defined recursively by: $a_0 = 0$, $a_1 = 1$, and for every $n \geq 2$, $a_n = a_{n-2} + a_{n-1}$. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Maclaurin series whose coefficients are the Fibonacci sequence.

(a) Prove that this is the Maclaurin series of the rational function $\frac{z}{1-z-z^2}$.

(b) What is the radius of convergence of the series?

(c) Find a closed formula for a_n .

Hint: (a) Try experimenting with linear combinations of $zF(z)$, $z^2F(z)$.

(c) Decompose into partial fractions: $\frac{z}{1-z-z^2} = \frac{A}{z-z_1} + \frac{B}{z-z_2}$, and obtain two geometric series. The formula you should obtain is: $a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}$ (Binet's formula).



41. Let $f(z)$ be analytic in a neighborhood of the point $z = 0$, and suppose that the series $\sum_{n=1}^{\infty} |f^{(n)}(0)|$ converges. Prove that f has an analytic continuation to an entire function.



42. Give an example of a two-sided series of complex numbers $\sum_{n=-\infty}^{\infty} w_n$ such that the limit

$\lim_{N \rightarrow \infty} \sum_{n=-N}^N w_n$ exists, but the sum $\sum_{n=-\infty}^{\infty} w_n$ does not exist.



Remark: What does this say about the precise definition of the sum of a two-sided series? How would you define such a sum?

43. Prove: An entire function $f(z)$ that takes only real values on the real axis and only imaginary values on the imaginary axis is an odd function (that is, $f(-z) = -f(z)$ for every $z \in \mathbb{C}$).

Hint: Use the Maclaurin series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Substitute real $z = x$ and imaginary $z = iy$, and use the assumptions $f(x) = \overline{f(x)}$, $f(iy) = -\overline{f(iy)}$.



44. The Weierstrass theorem states that every continuous real-valued function on the interval $[0, 1]$ is the uniform limit of a sequence of polynomials. Can this theorem be generalized to a continuous complex function on the closed disk $\overline{\mathbb{D}}(0, 1)$?

Hint: If $f(z)$ is continuous on $\overline{\mathbb{D}}(0, 1)$, then $\overline{f(z)}$ is also continuous there. Check whether polynomials have a similar property. See Mergelyan's theorem in Exercise 56, which is a possible generalization of the Weierstrass theorem.



45. Use power series to prove the generalized L'Hôpital rule: Let $f(z)$, $g(z)$ be functions analytic at the point z_0 , all of whose derivatives at z_0 of order less than n vanish, but $g^{(n)}(z_0) \neq 0$. Then

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f^{(n)}(z_0)}{g^{(n)}(z_0)}$$



46. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be the function defined by

$$f(z) = \int_0^1 t^2 e^{-zt} dt$$

- (a) Compute the Maclaurin series of f . What is its radius of convergence?
 (b) Compute $f'(0)$.



47. Prove that the series of functions $f(z) = \sum_{n=1}^{\infty} \frac{nz^n}{1-z^n}$ converges uniformly on every closed disk $|z| \leq r$, where $0 < r < 1$.



48. Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of analytic functions on a simply connected domain D , and let σ be a Jordan path contained in D . Suppose the sequence of functions converges uniformly on the path σ . Prove that it also converges uniformly on the interior of the path σ .



49. Prove that there is no sequence of polynomials $\{P_n(z)\}_{n=1}^{\infty}$ converging uniformly to the function $f(z) = 1/z^2$ on the annulus $1 < |z| < 3$.

Suggestion: Recall that $\oint_{|z|=2} dz/z = 2\pi i$, whereas for every n , $\oint_{|z|=2} P_n(z) dz = 0$. Therefore it is impossible that $P_n(z) \Rightarrow 1/z$.
 If $P_n(z) \Rightarrow 1/z^2$, then $zP_n(z) \Rightarrow 1/z \dots$





50. Bernoulli numbers B_n are defined by the coefficients of the Maclaurin series of the function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

Prove the following formula:

$$\frac{B_0}{n!0!} + \frac{B_1}{(n-1)!1!} + \frac{B_2}{(n-2)!2!} + \cdots + \frac{B_{n-1}}{1!(n-1)!} = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

Using this formula, compute the Bernoulli number B_n for every natural number n .



51. Prove that the series $\sum_{n=1}^{\infty} \left(\frac{z+i}{z-i}\right)^n$ converges to an analytic function on the open disk $|z+i| < 1$.

52. Let $f(z)$ be analytic in the annulus $R_1 < |z - z_0| < R_2$. Suppose that $|f(z)| \leq M$ on the annulus. Prove that all the Laurent-series coefficients of f , a_n , $0 \leq n$, satisfy $|a_n| \leq \frac{M}{R_2^n}$, and all the coefficients a_n , $n \leq 0$, satisfy $|a_n| \leq MR_1^n$.



53. Let $f(t)$ be a real-valued continuous function on the closed interval $[-1, 2]$. Define the function

$$F(z) = \int_{-1}^2 f(t) \sin(zt) dt$$

(a) Prove that $F(z)$ is entire and find its Maclaurin series.

(b) Prove that $F'(z) = \int_{-1}^2 t f(t) \cos(zt) dt$.

Hint: Use the Leibniz rule. See the appendix at the end of Chapter 5.



54. Let $f(z)$, $g(z)$ be two entire functions. Define a function of two variables $h: \mathbb{C}^2 \rightarrow \mathbb{C}$:

$$h(z_1, z_2) = \int_0^{2\pi} f(z_1 e^{i\theta}) g(z_2 e^{-i\theta}) d\theta$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, be the Maclaurin series of f , g . Compute the value of $h(z_1, z_2)$ for given z_1, z_2 .



55. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series converging in the disk $|z| < R$. Suppose that f has an analytic continuation $g(z)$ in the disk $D(0, R + \varepsilon)$, $\varepsilon > 0$, except for a single pole z_0 on the circle $|z| = R$. Prove that $\lim_{n \rightarrow \infty} a_n / a_{n+1} = z_0$.

Hint: After some thought, you may look at this solution, but it seems too cumbersome. There must be something more reasonable.



56. Prove that there exists a sequence of polynomials $\{P_n(z)\}_{n=1}^{\infty}$ that converges to 1 in the upper half-plane $\text{Im}(z) > 0$, converges to -1 in the lower half-plane $\text{Im}(z) < 0$, and converges to



0 on the real axis $\operatorname{Im}(z) = 0$.

Hint: Use Mergelyan's theorem: Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} - K$ is connected, and let K° be the interior of K . Then every continuous function $f : K \rightarrow \mathbb{C}$ that is analytic in the interior K° is the uniform limit of a sequence of polynomials.



57. Prove that there exists a sequence of rational functions $\{Q_n(z)\}_{n=1}^\infty$ converging to a function $f(z)$ such that $\operatorname{image}(f) = \mathbb{Z}$.

Hint: Construct a sequence of rational functions $P_n(z)$ converging to a "step" function. For example, if $\operatorname{Im}(z) > 0$, then $P_n(z) \rightarrow 1$. Otherwise $P_n(z) \rightarrow 0$. Take multiples and translates of $P_n(z)$ to construct $Q_n(z)$.



[Link to solutions](#)



Chapter 7

Applications

In the final chapter of the course we will present various applications of the material learned in the preceding chapters for computing integrals and characterizing special points of analytic functions.

7.1 Zeros and Singular Points

Understanding and using an analytic function correctly requires the ability to identify and understand its special points. The special points of an analytic function are divided into two principal types:

1. Zeros
2. Singular points

Each of these types is further divided into subtypes, and so on. The complete classification is shown in the tree diagram in Figure 7.2. But we still have some way to go before we can understand the diagram.

Zeros are points at which the function vanishes. The most familiar example is the fundamental theorem of algebra, which states that every polynomial of degree n has exactly n zeros, but this is of course not a precise formulation, since when counting the zeros one must also take the **multiplicity** of each zero into account.

Definition 7.1: A point z_0 is called a **zero of a function** $f(z)$ if $f(z_0) = 0$. The point z_0 is called a **zero of multiplicity k** (k is a positive natural number), if there exists a function $g(z)$ such that

$$f(z) = (z - z_0)^k g(z) \quad \text{and} \quad g(z_0) \neq 0$$

A zero of multiplicity 1 is also called a **simple zero**.

Example 1:

- A. The function $f(z) = (z - i)^3$ has one zero $z = i$, whose multiplicity is 3.
- B. The function $f(z) = (z - i)^3(z - 7)^2$ has one zero $z = i$ of multiplicity 3, and a second zero $z = 7$ of multiplicity 2. If we write f in the following form

$$f(z) = (z - i)(z - i)(z - i)(z - 7)(z - 7)$$



then we see that the expression $z - i$ occurs three times. Thus the term “multiplicity 3” indicates that, in a sense, the zero $z = i$ occurs three times in the list of zeros.

C. The function $f(z) = e^z$ has no zeros!

D. The zero function $f(z) = 0$ has infinitely many zeros: every complex number is a zero of f . But none of these zeros has a multiplicity!

i Proposition 7.1: Let $f(z)$ be a function analytic at the point z_0 . Then z_0 is a zero of multiplicity k if and only if $f^{(j)}(z_0) = 0$ for every $j < k$, but $f^{(k)}(z_0) \neq 0$.

✓ Proof: Direction 1: Assume that z_0 is a zero of multiplicity k . That is, $f(z) = (z - z_0)^k g(z)$, $g(z_0) \neq 0$. By Theorem 6.34, the function $g(z)$ has a Taylor series about the point z_0

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

and therefore

$$f(z) = (z - z_0)^k g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^{n+k} = \sum_{n=k}^{\infty} b_{n-k} (z - z_0)^n$$

Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be the Taylor series of f about the point z_0 . By the uniqueness of the Taylor series (Theorem 6.33), we obtain $a_j = 0$ for every $j < k$, $a_k = b_0$. Since $a_j = \frac{f^{(j)}(z_0)}{j!}$, we obtain $f^{(j)}(z_0) = 0$. Since $g(z_0) = b_0 \neq 0$, we obtain $f^{(k)}(z_0) \neq 0$.

Direction 2: Assume that $f^{(j)}(z_0) = 0$ for every $j < k$, but $f^{(k)}(z_0) \neq 0$. Therefore, by Taylor's theorem (6.34),

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \\ &= (z - z_0)^k \cdot \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^{n-k} \end{aligned}$$

If we set $g(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^{n-k}$ then $g(z_0) = \frac{f^{(k)}(z_0)}{k!} \neq 0$, and we have obtained the desired result. ■

*** Remark:** In the proposition, k is a positive natural number, and we used the convention $f^{(0)}(z_0) = f(z_0)$.

i Corollary 7.2: A point z_0 is a zero of multiplicity k of an analytic function $f(z)$ if and only if the first k coefficients of the Taylor series of f vanish, but $a_k \neq 0$.

$$f(z) = a_k (z - z_0)^k + a_{k+1} (z - z_0)^{k+1} + a_{k+2} (z - z_0)^{k+2} + \cdots \quad (a_k \neq 0)$$

i Definition 7.2: A point z_0 is called an **isolated zero** of a function $f(z)$ if there exists a neighborhood of z_0 in which it is the only zero of f .

i Proposition 7.3: A point z_0 is a nonisolated zero of f if and only if there exists an infinite sequence of zeros $\{z_n\}_{n=1}^{\infty}$ such that $z_0 = \lim_{n \rightarrow \infty} z_n$.

✓ Proof: If z_0 is a nonisolated zero, then for every n there exists a zero $z_n \neq z_0$ such that $|z_n - z_0| < \frac{1}{n}$. Clearly, the sequence of zeros $\{z_n\}_{n=1}^{\infty}$ converges to z_0 . The converse follows directly from the definition. ■

◇ Example 2:

A. The points $z = i$ and $z = 7$ are isolated zeros of the function $f(z) = (z - i)^3(z - 7)^2$.

B. The point $z = 0$ is a nonisolated zero of the function

$$f(z) = \begin{cases} z^2 \sin \frac{2\pi}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

because $f(0) = 0$, and for every n , the point $z_n = \frac{1}{n}$ is a zero of f . In the next proposition we will see that such a situation is impossible if f is analytic. In the present case, it is easy to check that f is continuous on the entire plane but is not analytic at the point $z = 0$.

i Theorem 7.4: Let $f(z)$ be an analytic function on a domain D . Then f has a nonisolated zero in D if and only if f vanishes throughout the domain D .

✓ Proof: Let z_0 be a nonisolated zero of f in the domain D , and let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be the Taylor series of f about the point z_0 . We will prove that all the coefficients of the series vanish. Otherwise, let a_k be the first coefficient that does not vanish. We can write $f(z) = (z - z_0)^k \sum_{n=k}^{\infty} a_n(z - z_0)^{n-k}$. Denote

$$g(z) = \sum_{n=k}^{\infty} a_n(z - z_0)^{n-k}$$

Then $g(z_0) = a_k \neq 0$. By the continuity of g , we conclude that there exists a neighborhood of z_0 in which $g(z) \neq 0$, but then also $f(z) = (z - z_0)^k g(z) \neq 0$ for every z in that neighborhood, except for the zero z_0 , of course. We have obtained that z_0 is an isolated zero, contrary to the assumption. Therefore all the coefficients of the Taylor series must vanish, and consequently there exists a neighborhood of z_0 on which f vanishes.

We will prove that in fact f vanishes throughout the domain D ! We use Figure 7.1. Let z_1 be any point in the domain D . Since D is connected, there exists a smooth path $\sigma: [a, b] \rightarrow D$, such that $\sigma(a) = z_0$, $\sigma(b) = z_1$.



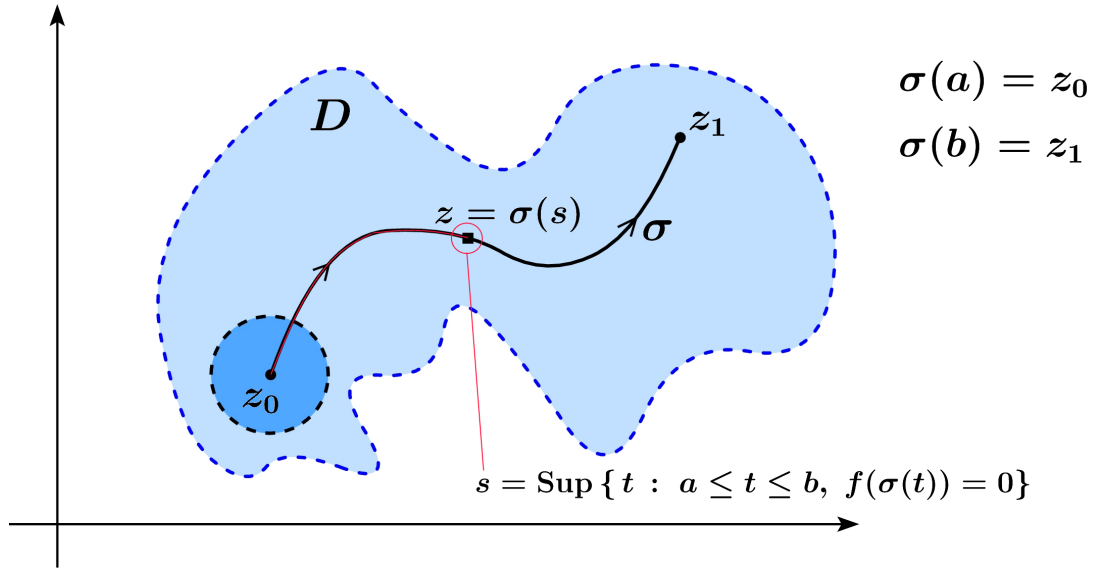


Fig. 7.1: D is a connected domain, and therefore there exists a smooth path σ between the point z_0 and an arbitrary point $z_1 \in D$

Define a subset I of the interval $[a, b]$

$$I = \{ t : t \in [a, b], \forall s \leq t : f(\sigma(s)) = 0 \}$$

The set I is nonempty because f vanishes in a neighborhood of $z_0 = \sigma(a)$, and hence vanishes on an interval $[a, a + \delta]$ of the path (as shown in Figure 7.1), and therefore $[a, a + \delta] \subseteq I$. The set I is bounded above by b , and therefore it has a supremum

$$s = \text{Sup } I$$

Clearly, $z = \sigma(s)$ is a nonisolated zero of f , because $f(z) = \lim_{t \rightarrow s^-} f(\sigma(t)) = 0$. We will prove that necessarily $s = b$. Suppose, to the contrary, that $s < b$. By the same argument used at the beginning of the proof, there exists a neighborhood of z on which f vanishes identically. Therefore there exists $\delta > 0$ such that $f(\sigma(t)) = 0$ for every t in the interval $[s - \delta, s + \delta]$. We have obtained $s + \delta \in I$, contradicting the fact that s is the supremum of I . Therefore $s = b$, and hence $f(z_1) = f(\sigma(b)) = 0$. This is true for every point $z_1 \in D$, and therefore $f \equiv 0$ identically throughout the domain D . ■

i Corollary 7.5: Let $f(z)$ be an analytic function on a domain D , and let σ be a path in D . If f vanishes on σ , then f vanishes throughout the domain D .

Hint: Choose a point z_0 on σ . Then z_0 is a nonisolated zero. By the preceding theorem, f vanishes on D .

i Theorem 7.6: (“Identity Theorem”) Let $f(z)$ and $g(z)$ be analytic functions on a domain D , and let S be an infinite set of points in D having an accumulation point $z_0 \in D$ that does not belong to S . If, for every $z \in S$, we have $f(z) = g(z)$, then $f(z) = g(z)$ for every $z \in D$.

❖ **Proof:** Review the definition of an accumulation point (Definition 2.22 on page 68). From the assumptions, we conclude that there exists an infinite sequence $\{z_n\}_{n=1}^{\infty}$ converging to the point z_0 . Therefore z_0 is a nonisolated zero of $f(z) - g(z)$, and hence $f(z) - g(z) = 0$ for every $z \in D$. ■

❖ **Exercise 7.1:** Let $f(z)$ be an analytic function in the disk $|z| < 2$. It is known that for every natural number n ,

$$f\left(\frac{1}{n}\right) = \frac{n}{2n+1}$$

Compute $f(i)$.

❖ **Solution:** If we set $z = \frac{1}{n}$, then $n = \frac{1}{z}$, and define the function

$$g(z) = \frac{\frac{1}{z}}{\frac{2}{z} + 1} = \frac{1}{2 + z}$$

By the construction of g , it is clear that for every n , $f(\frac{1}{n}) = g(\frac{1}{n})$, and therefore, by the preceding corollary, $f(z) = g(z)$ for every $|z| < 2$. Thus $f(i) = g(i) = \frac{1}{2+i} = \frac{2-i}{5}$.

Singular Points

❖ **Definition 7.3:** A point z_0 is called an **isolated singular point** of a function f if

- (a) f is analytic in a punctured neighborhood of z_0
- (b) f is not analytic or is not defined at z_0 .

❖ **Example 3:**

- A. The function $f(z) = \frac{1}{(z-i)^3}$ has one isolated singular point, $z = i$. The function is not defined at the point $z = i$, and it cannot be defined there so that it becomes analytic.
- B. The function $f(z) = \frac{1}{(z-i)^3(z-7)^2}$ has two isolated singular points: $z = i$, $z = 7$.
- C. The function $f(z) = e^z$ is an entire function, and therefore it has no singular points.

➤ In the present course we will focus only on isolated singular points, but we will briefly mention two other types of singular points:

A. A **nonisolated singular point** is a point at which the function is not analytic but which is a limit of isolated singular points. The canonical example of this type of singularity is the function

$$f(z) = \frac{1}{\sin \frac{2\pi}{z}}, \quad z \neq 0$$

It is easy to check that for every natural number n , the point $z_n = \frac{1}{n}$ is an isolated singular point of f . The sequence $\{z_n\}_{n=1}^{\infty}$ converges to the point $z = 0$, but the function is not analytic at the point $z = 0$, and therefore $z = 0$ is a nonisolated singular point of f .



B. Another type of singular point is a **branch point**, which has different definitions in different books and which we will not discuss. For example, the point $z = 0$ is a branch point of the logarithm function $\text{Log}(z)$. The point $z = 0$ is not an isolated singular point because $\text{Log}(z)$ is not analytic on the slit $(-\infty, 0]$, and in addition $z = 0$ is not a nonisolated singular point either. As noted, this type of point will not be treated in the present course.

i Definition 7.4: A point z_0 is called a **removable singular point** of a function f , if it is an isolated singular point of f and $f(z_0)$ can be defined so that the function f is analytic at the point z_0 .

➤ The meaning of a removable singular point is that the limit $w = \lim_{z \rightarrow z_0} f(z)$ exists, and after the “correction” $f(z_0) = w$, the function f will be analytic at the point z_0 .

◆ **Example 4:** The point $z = 0$ is a removable singular point of the function $f(z) = \frac{\sin z}{z}$. The function f is not defined at the point $z = 0$, but if we define $f(0) = \lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$, then f will be analytic at the point $z = 0$, and will become an entire function. This can be proved by using the Taylor series of $\sin z$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \implies \frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

The Taylor series of $\sin z$ begins with the term z , and therefore after dividing it by z , it remains a Taylor series that is defined and converges throughout the plane.

◆ **Example 5:** The point $z = 0$ is an isolated singular point of $f(z) = \frac{1}{z}$ that is not removable, because the limit of $f(z)$ at the point $z = 0$ does not exist. Therefore there is no possibility of correcting it so that it will be analytic at the point $z = 0$.

i Proposition 7.7: An isolated singular point of a function $f(z)$ is removable if and only if the Laurent series of f about z_0 is a Taylor series (that is, all the coefficients of the negative powers vanish).

✓ **Proof:** The assumption that z_0 is a singular point of f implies that f is defined in a punctured neighborhood of z_0 . By Laurent’s theorem 6.38, the function f has a Laurent series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ about the point z_0 . If z_0 is a removable singular point of f , then we can define $f(z_0)$ so that f is analytic at the point z_0 , and therefore, by Taylor’s theorem 6.34, it has a Taylor series in a neighborhood of z_0 , and hence the uniqueness of the Laurent series implies that all the coefficients of the negative powers must vanish.

The converse direction of the proposition is clear: If the coefficients of the negative powers in the Laurent series vanish, then the Laurent series becomes a Taylor series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and therefore, if we define $f(z_0) = a_0$, f will automatically become analytic. ■



- ❖ The series of negative powers $\sum_{n=-\infty}^{-1} a_n(z - z_0)^n$ is sometimes called the **principal part** of the Laurent series, $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$, and therefore we can also formulate the last proposition as follows: An isolated singular point of a function $f(z)$ is removable if and only if the principal part of the Laurent series of f about z_0 vanishes.

i Definition 7.5: An isolated singular point z_0 is called a **pole of order k** (k is a positive natural number) of a function f , if z_0 is a removable singular point of the function

$$g(z) = (z - z_0)^k f(z)$$

and, in addition, $g(z_0) = \lim_{z \rightarrow z_0} (z - z_0)^k f(z) \neq 0$. A pole of order **1** is also called a **simple pole**.

❖ **Example 6:**

A. The point $z = i$ is a pole of order **3** of the function $f(z) = \frac{1}{(z-i)^3}$. The function

$$g(z) = (z - i)^3 f(z) = 1$$

does not vanish at the point $z = i$.

B. The function $f(z) = \frac{1}{(z-i)^3(z-7)^2}$ has two poles: $z = i$ is a pole of order **3**, and $z = 7$ is a pole of order **2**.

i Proposition 7.8: Let z_0 be an isolated singular point of a function $f(z)$. Then z_0 is a pole of order k if and only if the Laurent series of f about z_0 contains a finite positive number of negative powers.

❖ **Proof:** If z_0 is a pole of order k , then the function $g(z) = (z - z_0)^k f(z)$ has a Taylor series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ (after its correction at the point z_0), and, in addition, $a_0 = g(z_0) \neq 0$. Therefore

$$f(z) = \frac{g(z)}{(z - z_0)^k} = \sum_{n=0}^{\infty} a_n(z - z_0)^{n-k} = \sum_{n=-k}^{\infty} a_{n+k}(z - z_0)^n$$

The series of negative powers is finite: $\sum_{n=-k}^{-1} a_{n+k}(z - z_0)^n$. We also note that the coefficient of the first negative power in the series, $(z - z_0)^{-k}$, is $a_0 \neq 0$.

The converse is easy: If the Laurent series of f about z_0 contains a finite positive number of negative powers, then it has the form $\sum_{n=-k}^{\infty} a_n(z - z_0)^n$, where k is a positive natural number.

We may assume that $a_{-k} \neq 0$. It is then clear that $g(z) = (z - z_0)^k f(z)$ satisfies the definition of a pole of order k . ■

i Definition 7.6: Every isolated singular point that is neither a pole nor removable is called an



essential singular point.

i Proposition 7.9: An isolated singular point z_0 of a function $f(z)$ is an essential singular point if and only if the Laurent series of f about z_0 contains infinitely many negative powers.

✓ Proof: The proposition follows immediately from the preceding proposition. If the series of negative powers were finite, then z_0 would be a pole or a removable singular point. Therefore the series of negative powers must be infinite. ■

◇ Example 7: One of the simplest ways to produce examples of essential singular points is to substitute $\frac{1}{z-z_0}$ into the formula of an analytic function. For example, $\sin z$ is an entire function. But the function $\sin \frac{1}{z}$ has an essential singular point at $z = 0$. This can be seen immediately from the Laurent series of $\sin \frac{1}{z}$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \implies \sin \frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n-1}}{(2n+1)!}$$

The entire Laurent series consists of infinitely many negative powers, and therefore $z = 0$ is an essential singular point.

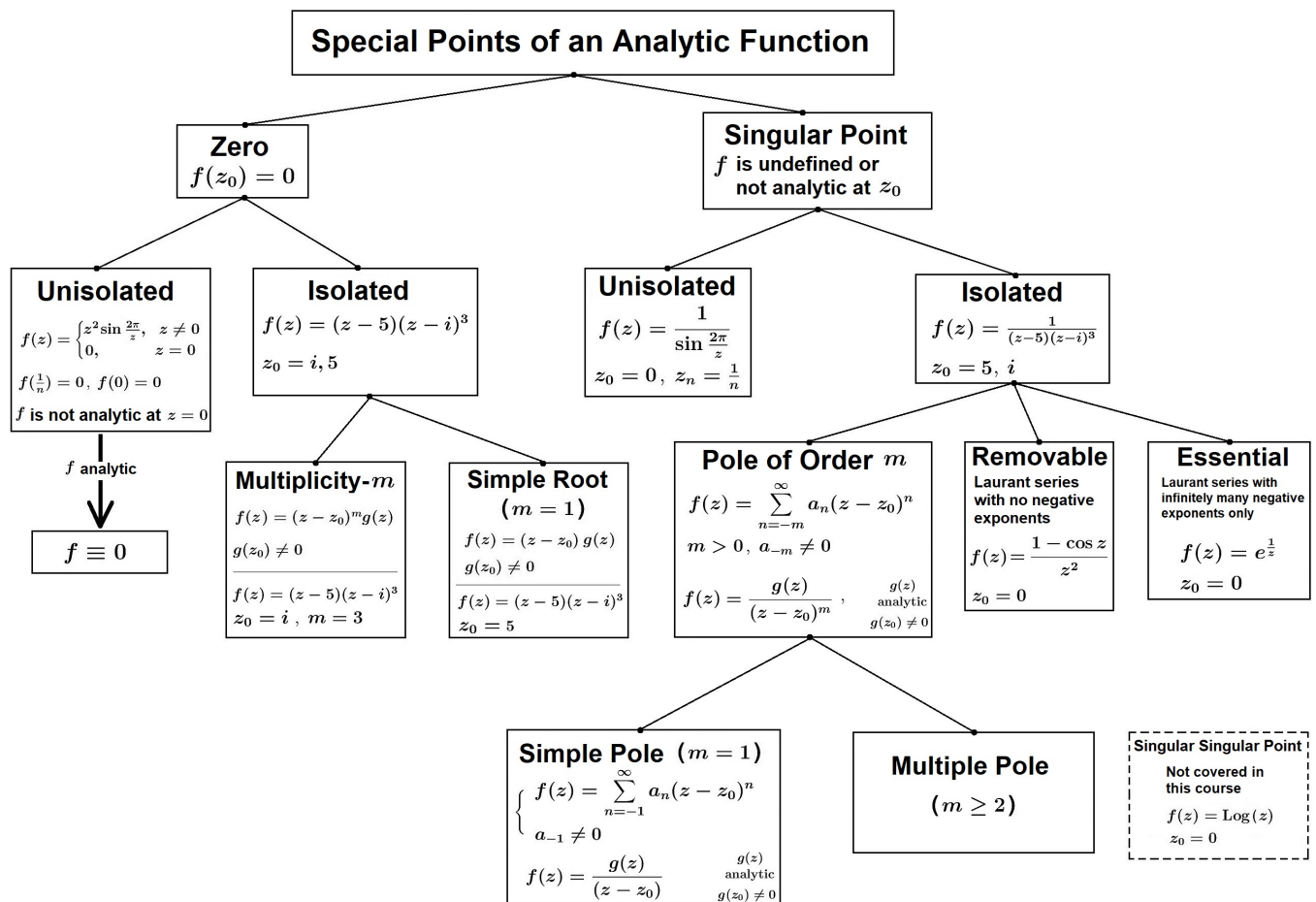


Fig. 7.2: A tree diagram of the special points of an analytic function

i Proposition 7.10: (Analyticity and boundedness imply removability)

Let $f(z)$ be a function analytic on a domain D except at an interior point z_0 of D . If $f(z)$ is bounded in a neighborhood of z_0 , then z_0 is a removable singular point of f .

Proof: By Laurent's theorem, there exists a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

of f about z_0 . We will prove that for every $n < 0$, $a_n = 0$. By Laurent's theorem, we can choose any sufficiently small $r > 0$ and obtain

$$a_n = \frac{1}{2\pi i} \oint_{\sigma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where σ_r is the circle $|z - z_0| = r$. Let $n < 0$. Then $m = -n$ is a positive natural number. By the integral triangle inequality,

$$\begin{aligned} |a_n| &= \frac{1}{2\pi} \left| \oint_{\sigma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \\ &\leq \frac{1}{2\pi} \oint_{\sigma_r} \frac{|f(z)|}{|z - z_0|^{n+1}} dz \\ &\leq \frac{1}{2\pi} \cdot 2\pi r \cdot \frac{M}{r^{n+1}} = Mr^m \xrightarrow{r \rightarrow 0} 0 \end{aligned}$$

The symbol M is the bound for $f(z)$ in a neighborhood of z_0 . We have obtained that for every $n < 0$, $a_n = 0$, and therefore z_0 is a removable singular point of f . ■

◆ **Example 8:** In the nonisolated-zero box of the tree diagram, we encounter the example

$$f(z) = \begin{cases} z^2 \sin \frac{2\pi}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

The function $f(z)$ has a sequence of isolated zeros at the points $z_n = \frac{1}{n}$, and a nonisolated zero at $z = 0$. It is easy to check that the function f is continuous at the point $z = 0$ but is not differentiable there. We also recall this example from Calculus 1, where one proves that the real function $f(x) = x^2 \sin \frac{2\pi}{x}$ has a removable point of nondifferentiability at $x = 0$. The example fails in the complex case because, if $f(z)$ were analytic at the point $z = 0$, we would obtain a contradiction to the identity theorem (Theorem 7.6)! Indeed, the zero function also vanishes on the sequence $\{z_n\}_{n=1}^{\infty}$.



i Proposition 7.11:

A. If z_0 is a zero of multiplicity m of $f(z)$, and a zero of multiplicity n of $g(z)$, then

- ◆ if $m < n$, then z_0 is a pole of order $n - m$ of $\frac{f(z)}{g(z)}$;
- ◆ if $m = n$, then z_0 is a removable singular point of $\frac{f(z)}{g(z)}$;
- ◆ if $m > n$, then z_0 is a zero of multiplicity $n - m$ of $\frac{f(z)}{g(z)}$.

B. If z_0 is a pole of order m of $f(z)$, and a pole of order n of $g(z)$, then

- ◆ if $m < n$, then z_0 is a zero of multiplicity $n - m$ of $\frac{f(z)}{g(z)}$;
- ◆ if $m = n$, then z_0 is a removable singular point of $\frac{f(z)}{g(z)}$;
- ◆ if $m > n$, then z_0 is a pole of order $n - m$ of $\frac{f(z)}{g(z)}$.

✓ Proof: (a) The definition of a zero implies that

$$\begin{aligned} f(z) &= (z - z_0)^m f_1(z) \\ g(z) &= (z - z_0)^n g_1(z) \end{aligned}$$

where f_1, g_1 , are analytic functions satisfying $f_1(z_0) \neq 0$, $g_1(z_0) \neq 0$, and therefore, for every $z \neq z_0$,

$$\frac{f(z)}{g(z)} = (z - z_0)^{m-n} \cdot \frac{f_1(z)}{g_1(z)}$$

If $m < n$, then we immediately obtain that z_0 is a pole of order $n - m$ of $\frac{f(z)}{g(z)}$. If $m = n$, then $\frac{f(z)}{g(z)} = \frac{f_1(z)}{g_1(z)}$, and therefore z_0 is a removable singular point. If $m > n$, then the expression $(z - z_0)^{m-n}$ has a positive power, and therefore z_0 is a zero of multiplicity $n - m$ of $\frac{f(z)}{g(z)}$.

(b) The definition of a pole implies that $f_1(z) = (z - z_0)^m f(z)$, $g_1(z) = (z - z_0)^n g(z)$, are analytic functions (after correcting a removable singular point) satisfying $f_1(z_0) \neq 0$, $g_1(z_0) \neq 0$, and therefore, for every $z \neq z_0$,

$$\frac{f(z)}{g(z)} = (z - z_0)^{n-m} \frac{f_1(z)}{g_1(z)}$$

If $m < n$, then we immediately obtain that z_0 is a zero of multiplicity $n - m$ of $\frac{f(z)}{g(z)}$. If $m = n$, then $\frac{f(z)}{g(z)} = \frac{f_1(z)}{g_1(z)}$, and therefore z_0 is a removable singular point. If $m > n$, then the expression $(z - z_0)^{n-m}$ has a negative power, and therefore z_0 is a pole of order $m - n$ of $\frac{f(z)}{g(z)}$. ■

i Proposition 7.12:

A. If z_0 is a zero of multiplicity m of $f(z)$, and a pole of order n of $g(z)$, then z_0 is a zero of multiplicity $m + n$ of $\frac{f(z)}{g(z)}$.

B. If z_0 is a pole of order m of $f(z)$, and a zero of multiplicity n of $g(z)$, then z_0 is a pole of order $m + n$ of $\frac{f(z)}{g(z)}$.

✦ **Exercise 7.2:** If z_0 is a zero or a pole of order m of a function $f(z)$, and in addition z_0 is a zero or a pole of order n of a function $g(z)$, what can you say about the type of the point z_0 with respect to the product $f(z)g(z)$? State all the possibilities and prove your answer.

7.2 The Residue Theorem

➤ The residue theorem is one of the most important and useful results in the theory of complex functions. It has many applications, including the computation of various kinds of integrals, including real integrals that cannot be solved by methods of real differential and integral calculus.

i **Definition 7.7:** Let $f(z)$ be a function analytic in a punctured neighborhood of z_0 . **The residue of f at the point z_0** is defined to be the coefficient a_{-1} in the Laurent series of f about z_0 .

Notation: $\text{Res}(f, z_0)$.

➤ That is: the residue $\text{Res}(f, z_0)$ is the coefficient of the first negative power in the Laurent series of f about z_0 .

➤ If the function is analytic at the point z_0 , then its residue at the point z_0 is 0.

➤ By formula (6.15) in Laurent's theorem 6.38, **The residue formula** for a function $f(z)$ at the point z_0

$$\text{Res}(f, z_0) = \frac{1}{2\pi i} \int_{\sigma} f(z) dz \quad (7.1)$$

where σ is a piecewise smooth Jordan path around z_0 .

✦ **Example 9:** The residue of $f(z) = \frac{1}{z}$ at the point $z = 0$ is 1. The residue of $f(z) = \frac{1}{z^2}$ at the point $z = 0$ is 0.

i **Theorem 7.13: (Residue Theorem)** Let $f(z)$ be a function analytic on a domain D , except at finitely many isolated singular points in the domain. Let σ be a piecewise smooth Jordan path in D enclosing the singular points $z_1, z_2, \dots, z_m$. Then

$$\oint_{\sigma} f(z) dz = 2\pi i \sum_{n=1}^m \text{Res}(f, z_n) \quad (7.2)$$



📌 **Proof:** Denote by $d(z, \sigma)$ the shortest distance from a point z to the path σ , and let

$$r_0 = \text{Min}\{d(z_1, \sigma), d(z_2, \sigma), \dots, d(z_m, \sigma)\}$$

Define

$$r = \frac{1}{3} \cdot \text{Min}\{|z_j - z_k| : 1 \leq j < k \leq n\} \cup \{r_0\}$$

This is the set of all distances between every two distinct singular points $z_j, z_k, j \neq k$. This is a finite set, and therefore it has a minimum value. The radius r is defined to be $\frac{1}{3}$ of this value in order to ensure that the disks $\mathbb{D}(z_k, r)$ are disjoint, as shown in Figure 7.3.

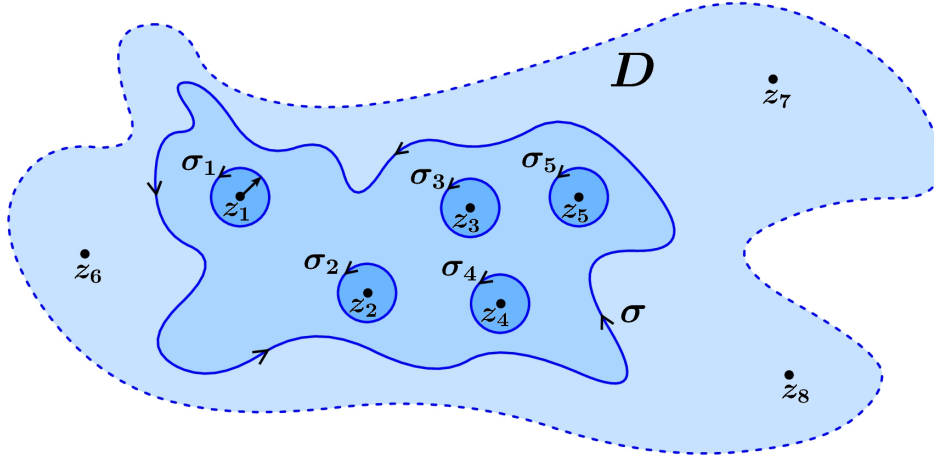


Fig. 7.3: The function $f(z)$ has five singular points inside the path σ and three singular points outside σ

Let $\sigma_1, \sigma_2, \dots, \sigma_m$ be circles of radius r around the singular points $z_1, z_2, \dots, z_m$. By Corollary 5.18 of the Cauchy-Goursat theorem,

$$\oint_{\sigma} f(z) dz = \oint_{\sigma_1} f(z) dz + \oint_{\sigma_2} f(z) dz + \dots + \oint_{\sigma_m} f(z) dz \quad (7.3)$$

Let $\sum_{n=-\infty}^{\infty} a_n(z - z_k)^n$ be the Laurent series of $f(z)$ about the singular point z_k . It is clear from the figure that the series converges absolutely in the domain $0 < |z - z_k| < R_k$, which contains the punctured disk $\dot{\mathbb{D}}(z_k, r)$. That is, the path σ_k is contained in the annulus of convergence $0 < |z - z_k| < R_k$. Therefore, by Laurent's theorem, we may use the path σ_k to compute the coefficients of the Laurent series.

$$a_n = \frac{1}{2\pi i} \oint_{\sigma_k} \frac{f(z)}{(z - z_k)^{n+1}} dz \quad (-\infty < n < \infty)$$

We now compute the integral $\oint_{\sigma_k} f(z)dz$ for $k = 1, 2, \dots, n$.

$$\begin{aligned}\oint_{\sigma_k} f(z)dz &= \oint_{\sigma_k} \left[\sum_{n=-\infty}^{\infty} a_n (z - z_k)^n \right] dz \\ &= \sum_{n=-\infty}^{\infty} a_n \oint_{\sigma_k} (z - z_k)^n dz \\ &= a_{-1} \oint_{\sigma_k} \frac{dz}{z - z_k} = 2\pi i \operatorname{Res}(f, z_k)\end{aligned}$$

Recall that the path σ_k is a closed and bounded subset of the annulus of convergence, and therefore the Laurent series converges uniformly on it, which permits interchange of the order of integration and summation. For every $n \neq -1$, $\oint_{\sigma_k} (z - z_k)^n dz = 0$ because the function $(z - z_k)^n$ has an analytic primitive (throughout the entire complex plane!). From equality (7.3) we obtain

$$\oint_{\sigma} f(z) = \sum_{k=1}^n \oint_{\sigma_k} f(z)dz = 2\pi i \sum_{k=1}^n \operatorname{Res}(f, z_k)$$

and this is, of course, the assertion of the theorem. ■

- Remark: In some books, the path σ need not be simple. It is easy to see that in this case each term in the sum must be multiplied by the index $\operatorname{Ind}(z_k, \sigma)$ (see Exercises 63–59 at the end of Chapter 5).
- The following proposition will often make it easier to compute the residue.

i Proposition 7.14: If z_0 is a pole of order k of a function $f(z)$, then

$$\operatorname{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{[(z - z_0)^k f(z)]^{(k-1)}}{(k-1)!}$$

- ✓ **Proof:** By the definition of a pole of order k , z_0 is a removable singular point of the function $g(z) = (z - z_0)^k f(z)$, and, in addition, after correction at the point z_0 ,

$$g(z_0) = \lim_{z \rightarrow z_0} (z - z_0)^k f(z) \neq 0$$

Multiplication by the factor $(z - z_0)^k$ moves the residue of f , which is the coefficient a_{-1} , to position $k - 1$ in the Taylor series of $g(z)$. Therefore, if we differentiate $g(z)$ $k - 1$ times, we obtain $(k - 1)!a_{-1}$. Therefore

$$\operatorname{Res}(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!} = \lim_{z \rightarrow z_0} \frac{[(z - z_0)^k f(z)]^{(k-1)}}{(k-1)!}$$



The reason why in the last step we had to use a limit is that the function $f(z)$ is not defined at the point z_0 . ■

- ✦ In Chapter 3, page 109, we presented a simple version of L'Hôpital's rule for complex functions (Theorem 3.27). In many cases it can be used to compute the limit required for computing the residue.

i Corollary 7.15: If z_0 is a simple pole of $f(z)$, then

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

- ✦ **Example 10:** It is not difficult to check that $z_0 = 1 + i$ is a simple pole of $f(z) = \frac{1}{z^4 + 4}$, and therefore, by the last corollary,

$$\begin{aligned} \text{Res}(f, z_0) &= \lim_{z \rightarrow z_0} (z - z_0) f(z) = \lim_{z \rightarrow 1+i} \frac{z - 1 - i}{z^4 + 4} = \frac{1}{4z^3} \Big|_{z=1+i} \\ &= \frac{1}{4(1+i)^3} = \frac{-1-i}{16} \end{aligned}$$

- ✦ **Example 11:** Compute the residue of the function $f(z) = \frac{e^z - 1 - z}{\sin(z) - z}$ at the pole $z = 0$.

- ✦ **Solution:** It is easy to see that $z = 0$ is a zero of multiplicity 2 of the numerator $e^z - 1 - z$ and a zero of multiplicity 3 of the denominator $\sin z$, and therefore (by Proposition 7.11) $z = 0$ is a simple pole of f . Therefore we can use the formula

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

to compute the residue of $f(z)$ at the point $z = 0$ (using L'Hôpital's rule).

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{z(e^z - 1 - z)}{\sin z - z} &= \lim_{z \rightarrow 0} \frac{(e^z - 1 - z) + z(e^z - 1)}{\cos(z) - 1} \\ &= \lim_{z \rightarrow 0} \frac{e^z - 1 + ze^z + e^z - 1}{-\sin(z)} \\ &= \lim_{z \rightarrow 0} \frac{3e^z + ze^z}{-\cos(z)} = -3 \end{aligned}$$

Second solution:



Another way to solve the problem is by using Taylor series.

$$\begin{aligned}
 e^z - 1 - z &= \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} \cdots \\
 &= z^2 \left(\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} \cdots \right) = z^2 g(z) \\
 \sin(z) - z &= -\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\
 &= z^3 \left(-\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots \right) = z^3 h(z)
 \end{aligned}$$

Clearly, $g(0) \neq 0$, $h(0) \neq 0$, and therefore $z = 0$ is a simple pole of $f(z) = \frac{g(z)}{zh(z)}$. By the formula for computing the residue at a simple pole,

$$\operatorname{Res}(f, 0) = \lim_{z \rightarrow 0} z f(z) = \lim_{z \rightarrow 0} z \frac{g(z)}{zh(z)} = \frac{g(0)}{h(0)} = \frac{\frac{1}{2!}}{-\frac{1}{3!}} = -3$$

◆ **Example 12:** Compute the integral $\oint_{|z|=1} z^5 e^{\frac{2}{z}}$.

🔴 **Solution:** The Laurent series of $e^{\frac{2}{z}}$ is

$$e^{\frac{2}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{2}{z} \right)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{-n}$$

and therefore the Laurent series of $z^5 e^{\frac{2}{z}}$ is

$$z^5 e^{\frac{2}{z}} = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{-n+5}$$

The function $f(z) = z^5 e^{\frac{2}{z}}$ has a single pole at the point $z = 0$. The residue of f at the point $z = 0$ is obtained for $n = 6$:

$$\operatorname{Res}(f, 0) = \frac{2^6}{6!} = \frac{64}{720} = \frac{4}{45}$$

Therefore

$$\oint_{|z|=1} z^5 e^{\frac{2}{z}} = 2\pi i \operatorname{Res}(f, 0) = \frac{8\pi i}{45}$$

◆ **Example 13:** Compute the integral $\oint_{|z|=1} \frac{(z^2 - 1)^2 dz}{z^2(z + \frac{1}{2})(z + 2)}$

🔴 **Solution:** Denote $f(z) = \frac{(z^2 - 1)^2}{z^2(z + \frac{1}{2})(z + 2)}$. We must use the residue theorem. We have two poles inside the disk $|z| = 1$: $z_0 = 0$, a pole of order 2, and $z_1 = -\frac{1}{2}$, a simple pole. Therefore, by

the residue theorem [7.13](#),

$$\oint_{|z|=1} \frac{(z^2 - 1)^2 dz}{z^2(z + \frac{1}{2})(z + 2)} = 2\pi i \left[\text{Res}(f, 0) + \text{Res}(f, -\frac{1}{2}) \right]$$

Denote

$$g(z) = z^2 f(z) = \frac{z^4 - 2z^2 + 1}{z^2 + \frac{5}{2}z + 1}$$

Then

$$\text{Res}(f, 0) = \lim_{z \rightarrow 0} \frac{g'(z)}{1!} = \frac{(-2z - 5/2)(z^4 - 2z^2 + 1)}{(z^2 + 5/2z + 1)^2} + \frac{4z^3 - 4z}{z^2 + 5/2z + 1} \Big|_{z=0} = -\frac{5}{2}$$

The pole $z_1 = -\frac{1}{2}$ is a simple pole, and therefore

$$\text{Res}(f, -\frac{1}{2}) = \frac{(z^2 - 1)^2}{z^2(z + 2)} \Big|_{z=-\frac{1}{2}} = \frac{9/16}{1/4 \cdot 3/2} = 3/2$$

Therefore

$$\oint_{|z|=1} \frac{(z^2 - 1)^2 dz}{z^2(z + \frac{1}{2})(z + 2)} = 2\pi i \left(-\frac{5}{2} + \frac{3}{2} \right) = -2\pi i$$

◆ **Example 14:** Compute the integral $\int_0^\pi \frac{d\theta}{5 + 4 \cos \theta}$.

🔗 **Solution:** The evenness of $\cos \theta$ implies that

$$\begin{aligned} \int_0^\pi \frac{d\theta}{5 + 4 \cos \theta} &= \frac{1}{2} \int_{-\pi}^\pi \frac{d\theta}{5 + 4 \cos \theta} = \frac{1}{2} \int_{-\pi}^\pi \frac{d\theta}{5 + 2e^{i\theta} + 2e^{-i\theta}} \\ &= \frac{1}{2i} \int_{-\pi}^\pi \frac{1}{e^{i\theta}} \cdot \frac{ie^{i\theta} d\theta}{5 + 2e^{i\theta} + 2e^{-i\theta}} = \frac{1}{2i} \oint_{|z|=1} \frac{1}{z} \cdot \frac{dz}{5 + 2z + 2z^{-1}} \\ &= \frac{1}{2i} \oint_{|z|=1} \frac{dz}{2(z + 2)(z + \frac{1}{2})} \\ &= \frac{1}{4i} \cdot 2\pi i \cdot \text{Res} \left(\frac{1}{(z + 2)(z + \frac{1}{2})}, -\frac{1}{2} \right) \\ &= \frac{\pi}{2} \cdot \frac{1}{-\frac{1}{2} + 2} = \frac{\pi}{3} \end{aligned}$$

We note that the expression $|z| = 1$ appearing in the lower limit of the path integral hides the parametrization of our circle, which is $z = e^{i\theta}$, $-\pi \leq \theta \leq \pi$.



➤ Using the substitutions $\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, one can solve many real trigonometric integrals of the form $\int_a^b f(\sin \theta, \cos \theta) d\theta$.

◆ **Example 15:** Compute the value of the integral $\int_0^{2\pi} \frac{d\theta}{2 + \sin \theta}$

🔑 **Solution:** We use the standard substitution $z = e^{i\theta}$ (and therefore $dz = ie^{i\theta} d\theta$, $d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$).

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} &= \int_{-\pi}^{\pi} \frac{d\theta}{2 + \frac{e^{i\theta} - e^{-i\theta}}{2i}} = \int_{-\pi}^{\pi} \frac{2id\theta}{4i + e^{i\theta} - e^{-i\theta}} \\ &= \int_{-\pi}^{\pi} \frac{2idz}{iz(4i + z - \frac{1}{z})} = 2 \int_{-\pi}^{\pi} \frac{dz}{z^2 + 4iz - 1} \\ &= 2 \oint_{|z|=1} \frac{dz}{(z - z_1)(z - z_2)} \end{aligned}$$

where z_1, z_2 are the roots of the quadratic equation $z^2 + 4iz - 1 = 0$.

$$z_1 = (-2 + \sqrt{3})i, \quad z_2 = (-2 - \sqrt{3})i$$

It is easy to check that the simple pole z_1 lies inside the circle $|z| = 1$, but the pole z_2 lies outside the circle. Therefore, by the residue theorem,

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} &= 2 \oint_{|z|=1} \frac{dz}{(z - z_1)(z - z_2)} = 2 \cdot 2\pi i \operatorname{Res} \left(\frac{1}{(z - z_1)(z - z_2)}, z_1 \right) \\ &= 4\pi i \cdot \frac{1}{z_1 - z_2} = \frac{4\pi i}{(-2 + \sqrt{3})i - (-2 - \sqrt{3})i} = \frac{2\pi}{\sqrt{3}} \end{aligned}$$

7.3 Computation of Improper Integrals

- Improper integrals are integrals in which at least one of the limits of integration is infinite. The computation of such integrals arises in the subject of integral transforms and is of great importance in electrical engineering and communications.
- Although in some cases these are real improper integrals, the use of the residue theorem and other theorems from complex function theory is critical for computing them. This brings us back to the discussion at the beginning of the chapter about the importance of “imaginary” numbers even for solving real problems.
- We begin by computing improper integrals of rational functions, $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are polynomials. As we recall from calculus, the computation of such integrals is in most



cases a difficult or unsolvable problem using the standard tools of Calculus 1.

i Proposition 7.16: If $P(z)$ and $Q(z)$ are a pair of complex polynomials satisfying $\deg(P) \leq \deg(Q) - 2$, and, in addition, the polynomial $Q(z)$ has no real zeros, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{n=1}^m \operatorname{Res} \left(\frac{P(z)}{Q(z)}, z_n \right)$$

where $z_1, z_2, \dots, z_m$ are the poles of $\frac{P(z)}{Q(z)}$ in the upper half-plane.

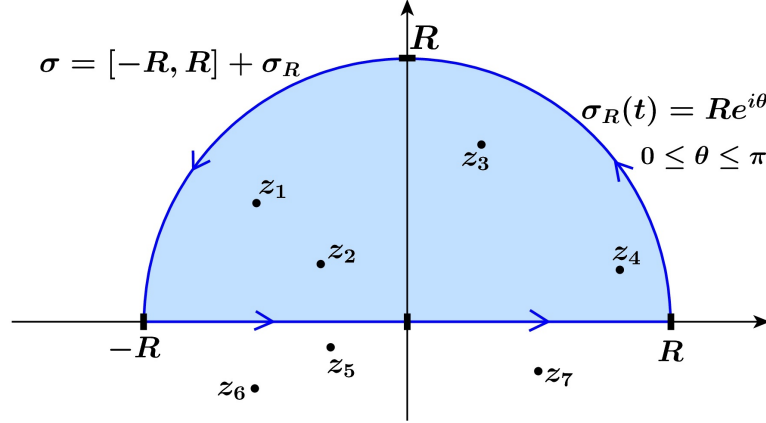


Fig. 7.4: A semicircular path $\sigma = [-R, R] + \sigma_R$, $\sigma_R(t) = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Some of the poles are enclosed by the path and some are outside it.

✓ Proof: Let $P(z)$ be a polynomial of degree m , let $Q(z)$ be a polynomial of degree n , and assume that $n \geq m + 2$. Then

$$\begin{aligned} \left| \frac{P(z)}{Q(z)} \right| &= \left| \frac{a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m}{b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n} \right| = \frac{|a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m|}{|b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n|} \\ &= \frac{|z|^m \cdot \left| \frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \frac{a_2}{z^{m-2}} + \dots + a_m \right|}{|z|^n \cdot \left| \frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \frac{b_2}{z^{n-2}} + \dots + b_n \right|} \\ &= \frac{\left| \frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \frac{a_2}{z^{m-2}} + \dots + a_m \right|}{|z|^{n-m} \cdot \left| \frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \frac{b_2}{z^{n-2}} + \dots + b_n \right|} \\ &\approx \frac{|a_m|}{|z|^{n-m} \cdot |b_n|} \leq \frac{|a_m|}{|z|^2 \cdot |b_n|} \xrightarrow{|z| \rightarrow \infty} \frac{|a_m|}{\infty \cdot |b_n|} = 0 \end{aligned}$$

Each of the fractions $\frac{a_k}{z^{m-k}}$, $\frac{b_k}{z^{n-k}}$, tends to zero as $|z| \rightarrow \infty$. Therefore the calculation proves that as $|z|$ tends to infinity, the expression $\left| \frac{P(z)}{Q(z)} \right|$ becomes small relative to $\frac{1}{|z|^2}$.

$$\oint_{\sigma} \frac{P(z)}{Q(z)} dz = \int_{[-R, R]} \frac{P(x)}{Q(x)} dx + \int_{\sigma_R} \frac{P(z)}{Q(z)} dz \quad (7.4)$$

We use the preceding result and the integral triangle inequality (Theorem 5.2)

$$\left| \int_{\sigma_R} \frac{P(z)}{Q(z)} dz \right| \leq \int_{\sigma_R} \left| \frac{P(z)}{Q(z)} \right| dz \leq \pi R \cdot \max_{|z|=R} \left| \frac{P(z)}{Q(z)} \right| \lesssim \pi R \cdot \frac{C}{R^2} = \frac{C}{R} \xrightarrow{R \rightarrow \infty} 0$$

and therefore

$$\lim_{R \rightarrow \infty} \int_{\sigma_R} \frac{P(z)}{Q(z)} dz = 0 \quad (7.5)$$

By the fundamental theorem of algebra, the polynomial $Q(x)$ has at most n distinct zeros, and therefore the rational function $\frac{P(x)}{Q(x)}$ has at most n poles. Let R_0 be a sufficiently large real number such that, for every $R \geq R_0$, the upper semicircle encloses all the poles in the upper half-plane. By the residue theorem, the value I of the integral $\int_{\sigma_R} \frac{P(z)}{Q(z)} dz$ will be constant for every $R \geq R_0$,

$$\forall R \geq R_0 : \quad I = \int_{\sigma_R} \frac{P(z)}{Q(z)} dz = 2\pi i \cdot \sum_{j=1}^k \text{Res} \left(\frac{P(z)}{Q(z)}, z_j \right)$$

where $z_1, \dots, z_k$ are the poles of $\frac{P(z)}{Q(z)}$ in the upper half-plane. Therefore, equality (7.4) and equality (7.5) imply that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{P(x)}{Q(x)} dx = \lim_{R \rightarrow \infty} \left[I - \int_{\sigma_R} \frac{P(z)}{Q(z)} dz \right] = I - 0 = I$$

We have therefore obtained the formula

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \cdot \sum_{j=1}^k \text{Res} \left(\frac{P(z)}{Q(z)}, z_j \right)$$

■

- ✦ In fact, the proposition can be generalized to every function $f(z)$ that tends to zero at a quadratic rate,

$$\lim_{|z| \rightarrow \infty} |z^2 f(z)| = 0$$

We leave the details to the student.

- ✦ The proposition is also true for the lower half-plane, but because the negative orientation of the path must be used, a minus sign must be added to the result.

- ◆ **Example 16:** Compute $\int_0^{\infty} \frac{dx}{x^4+4}$.

Solution: In this case $P(z) = 1$, $Q(z) = z^4 + 1$. Clearly, the conditions of Proposition 7.16 are satisfied. The poles of $f(z) = \frac{1}{z^4+1}$ are the roots of $z^4 + 4$,

$$z = (-4)^{\frac{1}{4}} = 4e^{\frac{(2k+1)\pi i}{4}} = 1 + i, -1 + i, 1 - i, -1 - i$$



and therefore

$$f(z) = \frac{1}{z^4 + 1} = \frac{1}{(z - 1 - i)(z - 1 + i)(z + 1 - i)(z + 1 + i)}$$

The poles lying in the upper half-plane are: $1 + i$, $-1 + i$, and therefore

$$\int_0^\infty \frac{dx}{x^4 + 4} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{x^4 + 4} = \pi i [\text{Res}(f, 1 + i) + \text{Res}(f, -1 + i)]$$

All the poles are simple, and therefore

$$\begin{aligned} \text{Res}(f, 1 + i) &= \lim_{z \rightarrow 1+i} (z - 1 - i)f(z) \\ &= \frac{1}{(z - 1 + i)(z + 1 - i)(z + 1 + i)} \Big|_{z=1+i} \\ &= \frac{1}{2i \cdot 2 \cdot (2 + 2i)} = \frac{1}{-8 + 8i} = \frac{-1 - i}{16} \end{aligned}$$

In the same manner,

$$\begin{aligned} \text{Res}(f, -1 + i) &= \lim_{z \rightarrow -1+i} (z + 1 - i)f(z) \\ &= \frac{1}{(z - 1 - i)(z - 1 + i)(z + 1 + i)} \Big|_{z=-1+i} \\ &= \frac{1}{-2 \cdot (-2 + 2i) \cdot 2i} = \frac{1}{8 + 8i} = \frac{1 - i}{16} \end{aligned}$$

Therefore

$$\int_0^\infty \frac{dx}{x^4 + 4} = \pi i \left[\frac{-1 - i}{16} + \frac{1 - i}{16} \right] = \frac{\pi}{8}$$

Remark: By using L'Hôpital's rule, the computation of the residues $\text{Res}(f, 1 + i)$ and $\text{Res}(f, -1 + i)$ can be greatly simplified. For example,

$$\text{Res}(f, 1 + i) = \lim_{z \rightarrow 1+i} \frac{z - 1 - i}{z^4 + 4} = \frac{1}{4z^3} \Big|_{z=1+i} = \frac{1}{4(1 + i)^3} = \frac{-1 - i}{16}$$

- To compute improper integrals involving rational and trigonometric functions, we will need the following lemma of Camille Jordan.

i Proposition 7.17: (Jordan's Lemma) Let $f(z)$ be a function analytic on $\mathbb{C}$ except at finitely



many poles, such that $\lim_{|z| \rightarrow \infty} |f(z)| = 0$. Then

$$\lim_{R \rightarrow \infty} \int_{\sigma_R} e^{i\alpha z} f(z) dz = 0$$

where $\alpha > 0$ is real and σ_R is an upper semicircle: $\sigma_R(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

✓ **Proof:** We use the inequality $\frac{2\theta}{\pi} \leq \sin \theta$ on the interval $0 \leq \theta \leq \frac{\pi}{2}$, which is easy to prove as an exercise and can be verified visually from Figure 7.5.

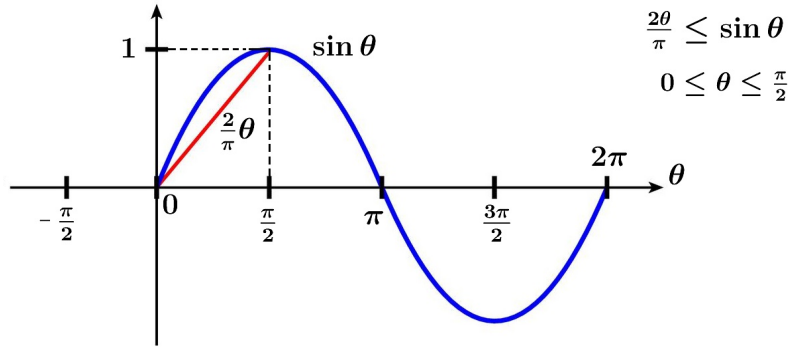


Fig. 7.5: A visual proof of the inequality for the semicircular path $\sigma_R(t) = Re^{i\theta}$, $0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned} \left| \int_{\sigma_R} e^{i\alpha z} f(z) dz \right| &= \left| \int_{\sigma_R} e^{i\alpha R \cos \theta - \alpha R \sin \theta} \cdot f(z) \cdot iRe^{i\theta} d\theta \right| \\ &\leq \int_0^\pi e^{-\alpha R \sin \theta} |f(Re^{i\theta})| \cdot R d\theta \\ &\leq R \cdot \text{Max}_{0 \leq \theta \leq \pi} |f(Re^{i\theta})| \cdot \int_0^\pi e^{-\frac{2\alpha R \theta}{\pi}} d\theta \\ &= 2R \cdot \text{Max}_{0 \leq \theta \leq \pi} |f(Re^{i\theta})| \cdot \int_0^{\frac{\pi}{2}} e^{-\frac{2\alpha R \theta}{\pi}} d\theta \quad (\text{symmetry}) \end{aligned}$$

We can easily evaluate the last integral:

$$\int_0^{\frac{\pi}{2}} e^{-\frac{2\alpha R \theta}{\pi}} d\theta = -\frac{\pi}{2\alpha R} e^{-\frac{2\alpha R \theta}{\pi}} \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{\pi}{2\alpha R} (1 - e^{-\alpha R})$$

In summary, we have obtained

$$\begin{aligned} \left| \int_{\sigma_R} e^{i\alpha z} f(z) dz \right| &\leq 2R \cdot \text{Max}_{0 \leq \theta \leq \pi} |f(Re^{i\theta})| \cdot \frac{\pi}{2\alpha R} (1 - e^{-\alpha R}) \\ &= \text{Max}_{0 \leq \theta \leq \pi} |f(Re^{i\theta})| \cdot \frac{\pi}{\alpha} (1 - e^{-\alpha R}) \xrightarrow{R \rightarrow \infty} 0 \end{aligned}$$

Because $\alpha > 0$, the expression $1 - e^{-\alpha R}$ tends to 1 as R tends to infinity, and therefore does

not affect the limit. ■

i Corollary 7.18: If $P(z)$ and $Q(z)$ are a pair of complex polynomials satisfying $\deg(P) < \deg(Q)$, then for every $\alpha > 0$,

$$\lim_{R \rightarrow \infty} \int_{\sigma_R} \frac{P(z)}{Q(z)} e^{i\alpha z} dz = 0$$

i Corollary 7.19: If $P(z)$ and $Q(z)$ are a pair of complex polynomials satisfying $\deg(P) < \deg(Q)$, and, in addition, the polynomial $Q(z)$ has no real zeros, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x} dx = 2\pi i \sum_{n=1}^m \operatorname{Res} \left(\frac{P(z)}{Q(z)} e^{i\alpha z}, z_n \right)$$

where $z_1, z_2, \dots, z_m$ are the poles of $\frac{P(z)}{Q(z)} e^{i\alpha z}$ in the upper half-plane.

➤ The proof is identical to the proof of Proposition 7.16, and we leave its details to the student (Figure 7.4 should be used). We note that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{i\alpha x} dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \alpha x dx + i \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \alpha x dx$$

and therefore, if

$$2\pi i \sum_{k=1}^n \operatorname{Res}(f, z_k) = A + Bi$$

then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(x) dx &= A \\ \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(x) dx &= B \end{aligned}$$

◆ **Example 17:** Prove that $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = -\frac{\pi \sin(2)}{e}$.

Solution: We compute

$$\oint_{\sigma_R} \frac{e^{iz}}{z^2 + 4z + 5} dz = \oint_{\sigma_R} \frac{\cos z}{z^2 + 4z + 5} dz + i \oint_{\sigma_R} \frac{\sin z}{z^2 + 4z + 5} dz$$

Clearly, here $P(z) = 1$, $Q(z) = z^2 + 4z + 5$ satisfy the conditions of Corollary 7.19. The poles of the function $f(z) = \frac{P(z)}{Q(z)} e^{iz}$ are the roots of the polynomial $z^2 + 4z + 5$,

$$z_1, z_2 = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 + i, -2 - i$$



We have a single pole $z_1 = -2 + i$ that lies in the upper half-plane, and therefore

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 4x + 5} dx = 2\pi i \cdot \text{Res} \left(\frac{e^{iz}}{z^2 + 4z + 5}, -2 + i \right)$$

This is a simple pole, and therefore

$$\begin{aligned} \text{Res} \left(\frac{e^{iz}}{z^2 + 4z + 5}, -2 + i \right) &= \lim_{z \rightarrow -2+i} \frac{e^{iz}}{z^2 + 4z + 5} (z + 2 - i) \\ &= \lim_{z \rightarrow -2+i} \frac{e^{iz}}{z + 2 + i} \\ &= \frac{e^{-1-2i}}{2i} \\ &= \frac{\sin(-2)}{e} - \frac{i \cos(-2)}{e} \end{aligned}$$

Therefore

$$\begin{aligned} \oint_{\sigma_R} \frac{e^{iz}}{z^2 + 4z + 5} dz &= 2\pi i \cdot \left[\frac{\sin(-2)}{e} - \frac{i \cos(-2)}{e} \right] \\ &= \frac{\pi \cos(2)}{e} - \frac{\pi \sin(2)}{e} i \end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = -\frac{\pi \sin(2)}{e}$$

➤ Remark: The polynomials $P(x)$ and $Q(x)$, which appear with the “real” variable x , may have complex coefficients!

7.4 The Argument Principle and Rouché’s Theorem

➤ Locating the zeros and poles of complex functions in given domains is highly important for the development and acceleration of computational algorithms in various fields. Sometimes, knowing that a given function has a known number of zeros in a given domain (without finding the exact zeros) is algorithmically important. The argument principle and Rouché’s theorem are the fundamental tools that help accomplish this.

i Definition 7.8: A function $f(z)$ is called **meromorphic** on a domain D if it is analytic on D except at poles.

i Definition 7.9: The **logarithmic derivative** of an analytic function $f(z)$ is defined to be

the quotient $\frac{f'(z)}{f(z)}$.

- The name comes from the fact that the expression $\frac{f'(z)}{f(z)}$ is the derivative of the function $\log f(z)$. If $f(z)$ is analytic on a domain D , then its logarithmic derivative $\frac{f'(z)}{f(z)}$ is a meromorphic function on the domain D . It is analytic at every point of D , except at the zeros of $f(z)$, where the function $\frac{f'(z)}{f(z)}$ has poles.

i Proposition 7.20: The logarithmic derivative of a product of functions equals the sum of their logarithmic derivatives.

$$\frac{[f(z)g(z)]'}{f(z)g(z)} = \frac{f'(z)}{f(z)} + \frac{g'(z)}{g(z)}$$

✓ **Proof:**

$$\frac{[f(z)g(z)]'}{f(z)g(z)} = \frac{f'(z)g(z) + f(z)g'(z)}{f(z)g(z)} = \frac{f'(z)}{f(z)} + \frac{g'(z)}{g(z)}$$

■

- By using mathematical induction, one can prove a similar formula for a product of n functions.

i Corollary 7.21: The logarithmic derivative of a product of n functions equals the sum of their logarithmic derivatives.

$$\frac{[f_1(z)f_2(z)\cdots f_n(z)]'}{f_1(z)f_2(z)\cdots f_n(z)} = \frac{f_1'(z)}{f_1(z)} + \frac{f_2'(z)}{f_2(z)} + \cdots + \frac{f_n'(z)}{f_n(z)} \quad (7.6)$$

i Proposition 7.22: Let $f(z)$ be an analytic function on a domain D , with m zeros $z_1, z_2, \dots, z_m$ of multiplicities $k_1, k_2, \dots, k_m$. Then there exists an analytic function $g(z)$ that does not vanish on the domain D , such that

$$f(z) = (z - z_1)^{k_1}(z - z_2)^{k_2}\cdots(z - z_m)^{k_m}g(z) \quad (7.7)$$

and the logarithmic derivative of f is given by

$$\frac{f'(z)}{f(z)} = \frac{k_1}{z - z_1} + \frac{k_2}{z - z_2} + \cdots + \frac{k_m}{z - z_m} + \frac{g'(z)}{g(z)} \quad (7.8)$$

✓ **Proof:** By the definition of a zero, there exists an analytic function $g_1(z)$ such that

$$f(z) = (z - z_1)^{k_1}g_1(z)$$

and, in addition, $g_1(z_1) \neq 0$. Therefore the zeros $z_2, z_3, \dots, z_m$ pass to the function $g_1(z)$ with the same multiplicities as before. After m steps of this kind, we obtain equality (7.7).



Equality (7.8) is an immediate consequence of the preceding corollary. We only note that the logarithmic derivative of $(z - c)^k$ is

$$\frac{[(z - c)^k]'}{(z - c)^k} = \frac{k(z - c)^{k-1}}{(z - c)^k} = \frac{k}{z - c}$$

i Proposition 7.23: Let $f(z)$ be a meromorphic function on a domain D , with m poles $z_1, z_2, \dots, z_m$ of orders $k_1, k_2, \dots, k_m$. Then there exists a function $g(z)$ analytic throughout the domain D , such that

$$f(z) = \frac{g(z)}{(z - z_1)^{k_1}(z - z_2)^{k_2} \cdots (z - z_m)^{k_m}} \quad (7.9)$$

and the logarithmic derivative of f is given by

$$\frac{f'(z)}{f(z)} = -\frac{k_1}{z - z_1} - \frac{k_2}{z - z_2} - \cdots - \frac{k_m}{z - z_m} + \frac{g'(z)}{g(z)} \quad (7.10)$$

✓ Proof: If z_1 is a pole of order k_1 of $f(z)$, then there exists a function $g_1(z)$ that is analytic at z_1 , such that $f(z) = \frac{g_1(z)}{(z - z_1)^{k_1}}$. Clearly, z_2 is now a pole of order k_2 of $g_1(z)$, and therefore there exists a function analytic at the point z_2 such that $g_1(z) = \frac{g_2(z)}{(z - z_2)^{k_2}}$, and therefore

$$f(z) = \frac{g_1(z)}{(z - z_1)^{k_1}} = \frac{g_2(z)}{(z - z_1)^{k_1}(z - z_2)^{k_2}}$$

After m steps of this kind, we obtain equality (7.9). We can also write it in the following form:

$$f(z) = (z - z_1)^{-k_1}(z - z_2)^{-k_2} \cdots (z - z_m)^{-k_m} g(z)$$

and the transition to equality (7.10) again follows immediately from formula (7.6). ■

i Theorem 7.24: (Argument Principle) Let $f(z)$ be a meromorphic function on a domain D , and let σ be a piecewise smooth Jordan path in D that does not pass through any zeros or poles of f . Let $Z(f, \sigma)$ be the number of zeros of f that lie inside σ , counted according to the multiplicity of each zero. Let $P(f, \sigma)$ be the number of poles of f that lie inside σ , counted according to the order of each pole. Then

$$\frac{1}{2\pi i} \oint_{\sigma} \frac{f'(z)}{f(z)} dz = Z(f, \sigma) - P(f, \sigma) \quad (7.11)$$

✓ Proof: Let $z_1, z_2, \dots, z_m$ be the zeros inside σ , of multiplicities $k_1, k_2, \dots, k_m$. Let $w_1, w_2, \dots, w_j$ be the poles inside σ , of orders $l_1, l_2, \dots, l_j$. In Exercise 7.3, the student

is asked to prove that there are only finitely many such points inside any Jordan path. By the preceding two propositions, there exists a function analytic in a neighborhood of the path σ such that

$$\frac{f'(z)}{f(z)} = \frac{k_1}{z - z_1} + \frac{k_2}{z - z_2} + \cdots + \frac{k_m}{z - z_m} - \frac{l_1}{z - w_1} - \frac{l_2}{z - w_2} - \cdots - \frac{l_j}{z - w_j} + \frac{g'(z)}{g(z)}$$

By Corollary 5.17 (see page 189), if z_0 is a point inside the path σ , then

$$\oint_{\sigma} \frac{1}{z - z_0} dz = 2\pi i$$

and therefore

$$\oint_{\sigma} \frac{f'(z)}{f(z)} dz = 2\pi i(k_1 + k_2 + \cdots + k_m - l_1 - l_2 - \cdots - l_j) + \oint_{\sigma} \frac{g'(z)}{g(z)} dz$$

By the Cauchy-Goursat theorem (Theorem 5.8 on page 177), the integral $\oint_{\sigma} \frac{g'(z)}{g(z)} dz$ vanishes because $\frac{g'(z)}{g(z)}$ is analytic inside the path σ and also on the path itself (because the denominator $g(z)$ has no zeros or poles on the path). Therefore we obtain

$$\oint_{\sigma} \frac{f'(z)}{f(z)} dz = 2\pi i(Z(f, \sigma) - P(f, \sigma))$$

and we are done. ■

- The name “argument principle” originates in the geometric interpretation of the expression $\oint_{\sigma} \frac{f'(z)}{f(z)} dz$ as a measurement of the change in the argument of $f(z)$ as z moves along the path σ , since $\frac{f'(z)}{f(z)}$ is the derivative of $\log f(z) = \ln |f(z)| + i \arg(f(z))$.
- The argument principle is useful in algorithms for locating regions in which a given function may have zeros or poles. These are usually numerical functions that cannot be treated by standard algebraic tools. In algorithms of this kind, one first computes the integral $\oint_{\sigma} \frac{f'(z)}{f(z)} dz$ (using numerical methods) over a large circle that covers all the regions of interest, and then computes the integral over smaller and smaller regions (by various bisection methods) until reaching regions that contain a single zero or pole.
- One of the well-known consequences of the argument principle is Rouché's theorem, (Eugène Rouché 1832-1910) which is also useful for locating zeros and poles of meromorphic functions.

i Theorem 7.25: (Rouché 1832-1910) Let $f(z)$ and $g(z)$ be two analytic functions on a domain D , and let σ be a piecewise smooth Jordan path in D , whose interior is also contained in D . If $|f(z)| > |g(z)|$ at every point on the path σ , then

$$Z(f + g, \sigma) = Z(f, \sigma)$$



That is, the number of zeros of $f(z) + g(z)$ inside the path σ is the same as that of $f(z)$.

✔ **Proof:** We use the argument principle and Proposition 7.20. The term that counts the number of poles does not appear, because these are analytic functions without poles.

$$\begin{aligned}
 Z(f + g, \sigma) &= \frac{1}{2\pi i} \oint_{\sigma} \frac{[f(z) + g(z)]'}{f(z) + g(z)} = \frac{1}{2\pi i} \oint_{\sigma} \frac{\left[f(z) \left(1 + \frac{f(z)}{g(z)} \right) \right]'}{f(z) \left(1 + \frac{f(z)}{g(z)} \right)} \\
 &= \frac{1}{2\pi i} \oint_{\sigma} \left[\frac{f'(z)}{f(z)} + \frac{\left(1 + \frac{f(z)}{g(z)} \right)'}{1 + \frac{f(z)}{g(z)}} \right] dz \\
 &= Z(f, \sigma) + \frac{1}{2\pi i} \oint_{\sigma} \frac{\left(1 + \frac{f(z)}{g(z)} \right)'}{1 + \frac{f(z)}{g(z)}} dz
 \end{aligned}$$

By assumption, $\left| \frac{g(z)}{f(z)} \right| < 1$ at every point of the path σ , and therefore the function $1 + \frac{g(z)}{f(z)}$ cannot take a nonpositive real value. That is, the image of the path under the function $1 + \frac{g(z)}{f(z)}$ is contained in the slit plane on which the principal branch of the logarithm function $\text{Log}(z)$ is analytic. Therefore the function $\text{Log}\left(1 + \frac{g(z)}{f(z)}\right)$ is well-defined and analytic on the path σ . Its derivative is

$$\left[\text{Log}\left(1 + \frac{g(z)}{f(z)}\right) \right]' = \frac{\left(1 + \frac{f(z)}{g(z)}\right)'}{1 + \frac{f(z)}{g(z)}}$$

Therefore, by the Cauchy-Goursat theorem (Theorem 5.8),

$$\oint_{\sigma} \frac{\left(1 + \frac{f(z)}{g(z)}\right)'}{1 + \frac{f(z)}{g(z)}} dz = 0$$

and hence we obtain $Z(f + g, \sigma) = Z(f, \sigma)$. ■

♦ **Example 18:** Prove that all the zeros of the polynomial $P(z) = z^5 + z^4 + z^3 + z^2 + z + 1$ are contained in the disk $|z| < 2$.

🔗 **Solution:** First, we note that there is no algebraic formula for finding the zeros of polynomials of degree 5 or higher. Therefore numerical methods are required to obtain approximations of the zeros. Set $f(z) = z^5$, $g(z) = z^4 + z^3 + z^2 + z + 1$ in Rouché's theorem. Our path is the circle $|z| = 2$. It is not difficult to see that on this circle,

$$|g(z)| \leq |z|^4 + |z|^3 + |z|^2 + |z| + 1 = 16 + 8 + 4 + 2 + 1 = 31 < 32 = |z|^5$$

and therefore $|g(z)| < |f(z)|$ on the circle $|z| = 2$. Thus the conditions of Rouché's theorem are satisfied, and therefore the function $f(z) + g(z)$ (which is our polynomial) has the same



number of zeros as the polynomial $f(z) = z^5$ inside the disk $|z| < 2$. The polynomial z^5 has a single zero $z = 0$ of multiplicity 5, and therefore all the zeros of our polynomial must lie inside the disk $|z| < 2$.

◈ **Exercise 7.3:** If $f(z)$ is a meromorphic function that is not identically zero on a compact set, then the number of zeros and poles in the set is finite.

7.5 Further Applications

➤ In the present section we will give more advanced examples of applications of the residue theorem and the argument principle.

◈ **Example 19:** Prove that

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = 0, \quad \int_0^\infty \frac{\ln^2 x}{1+x^2} dx = \frac{\pi^3}{8}$$

by using the path shown in Figure 7.6.

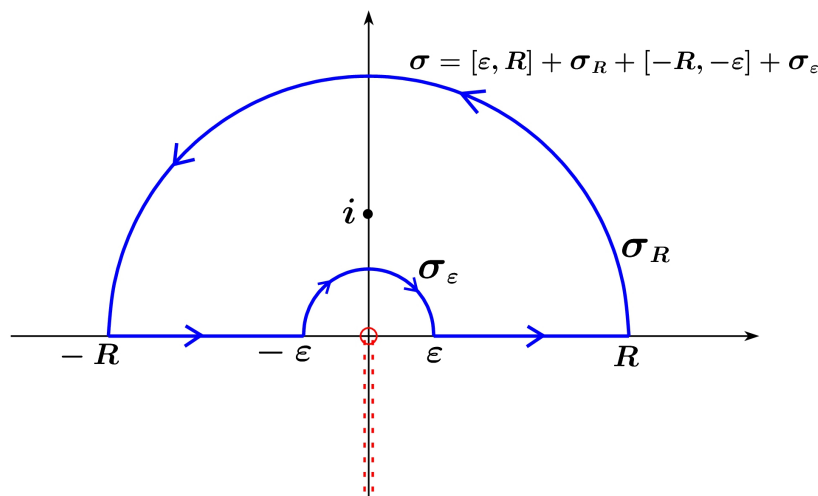


Fig. 7.6: The path σ bypasses the singular point $z = 0$ and the slit corresponding to the branch $[-\frac{\pi}{2}, \frac{3\pi}{2}]$

🔴 **Solution:** We present a mathematical proof of the first integral without using complex functions,

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = \int_0^1 \frac{\ln x}{1+x^2} dx + \int_1^\infty \frac{\ln x}{1+x^2} dx$$

We solve the second integral using the substitution $x = \frac{1}{t}$.

$$\int_1^\infty \frac{\ln x}{1+x^2} dx = \int_1^0 \frac{-\ln t}{1+\frac{1}{t^2}} \cdot \left(-\frac{1}{t^2}\right) dt = - \int_0^1 \frac{\ln t}{1+t^2} dt$$

Therefore

$$\int_0^\infty \frac{\ln x}{1+x^2} dx = \int_0^1 \frac{\ln x}{1+x^2} dx + \int_1^\infty \frac{\ln x}{1+x^2} dx = \int_0^1 \frac{\ln x}{1+x^2} dx - \int_0^1 \frac{\ln x}{1+x^2} dx = 0$$

Define a complex function

$$f(z) = \frac{\log^2(z)}{1+z^2}$$

where $\log(z)$ is based on the branch $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ (whose slit, drawn in red in Figure 7.6, does not intersect the path σ). Assume that $R > 1$, $\varepsilon < 1$, and therefore the simple pole $z = i$ of $f(z)$ lies inside the path σ . By the residue theorem,

$$\begin{aligned} \oint_{\sigma} f(z) dz &= \int_{[\varepsilon, R]} f(z) dz + \int_{\sigma_R} f(z) dz + \int_{[-R, -\varepsilon]} f(z) dz + \int_{\sigma_\varepsilon} f(z) dz \\ &= 2\pi i \cdot \text{Res}(f, i) = 2\pi i \cdot \frac{\log^2(z)}{z+i} \Big|_{z=i} \\ &= 2\pi i \cdot \frac{\log^2(i)}{2i} = 2\pi i \cdot \frac{(\pi i/2)^2}{2i} = -\frac{\pi^3}{4} \end{aligned}$$

We estimate the magnitude of each of the four integrals.

$$\left| \int_{\sigma_R} f(z) dz \right| \leq \int_{\sigma_R} \frac{|z|^{\frac{1}{2}}}{|z|^2 - 1} \leq \pi R \cdot \frac{R^{\frac{1}{2}}}{R^2 - 1} = \frac{\pi R^{\frac{3}{2}}}{R^2 - 1} \xrightarrow{R \rightarrow \infty} 0$$

Similarly, the integral over the small circle tends to zero as $\varepsilon \rightarrow 0$.

$$\begin{aligned} \left| \int_{\sigma_\varepsilon} f(z) dz \right| &= \left| \int_{\sigma_\varepsilon} \frac{[\ln |z| + i \arg(z)]^2}{1+z^2} dz \right| \leq \int_{\sigma_\varepsilon} \frac{|\ln |z| + i \arg(z)|^2}{|1+z^2|} dz \\ &\leq \pi \varepsilon \cdot \frac{(\ln \varepsilon + 2\pi)^2}{1 - \varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

Two integrals over the real intervals remain.

$$\begin{aligned} \int_{[\varepsilon, R]} f(z) dz &= \int_{\varepsilon}^R \frac{\ln^2 x}{1+x^2} dx \\ \int_{[-R, -\varepsilon]} f(z) dz &= \int_{-R}^{-\varepsilon} \frac{[\ln(-x) + i\pi]^2}{1+x^2} dx = \int_{\varepsilon}^R \frac{[\ln x + i\pi]^2}{1+x^2} dx \\ &= \int_{\varepsilon}^R \frac{\ln^2 x + 2\pi i \ln x - \pi^2}{1+x^2} dx \end{aligned}$$



$$= \int_{\varepsilon}^R \frac{\ln^2 x}{1+x^2} dx + 2\pi \int_{\varepsilon}^R \frac{\ln x}{1+x^2} dx - \pi^2 \int_{\varepsilon}^R \frac{1}{1+x^2} dx$$

In summary, we have obtained that as $R \rightarrow \infty$, $\varepsilon \rightarrow 0$,

$$\begin{aligned} \oint_{\sigma} f(z) dz &= 2 \int_0^{\infty} \frac{\ln^2 x}{1+x^2} dx + 2\pi \int_0^{\infty} \frac{\ln x}{1+x^2} dx - \pi^2 \int_0^{\infty} \frac{1}{1+x^2} dx \\ &= 2 \int_0^{\infty} \frac{\ln^2 x}{1+x^2} dx + 2\pi \cdot 0 - \pi^2 \cdot \frac{\pi}{2} = -\frac{\pi^3}{4} \end{aligned}$$

Therefore

$$2 \int_0^{\infty} \frac{\ln^2 x}{1+x^2} dx = -\frac{\pi^3}{4} + \frac{\pi^3}{2} = \frac{\pi^3}{4}$$

and therefore

$$\int_0^{\infty} \frac{\ln^2 x}{1+x^2} dx = \frac{\pi^3}{8}$$

◆ **Example 20:** Prove that for every $0 < a < 2$,

$$\int_0^{\infty} \frac{x^{a-1}}{1+x^2} dx = \frac{\pi}{2 \sin \frac{\pi a}{2}}$$

by using the residue theorem on the path in Figure 7.7.

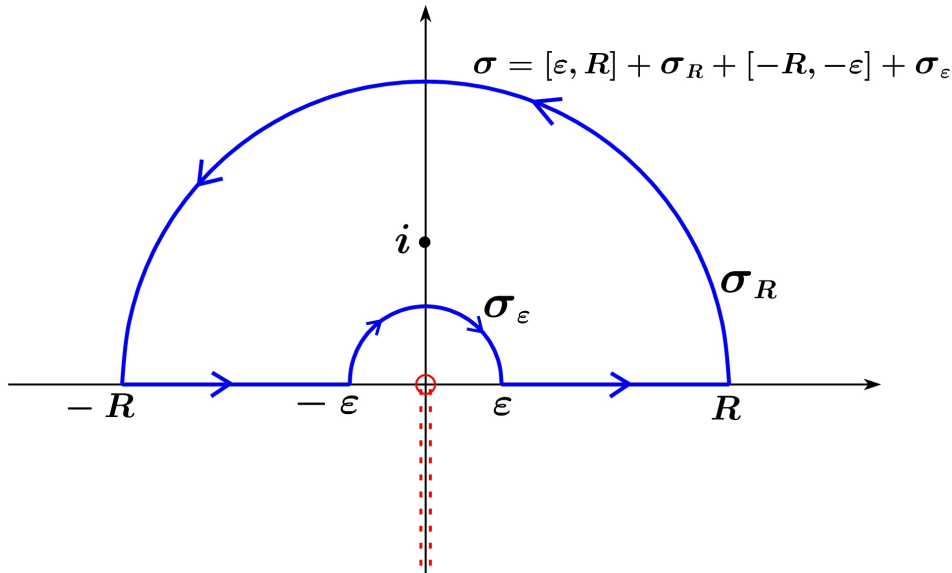


Fig. 7.7: The path σ bypasses the singular point $z = 0$ and the slit corresponding to the branch $[-\frac{\pi}{2}, \frac{3\pi}{2}]$

🔴 **Solution:** Define a complex function

$$f(z) = \frac{z^{a-1}}{1+z^2}$$

where the logarithm function in the expression $z^{a-1} = e^{(a-1)\log z}$ is based on the branch $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ (whose slit, drawn in red in Figure 7.7, does not intersect the path σ). Assume that $R > 1$, $\varepsilon < 1$, and therefore the simple pole $z = i$ of $f(z)$ lies inside the path σ . By the residue theorem,

$$\begin{aligned} \oint_{\sigma} f(z) dz &= \int_{[\varepsilon, R]} f(z) dz + \int_{\sigma_R} f(z) dz + \int_{[-R, -\varepsilon]} f(z) dz + \int_{\sigma_{\varepsilon}} f(z) dz \\ &= 2\pi i \cdot \text{Res}(f, i) = 2\pi i \cdot \frac{z^{a-1}}{z+i} \Big|_{z=i} \\ &= 2\pi i \cdot \frac{(2i)^{a-1}}{2i} = \pi e^{(a-1)\log(2i)} = \pi e^{\frac{(a-1)\pi i}{2}} \end{aligned}$$

We estimate the magnitude of each of the four integrals.

$$\left| \int_{\sigma_R} \frac{z^{a-1}}{1+z^2} dz \right| \leq \int_{\sigma_R} \frac{|z|^{a-1}}{|z|^2-1} \leq \pi R \cdot \frac{R^{a-1}}{R^2-1} = \frac{\pi R^a}{R^2-1} \xrightarrow{R \rightarrow \infty} 0$$

Similarly, the integral over the small circle tends to zero as $\varepsilon \rightarrow 0$.

$$\begin{aligned} \left| \int_{\sigma_{\varepsilon}} \frac{z^{a-1}}{1+z^2} dz \right| &= \int_{\sigma_{\varepsilon}} \frac{\varepsilon^{a-1}}{|1+z^2|} dz \leq \int_{\sigma_{\varepsilon}} \frac{\varepsilon^{a-1}}{1-\varepsilon^2} dz \\ &\leq \pi \varepsilon \cdot \frac{\varepsilon^{a-1}}{1-\varepsilon^2} = \frac{\pi \varepsilon^a}{1-\varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

Two integrals over the real intervals remain.

$$\begin{aligned} \int_{[\varepsilon, R]} \frac{z^{a-1}}{1+z^2} dz &= \int_{\varepsilon}^R \frac{x^{a-1}}{1+x^2} dx \quad (x \in \mathbb{R}) \\ \int_{[-R, -\varepsilon]} \frac{z^{a-1}}{1+z^2} dz &= \int_{-R}^{-\varepsilon} \frac{e^{(a-1)(\ln(-x)+i\pi)}}{1+x^2} dx = \int_{\varepsilon}^R \frac{e^{(a-1)(\ln(x)+i\pi)}}{1+x^2} dx \\ &= \int_{\varepsilon}^R \frac{x^{a-1} \cdot e^{(a-1)\pi i}}{1+x^2} dx = e^{(a-1)\pi i} \int_{\varepsilon}^R \frac{x^{a-1}}{1+x^2} dx \end{aligned}$$

In summary, we have obtained that as $R \rightarrow \infty$, $\varepsilon \rightarrow 0$,

$$\begin{aligned} \oint_{\sigma} \frac{z^{a-1}}{1+z^2} dz &= \int_{\varepsilon}^R \frac{x^{a-1}}{1+x^2} dx + e^{(a-1)\pi i} \int_{\varepsilon}^R \frac{x^{a-1}}{1+x^2} dx \\ &= (1 + e^{(a-1)\pi i}) \int_{\varepsilon}^R \frac{x^{a-1}}{1+x^2} dx = \pi e^{\frac{(a-1)\pi i}{2}} \end{aligned}$$



Therefore

$$\int_{\varepsilon}^R \frac{x^{a-1}}{1+x^2} dx = \frac{\pi e^{\frac{(a-1)\pi i}{2}}}{1 + e^{(a-1)\pi i}} = \frac{\pi}{e^{-\frac{(a-1)\pi i}{2}} + e^{\frac{(a-1)\pi i}{2}}} = \frac{\pi}{2 \cos \frac{(a-1)\pi}{2}} = \frac{\pi}{2 \sin \frac{\pi a}{2}}$$

- ◆ **Example 21:** (a) Find the poles (including their orders) of the function $f(z) = \frac{1}{(e^z + e^{-z})^3}$.
 (b) Compute the residue of the function at each of the poles you found in the preceding part.
 (c) Compute the integral $\int_{-\infty}^{\infty} \frac{dx}{(e^x + e^{-x})^3}$.

🔴 **Solution:**

(a) The poles of $f(z)$ are the roots of the equation

$$e^z + e^{-z} = 0$$

Multiplying both sides by e^z , we obtain $e^{2z} + 1 = 0$, and therefore

$$e^{2z} = -1 = e^{(2k+1)\pi i}, \quad k \in \mathbb{Z}$$

and we obtain infinitely many poles: $z = \frac{(2k+1)\pi i}{2}$. We will be interested only in the following two poles: $z_1 = \frac{\pi i}{2}$, $z_2 = -\frac{\pi i}{2}$.

(b) We will treat only the pole z_1 , since it is relevant to the next part. The treatment of the second pole is similar. By Proposition 7.14, it is enough to find a natural number m such that the limit $\lim_{z \rightarrow z_1} (z - z_1)^m f(z) \neq 0$ exists. It is not difficult to guess that $m = 3$ will work. We use L'Hôpital's rule to compute the limit:

$$\begin{aligned} \lim_{z \rightarrow z_1} \frac{(z - z_1)^3}{(e^z + e^{-z})^3} &= \lim_{z \rightarrow z_1} \frac{3(z - z_1)^2}{3(e^z + e^{-z})^2(e^z - e^{-z})} \\ &= \lim_{z \rightarrow z_1} \frac{6(z - z_1)}{6(e^z + e^{-z})(e^z - e^{-z})^2 + 3(e^z + e^{-z})^3} \\ &= \lim_{z \rightarrow z_1} \frac{6}{6(e^z - e^{-z})^3 + 12(e^z + e^{-z})^2(e^z - e^{-z}) + 9(e^z + e^{-z})^2(e^z - e^{-z})} \\ &= \frac{6}{6 \cdot (\frac{i}{2} + \frac{i}{2}) + 0 + 0} = \frac{6}{12i} = -\frac{i}{2} \end{aligned}$$

In the last step, we used the equality $e^{z_1} + e^{-z_1} = 0$ and the equality $e^{z_1} - e^{-z_1} = 2i$.

We have obtained that the function $g(z) = (z - z_1)^3 f(z)$ satisfies the conditions of Proposition 7.14, and therefore

$$\text{Res}(f, z_1) = \lim_{z \rightarrow z_1} \frac{g(2)(z)}{2!}$$



We compute $g^{(2)}(z)$:

$$\begin{aligned} \frac{d^2}{dz^2} \left[\frac{(z - z_1)^3}{(e^z + e^{-z})^3} \right] &= \frac{d}{dz} \left[\frac{3(z - z_1)^2(e^z + e^{-z})^3 - 3(z - z_1)^3(e^z + e^{-z})^2(e^z - e^{-z})}{(e^z + e^{-z})^6} \right] \\ &= \frac{d}{dz} \left[\frac{3(z - z_1)^2}{(e^z + e^{-z})^3} - \frac{3(z - z_1)^3(e^z - e^{-z})}{(e^z + e^{-z})^4} \right] \\ &= \frac{3(z - z_1)}{(e^z + e^{-z})^3} \left[\frac{4(z - z_1)^2(e^z - e^{-z})^2}{(e^z + e^{-z})^2} - (z - z_1)^2 - \frac{6(z - z_1)(e^z - e^{-z})}{e^z + e^{-z}} + 2 \right] \end{aligned}$$

With a little effort, one can obtain

$$\lim_{z \rightarrow z_1} \frac{d^2}{dz^2} \left[\frac{(z - z_1)^3}{(e^z + e^{-z})^3} \right] = -\frac{i}{8}$$

and therefore

$$\text{Res}(f, z_1) = \lim_{z \rightarrow z_1} \frac{g^{(2)}(z)}{2!} = -\frac{i}{16}$$

(c) We use the residue theorem and a rectangular path σ shown in Figure 7.8.

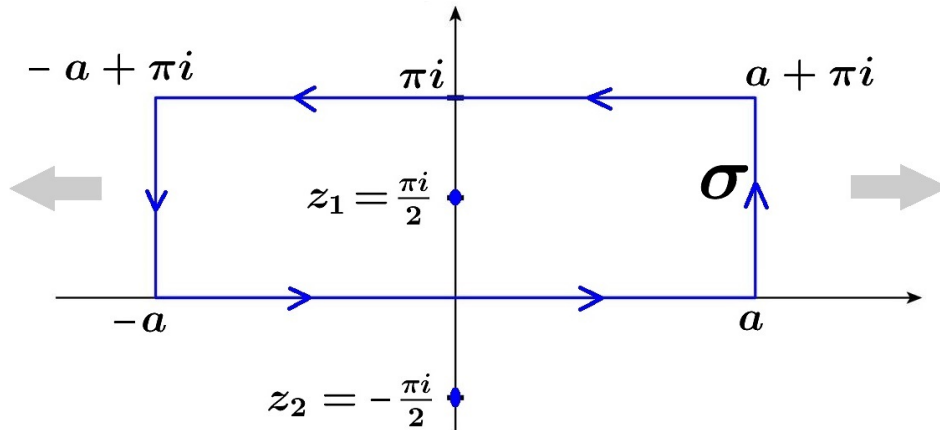


Fig. 7.8: A rectangular path around the pole $z_1 = \frac{\pi i}{2}$. As a tends to infinity, an infinite strip of height π is obtained

Let $a > 0$ be real.

$$\sigma = [-a, a] + [a, a + \pi i] + [a + \pi i, -a + \pi i] + [-a + \pi i, -a]$$

Denote the four integrals along the path σ by

$$I_1 = \int_{-a}^a f(z) dz, \quad I_2 = \int_a^{a+\pi i} f(z) dz, \quad I_3 = \int_{a+\pi i}^{-a+\pi i} f(z) dz, \quad I_4 = \int_{-a+\pi i}^{-a} f(z) dz$$

Therefore

$$\oint_{\sigma} f(z) dz = I_1 + I_2 + I_3 + I_4 = 2\pi i \text{Res}(f, z_1) = 2\pi i \cdot -\frac{i}{16} = \frac{\pi}{8}$$

We compute the value of each integral as $a \rightarrow \infty$.

$$I_1 = \int_{-a}^a f(z) dz = \int_{-a}^a \frac{dx}{(e^x + e^{-x})^3} \xrightarrow{a \rightarrow \infty} \int_{-\infty}^{\infty} \frac{dx}{(e^x + e^{-x})^3}$$

The integral along the right side of the rectangle: $z = a + ti$, $t \in [0, \pi]$.

$$I_2 = \int_a^{a+\pi i} \frac{dz}{(e^z + e^{-z})^3} = \int_a^{a+\pi i} \frac{dt}{(e^{a+ti} + e^{-a-ti})^3} = \int_a^{a+\pi i} \frac{dt}{(e^a e^{ti} + e^{-a} e^{-ti})^3}$$

By the triangle inequality,

$$|e^a e^{ti} + e^{-a} e^{-ti}| \geq |e^a e^{ti}| - |e^{-a} e^{-ti}| = e^a - e^{-a}$$

and therefore Similarly, we prove that the integral along the left side of the rectangle also tends to zero.

$$|I_2| \leq \int_a^{a+\pi i} \left| \frac{1}{(e^a e^{ti} + e^{-a} e^{-ti})^3} \right| dt \leq \int_a^{a+\pi i} \frac{1}{(e^a - e^{-a})^3} dt = \frac{a}{(e^a - e^{-a})^3} \xrightarrow[a \rightarrow \infty]{\text{L'Hôpital}} 0$$

Similarly, we prove that the integral along the left side of the rectangle also tends to zero.

$$|I_4| \leq \int_{-a+\pi i}^{-a} \left| \frac{1}{(e^{-a} e^{ti} + e^a e^{-ti})^3} \right| dt \leq \int_{-a+\pi i}^{-a} \frac{1}{(e^a - e^{-a})^3} dt = \frac{a}{(e^a - e^{-a})^3} \xrightarrow[a \rightarrow \infty]{\text{L'Hôpital}} 0$$

It remains to examine the integral along the upper side of the rectangle: $z = t + \pi i$, $t \in [\pi, -\pi]$.

$$I_3 = \int_{a+\pi i}^{-a+\pi i} \frac{dz}{(e^z + e^{-z})^3} = \int_a^{-a} \frac{dt}{(e^{t+\pi i} + e^{-t-\pi i})^3} = \int_a^{-a} \frac{dt}{(-e^t - e^{-t})^3} = \int_{-a}^a \frac{dt}{(e^t + e^{-t})^3} = I_1$$

In summary, we have obtained that

$$\frac{\pi}{8} = \lim_{a \rightarrow \infty} [I_1 + I_2 + I_3 + I_4] = \lim_{a \rightarrow \infty} 2 \int_{-a}^a \frac{dt}{(e^t + e^{-t})^3} = 2 \int_{-\infty}^{\infty} \frac{dt}{(e^t + e^{-t})^3}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{dt}{(e^t + e^{-t})^3} = \frac{\pi}{16}$$



7.6 Software Tools and Applications

The following list gives sources for applets (applications for cellular phones) and software tools available online that can be used to plot graphs of functions and equations, and to perform symbolic computations such as evaluating integrals and solving equations.

-  [GeoGebra Graphing Calculator](#)
-  [GeoGebra Neat Examples](#)
-  [Desmos Graphing Calculator](#)
-  [Desmos - Android App](#)
-  [3D Plot Grapher - Android App](#)
-  [CalcPlot3D](#)
-  [3D Surface Plotter](#)
-  [3D Surface Plotter - Android App](#)
-  [WolframAlpha Computational Intelligence](#)
-  [Symbolab solver: Algebra, Calculus, Trigonometry, Equations Solver](#)
-  [Matlab Online](#)
-  [Sympy Live \(Python Programming\)](#)
-  [Plotly Matplotlib Library](#)
-  [MIT Tech tools for teaching and learning](#)

- This is only a partial list, and many more excellent software tools and applications for studying differential calculus, complex functions, and mathematics in general can be found on the Internet. We would be very pleased to receive additional suggestions that should be included in the above list.
- The purpose of the present section is to provide a small taste of the great abundance of software tools and code available on the Internet for solving various problems concerning complex functions. As needed, additional software tools and code packages can be found for nearly every topic covered in this booklet by skillfully using Google's search box.
- Users of the **Python** programming language are advised to use the **SymPy** web server, which can be reached through the following link:

<http://live.sympy.org>

- ❖ **Example 22:** For example, we compute the integral in Example [14](#) and Exercise [10](#) at the end of the chapter. Although we obtain only the final answer, it is useful for verifying the full solution.

```

from sympy import *
    t = Symbol('t')
f1 = 1/(5+4*cos(t))
integrate(f1, (t,0,pi))
=> pi/3
f2 = 1/(13+5*cos(t))
integrate(f2, (t,0,2*pi))
=> pi/6

```

❖ **Example 23:** The **SymPy** package also allows symbolic computation of improper integrals. For example, we compute the integrals from Examples [16](#), [17](#).

```

from sympy import *
    x = Symbol('x')
f1 = 1/(x**4+4)
integrate(f1, (x,0,oo))
=> pi/8
f2 = sin(x)/(x**2+4*x+5)
Integral(f2, (x, -oo, oo)).evalf()
=> -1.0509
float(-pi*sin(2)/np.e)
=> -1.0508999052769497

```

❖ **Example 24:** Additional examples from the exercises at the end of the chapter:

```

from sympy import *
f = 1/((x**2+1)*(x**2+4))
integrate(f, (x,-oo,oo))
=> pi/6
f = (x**4+1)/(x**6 + 1)
integrate(f, (x,0,oo))
=> 2*pi/3
f = (x*sin(2*x))/(x**2 + 9)
integrate(f, (x,0,oo))
=> sqrt(pi)*(-sqrt(pi)*sinh(6) + sqrt(pi)*cosh(6))/2
r = integrate(f, (x,0,oo))

```



```
simplify(r)
=> pi*exp(-6)/2
f = (1-cos(x))/(x**2)
integrate(f, (x,0,oo))
=> pi/2
f = sin(x**2)
integrate(f, (x,0,oo))
=> sqrt(2)*sqrt(pi)/4
f = cos(x**2)
integrate(f, (x,0,oo))
=> sqrt(2)*sqrt(pi)/4
```

❖ **Example 25:** The **SymPy** package also allows computation of polynomial roots over the field of complex numbers:

```
f = z**6-5*z**4+3*z**2-1
roots = solve(f, z)
[abs(r)<1 for r in roots]
=> [False, False, True, True, True, True]
[complex(r) for r in roots]
=> [(-2.0893132875215477+0j),
    (2.0893132875215477+0j),
    (-0.6308768323858478+0.2839376416543773j),
    (-0.6308768323858478-0.2839376416543773j),
    (0.6308768323858478+0.2839376416543773j),
    (0.6308768323858478-0.2839376416543773j)]
=> 4 roots in disk |z|<1
f = z**9 +5*z**2 + 3
roots = solve(f, z)
Roots = [complex(r) for r in roots]
f = z**9 +5*z**2 + 3
[r.real>0 and r.imag>0 for r in Roots]
=> [False, False, False, False, False, False, True, False, True]
=> Only 2 root in quadrant I
```



❖ **Example 26:** To compute the residues, poles, and partial-fraction decomposition of a rational function $\frac{P(z)}{Q(z)}$, the **scipy** package must be used. A polynomial $P(z) = a_0z^n + a_1z^{n-1} + \dots + a_n$ is represented by an array $(a_0, a_1, \dots, a_n)$. For example, the polynomial $z^2 + 1$ is represented by the array $(1, 0, 1)$, and the polynomial $z^4 + 1$ is represented by the array $(1, 0, 0, 0, 1)$. Therefore the residues and poles of the rational function $\frac{z^2+1}{z^4+1}$ are computed by the following code:

```
from scipy.signal import residue
residue((1,0,1), (1,0,0,0,1))
=>
(array([-5.88784672e-17-0.35355339j, -5.88784672e-17+0.35355339j,
        9.81307787e-17-0.35355339j, 9.81307787e-17+0.35355339j]),
 array([ 0.70710678+0.70710678j, 0.70710678-0.70710678j,
        -0.70710678+0.70710678j, -0.70710678-0.70710678j]),
 array([], dtype=float64))
```

The result consists of three arrays: the first array is the list of the residues corresponding to each of the poles. The second array is the list of poles, and the third array is the polynomial obtained as the quotient of the division. From the result we learn that there are four simple poles, and that the quotient is zero, because the degree of the polynomial in the numerator is smaller than the degree of the polynomial in the denominator.

❖ **Example 27:** Using the **matplotlib** graphics package, one can plot four-dimensional graphs of complex functions. The x -axis and y -axis represent the real and imaginary parts of the argument $z = x + iy$. The z -axis represents the real part of the function value $\text{Re}(f(z))$, and the imaginary part $\text{Im}(f(z))$ is represented by a color from a color map that can be specified in advance. Figure 7.9 shows the graph of $f(z) = \sin(z)$.

The **Python** code needed to create the graph is

```
import matplotlib.pyplot as plt # graphing module
from matplotlib import cm      # colormap module
import numpy as np             # numeric module

fig = plt.figure()
ax = fig.gca(projection='3d')
a = 4 ; b = 2 ; dx = 0.05 ; dy = 0.05
X = np.arange(-a*3.14, a*3.14, dx)
Y = np.arange(-b, b, dy)
X, Y = np.meshgrid(X, Y)
```



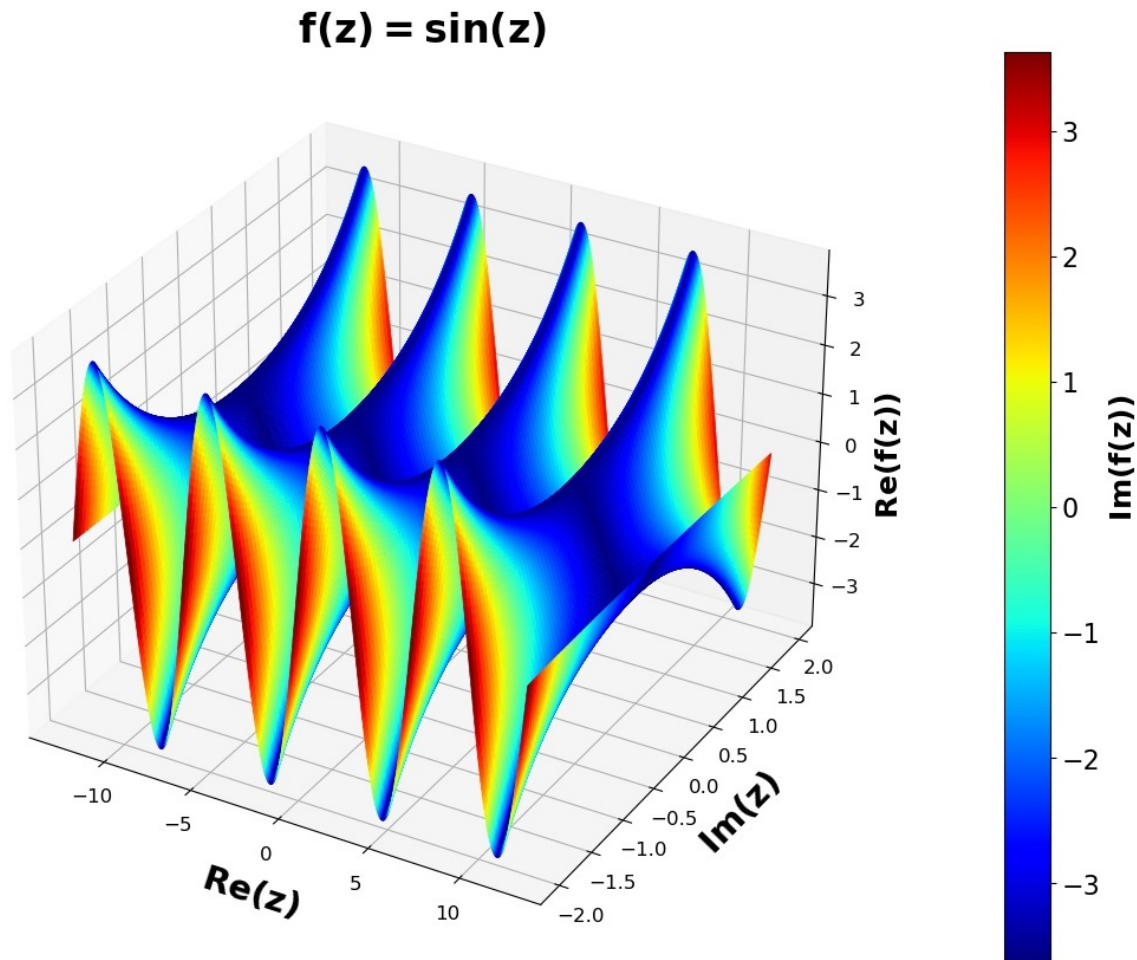


Fig. 7.9: A “four-dimensional” graph of $f(z) = \sin z$; the “fourth dimension” $\text{Im}(f(z))$ is represented by color

```
R = np.sin(X + 1j*Y)
Z = R.real
T = R.imag

N = np.abs(T/T.max()) # normalize 0..1
plt.title('  $\mathbf{f(z)=sin(z)}$  ', fontsize=18)
plt.xlabel('  $\mathbf{Re(z)}$  ', fontsize=18)
plt.ylabel('  $\mathbf{Im(z)}$  ', fontsize=18)
surf = ax.plot_surface(
    X, Y, Z, rstride=1, cstride=1,
    facecolors=cm.jet(N),
    linewidth=0.5, edgecolors='0.5',
    antialiased=True, shade=False,
)
m = cm.ScalarMappable(cmap=cm.jet, norm=surf.norm)
m.set_array(T)
p = plt.colorbar(m)
```



```
p.set_label(' $\mathbf{Im}(f(z))$', fontsize=18)
fig.set_size_inches(12,10)
plt.savefig('sine1.jpg', bbox_inches='tight')
plt.show()
```

- Additional examples of **Python** code for complex functions can be found on the Internet by entering suitable words in Google's search box. At the beginning of the section, we presented several examples of programs and applications for performing symbolic computations, which are developed and updated daily, and therefore only a good search can yield up-to-date results.

7.7 Finished, but Not Complete

This course booklet was published in May 2021 (in Hebrew), and since then it has undergone quite a few reforms, changes, and corrections. Thanks to all the students who sent comments, ideas, and suggestions for improving the earlier versions. Further effort will probably be required to improve and refine the booklet. We would be very pleased to receive comments, corrections, and suggestions for improvement (the email address appears at the bottom of every page). Full solutions to exercises will be gratefully received, and if enough of them accumulate, they will be published in a suitable solutions booklet.

Exercises for chapter 7

1. Let $f(z)$ be an analytic function on a domain D , with finitely many zeros in the domain D (counted according to multiplicity). Prove that there exist an analytic function $g(z)$ with no zeros in the domain D , and a polynomial $P(z)$, such that $f(z) = g(z)P(z)$ for every $z \in D$.
2. Let $f(z)$ be an analytic function on a domain D , and let $z_0 \in D$. Assume that $f^{(n)}(z_0) = 0$ for every $n \geq 1$. Prove that f is a constant function on D .
3. Let $f(z)$ be an entire function satisfying $|f(z)| \leq a + b|z|^n$ for every $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most n .



4. Let $f: \mathbb{C} \rightarrow \mathbb{C}_\infty$ be a meromorphic function with finitely many poles, and assume that there exist real constants $c, R > 0$ and a natural number n such that $|f(z)| \leq c|z|^n$ for every $|z| > R$. Prove that f must be a rational function.

Hint: Note that the range of the function is $\mathbb{C}_\infty$, and therefore at pole points z , $f(z) = \infty$. Try to use Exercise 3.



5. For each of the following functions, find its singular points and their types. Write the principal part of the function at each singular point you found, and compute the residue at each singular



point.

$$\begin{array}{llll} \text{(a)} \frac{1}{\cos^2 z} & \text{(b)} (1 - z^3)e^{1/z} & \text{(c)} \frac{\sin z}{z^{2021}} & \text{(d)} \frac{e^z}{1 - z^2} \\ \text{(e)} \frac{1}{\sin^2 z} & \text{(f)} \frac{1 - \cos z}{z^2} & \text{(g)} \frac{e^z}{z(z - 1)^2} & \text{(h)} (1 - z^2) \sin \frac{1}{z} \\ \text{(i)} \frac{e^z}{z^{2021}} & \text{(j)} \frac{\cos z}{z^2 - z^3} \end{array}$$

6. Let $f(z) = \frac{\sin z}{\cos^3 z - 1}$. Find the type of singularity at the point $z = 0$ and compute the residue of f at the point $z = 0$.

7. Prove that the point $z = 0$ is a nonisolated singular point of $f(z) = \frac{1}{\sinh^2(\frac{1}{z})}$.

8. Prove that the series $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z)}$ converges to a meromorphic function; find its set of poles and the order of each pole.

9. Consider the function $f(z) = \frac{1}{(2 \cos z - 2 + z^2)^2}$. Prove that $z = 0$ is a pole of f , and find its order.

Hint: It is preferable to work with the function $1/f(z)$.

10. Compute the integral $\int_0^{2\pi} \frac{dt}{13 + 5 \cos t}$.

Answer: $\frac{\pi}{6}$



11. Let z_0 be an isolated singular point of $f(z)$ that is not removable. Prove that z_0 is an essential singular point of $e^{f(z)}$.

12. Let z_0 be an isolated singular point of $f(z)$. Suppose that the function $g(z) = (z - z_0)^n f(z)$ is bounded in a punctured neighborhood of z_0 . Prove that z_0 is a removable singular point of f or a pole of order n of f .



13. Prove: If z_0 is a pole of order n of $f(z)$, then z_0 is a pole of order $n + 1$ of $f'(z)$.



14. Let $f(z)$ be a function analytic on $\mathbb{C}$ except at the natural numbers $z = n$. Suppose that f is bounded. Prove that f must be a constant function.

Hint: Exercise 12. Proposition 7.10 on page 281 may also help.



15. Does there exist an analytic function satisfying $f(\frac{1}{2n}) = f(\frac{1}{2n+1}) = \frac{1}{2n}$ for every natural number n ?

Hint: Review Exercise 7.1 at the beginning of the chapter.



16. Give an example of a function $f(z)$ that has a removable singular point at $z = 0$, a pole of order 5 at $z = -i$, and an essential singular point at $z = 3$.



17. Give an example of a function $f(z)$ that has a pole of order 2 at $z = -i$, and two essential



singular points, at $z = i$ and at $z = 2$.

18. :elpmaxeretnuoc a yb evorpsid ro evorP If z_0 is a pole of $f(z)$ and an essential singular point of $g(z)$, then z_0 is a pole of $f(z)g(z)$.



19. :elpmaxeretnuoc a yb evorpsid ro evorP Suppose that $f(z)$ is analytic in the punctured closed disk $D: 0 < |z| \leq 1$. If the function $z^n f(z)$ is unbounded on D for every natural number n , then $z = 0$ is an essential singular point of f .



20. Let $f(z)$ be an entire function. Suppose that for every point $z \in \mathbb{C}$ there exists a natural number n such that $f^{(n)}(z) = 0$. Prove that f is a polynomial.

Hint: Show that inside the unit disk there are infinitely many distinct points z_k for which $f^{(n)}(z_k) = 0$ for the same natural number n . The pigeonhole principle of set theory should be used. The identity theorem should be used (Theorem 7.6).

21. Let $f(z)$ be an entire function satisfying $f(n) = n$ for every natural number n , and, in addition,

$$\lim_{|z| \rightarrow \infty} |f(z)| = \infty$$

Prove that $f(z) = z$.



22. Prove that for every pair of real numbers $a, b \in \mathbb{R}$ satisfying $a > |b|$,

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{a^2 - b^2}$$

23. If $f(z)$ is an entire function that has no limit at infinity, then $z = 0$ is an essential singular point of the function $g(z) = f(\frac{1}{z})$.

Hint: Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of $f(z)$ about the point $z = 0$. Then $\sum_{n=0}^{\infty} a_n z^{-n}$ is the Laurent series of $f(\frac{1}{z})$.

The nonexistence of a limit at infinity implies that $f(z)$ is not a polynomial (see Proposition 3.18 on page 102). Therefore infinitely many coefficients a_n do not vanish. See also Proposition 7.9 on page 280.

24. Let $f: \mathbb{C} - \overline{\mathbb{D}}(0, r) \rightarrow \mathbb{C}$ be a function analytic on the plane with a hole $|z| > r$. We say that the point $z = \infty$ is a zero of multiplicity m of f if the point $z = 0$ is a pole of order m of the function $g(z) = f(1/z)$. Prove that in this case the Laurent series of f about $z = 0$ is a series consisting only of negative powers, $\sum_{n=-m}^{\infty} a_n z^n$, such that $a_{-m} \neq 0$, and that the series converges on the entire plane with a hole $|z| > r$.

Illumination: This exercise, together with Exercise 82, illuminates an interesting duality between the point $z = 0$ and the point $z = \infty$. In a deep sense, these two points are mirror images of one another under the “reflection” function $1/z$. In this sense, a Laurent series is the “mirror image” of a Maclaurin series, and a pole is the mirror image of a zero. It is recommended to solve these two exercises together.



25. Prove: There is no analytic function $f(z)$ in the punctured unit disk $0 < |z| < 1$ such that its derivative $f'(z)$ has a simple pole at the point $z = 0$.

Hint: First solve Exercise 37 in Chapter 4.



26. Prove that $\oint_{|z|=4} \frac{e^z dz}{(z^2 + \pi^2)^2} = \frac{i}{\pi}$.

27. Compute the following integrals:

(a) $\oint_{|z|=3} \frac{e^z}{\sinh z} dz$ (b) $\oint_{|z|=3} \frac{e^z}{(z+2)^2 \sin z} dz$ (c) $\oint_{|z-\pi|=1} \frac{e^z}{\sin z \cdot \cos z} dz$



28. Compute the integral

$$\oint_{|z|=2} \frac{z^{2021} dz}{z^{2021} + z^{2020} + z^{2019} + 1}$$

Hint: Prove that all the zeros of $z^{2021} + z^{2020} + z^{2019} + 1$ lie inside the circle $|z| = 2$. Therefore, if we replace the path $|z| = 2$ in the integral by any path $|z| = R$, $R > 2$, the value of the integral will not change. Let R tend to infinity.

29. Prove that

$$\oint_{|z|=1} e^{\frac{z+1}{z}} dz = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$$

30. Use the residue theorem to provide a shorter solution to Exercise 39 in Chapter 6.

Hint: Prove that $\text{Res}\left(\frac{1}{f(z)-f(z_0)}, z_0\right) = \frac{1}{f'(z_0)}$.

31. Prove that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$.

Hint: Use the residue theorem on an upper semicircle. There are two simple poles in the upper half-plane: $z = i, 2i$. The integral over the upper semicircle tends to zero by a simple estimate of the path integral.

32. Assume that a meromorphic function $f(z)$ has an isolated singular point z_0 . Prove that z_0 is also an isolated singular point of $f'(z)$, and compute $\text{Res}(f', z_0)$.



33. Compute the following integrals:

(a) $\int_{-\infty}^{\infty} \frac{\cos x}{x^4+1} dx$ (b) $\int_0^{\infty} \frac{x^4+1}{x^6+1} dx$ (c) $\int_0^{\infty} \frac{x^2}{x^6+1} dx$
 (d) $\int_0^{\infty} \frac{x \sin 2x}{9+x^2} dx$ (e) $\int_0^{\infty} \frac{1-\cos x}{x^2} dx$ (f) $\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)(x+4)} dx$

Answer: (b) $\frac{2\pi}{3}$, (c) $\frac{\pi}{6}$, (d) $\frac{\pi}{2e^6}$, (e) $\frac{\pi}{2}$, (f) $\frac{\pi}{6}$

34. Prove that $\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1} dx = \sqrt{2}\pi$.

35. Compute the pair of integrals $\int_0^{\infty} \sin(x^2) dx$, $\int_0^{\infty} \cos(x^2) dx$.

Hint: Compute the integral $\oint_{\sigma} e^{iz^2} dz$ along the boundary of the sector. $0 \leq \text{Arg}(z) \leq \frac{\pi}{4}$, $|z| \leq r$. **Answer:** $\frac{\sqrt{2}\sqrt{\pi}}{4}$.



36. Prove that for every natural number n ,

$$\frac{1}{2\pi i} \oint_{|z|=n} \frac{dz}{z^2 \sin z} = \frac{1}{6} + 2 \sum_{k=1}^n \frac{(-1)^k}{k^2 \pi^2}$$



Deduce from this the following summation formula:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$$



37. The function $f(z) = \frac{\cot \pi z}{z^2 + a^2}$, where $a \neq 0$ is real, is given.

(a) Find the poles of the function.

(b) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2}$.

Hint: $f(z) = \frac{\cos \pi z}{(z^2 + a^2) \sin \pi z}$. There are infinitely many poles. Apply the residue theorem to a rectangle $[-N - 0.5, N + 0.5] \times [-b, b]$ that encloses finitely many poles, and enlarge it to infinity. **Answer:** (a) $\pm a, n \in \mathbb{Z}$, simple poles.

38. Compute the integral $\int_0^{\infty} \frac{dx}{1+x^n}$ for every natural number $n \geq 2$.

Hint: Use the path shown in Figure 7.10. **Answer:** $\frac{\pi/n}{\sin(\pi/n)}$.

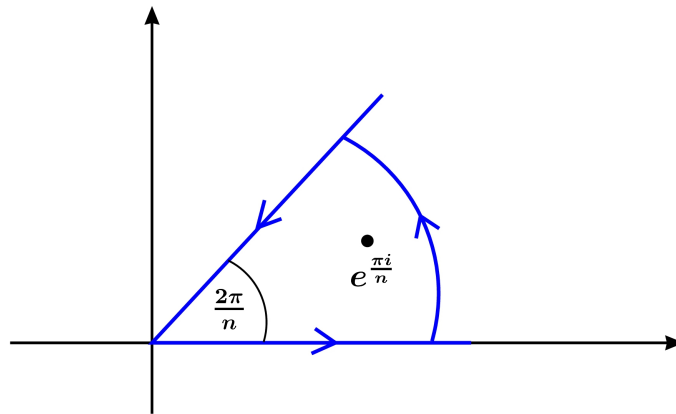


Fig. 7.10: The path σ for Exercise 38

39. Compute the integral

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)}$$

Answer: π

Hint: Use the path σ shown in Figure 7.11, where $0 < r < 1 < R$. Compute the integral of $f(z) = \frac{1}{z^{1/2}(1+z)}$ along the path σ . Choose an analytic branch of $z^{1/2}$ in the slit plane $\mathbb{C} - [0, \infty)$. Examine what happens as $R \rightarrow \infty$, $r \rightarrow 0^+$, and the angle $\varepsilon \rightarrow 0^+$. The function $f(z)$ has a simple pole at the point $z = -1$.



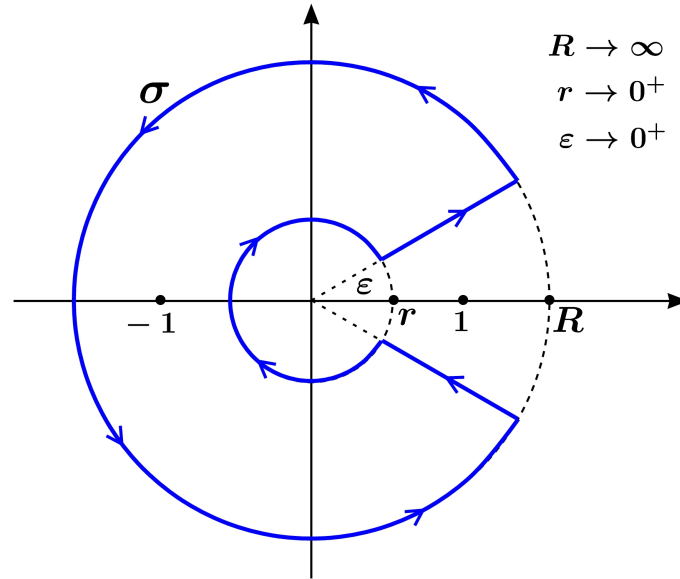
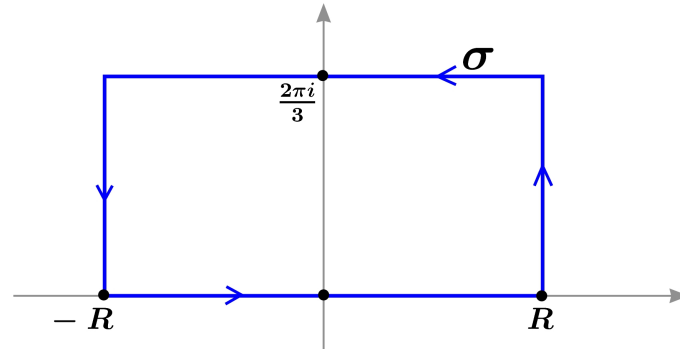
40. Compute the integral

$$\int_{-\infty}^{\infty} \frac{e^x dx}{1+e^{3x}}$$

Answer: $\frac{2\pi}{3\sqrt{3}}$

Hint: Use the path σ shown in Figure 7.12. Compute an integral of $f(z) = \frac{e^z}{1+e^{3z}}$ along the path σ . Show that the integral along the three upper sides of the rectangle tends to zero as $R \rightarrow \infty$.



Fig. 7.11: The path σ for Exercise 39Fig. 7.12: The path σ for Exercise 40

41. For every natural number $n \geq 2$ and every real number $a > 0$, compute the integral

$$\int_0^\infty \frac{\sqrt[n]{x} dx}{x^2 + a^2}$$

Hint: Use a path similar to the one in Figure 7.6. Choose an appropriate analytic branch of $f(z) = z^{1/n}$.

42. Prove that $\int_0^\infty x^{4n+3} e^{-x} \sin x dx = 0$, for $n = 0, 1, 2, 3, \dots$

Hint: Use a path integral along the boundary of the sector $0 \leq |z| \leq r$, $0 \leq \text{Arg}(z) \leq \pi/4$.



43. Prove that for every positive natural number n ,

$$\int_0^{2\pi} e^{\cos \theta} \cos(n\theta - \sin \theta) d\theta = \frac{2\pi}{n!}$$

Hint: After reaching the breaking point, you may google: Whittaker and Watson, Modern Analysis, 6.21.





44. Prove that for every complex number $z \in \mathbb{C}$, if $\operatorname{Re}(z) > 0$, then

$$\int_0^\infty (e^{-t} - e^{-tz}) \frac{dt}{t} = \log z$$

Hint: After reaching the breaking point, you may google: Whittaker and Watson, *Modern Analysis*, 6.22.



45. Let $\sum_{n=0}^\infty a_n z^n$ be a power series with radius of convergence $R = 1$, such that all the points on the circle $|z| = 1$ at which it does not converge are simple poles. Prove: The sequence of coefficients $\{a_n\}_{n=0}^\infty$ is bounded.



46. Let $f(z)$ be an entire function that takes only real values. Prove that f must be a constant function.

47. Let $f(z)$ be a function analytic in the unit disk $|z| < 1$, such that $|f(z)| \leq 1$ for every z in the unit disk. Prove: If $z = 0$ is a zero of multiplicity m of f , then $|f(z)| \leq |z|^m$.



48. Assume that the function $f(z)$ has only one zero z_0 , of multiplicity 1, inside a circle σ . Prove that

$$z_0 = \frac{1}{2\pi i} \oint_\sigma \frac{z f'(z)}{f(z)} dz$$

49. Let σ be a positive path around the right half of the disk $|z - 1| = 3$ (see Figure 7.13). Let $f(z)$ be an entire function with no zeros on the path σ , satisfying

$$\frac{1}{2\pi i} \oint_\sigma \frac{z f'(z)}{f(z)} dz = 3, \quad \frac{1}{2\pi i} \oint_\sigma \frac{z^2 f'(z)}{f(z)} dz = \frac{5}{2}$$

Find the zeros of f inside the path σ .

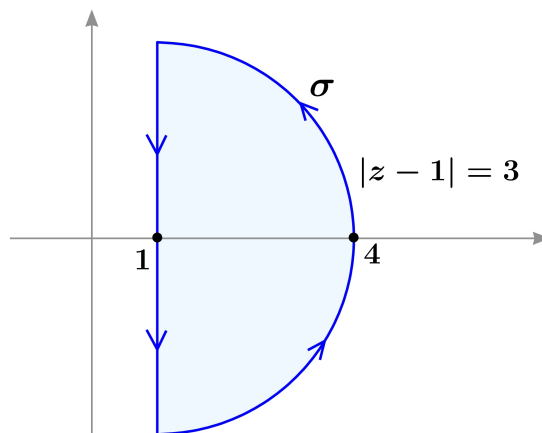


Fig. 7.13: The path σ for Exercise 49

50. Let D be a domain containing the closed unit disk $|z| \leq 1$, and let f be a meromorphic function having finitely many simple poles $z_1, z_2, \dots, z_k$ on the unit circle $|z| = 1$. Let



$f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Maclaurin series of f in the domain $|z| < 1$. Prove that there exists a real number $M > 0$ such that $|a_n| \leq M$ for every n .



51. Let $P(z)$ and $Q(z)$ be two nonconstant polynomials such that $\deg(P) \leq \deg(Q) - 2$, and such that all the zeros of Q are simple. Prove that

$$\sum_{Q(z)=0} \frac{P(z)}{Q'(z)} = 0$$

This is a finite sum: the variable z ranges only over the zeros of Q .

Hint: Use the residue theorem on a circle $|z| = R$, for sufficiently large R , and estimate the magnitude of the integral $\oint_{|z|=R} \frac{P(z)}{Q(z)} dz$ as $R \rightarrow \infty$.

52. Compute the number of zeros of $f(z) = 3e^z - z$ in the closed unit disk $|z| \leq 1$.
53. Compute the number of zeros of $f(z) = z^6 - 5z^4 + 3z^2 - 1$ in the closed unit disk $|z| \leq 1$.

Answer: 4

54. Compute the number of zeros of $f(z) = 2z^5 - 6z^2 + z + 1$ in the annulus $1 \leq |z| < 2$.

Answer: 3

55. Compute the number of zeros of $f(z) = z^9 + 5z^2 + 3$ in the first quadrant of $\mathbb{C}$.

Hint: Apply the argument principle to the quarter-disk $|z| \leq r$, $0 \leq \text{Arg}(z) \leq \frac{\pi}{2}$, and let r tend to infinity. **Answer:** 2.



56. Prove that for every real $r > 0$, there exists a natural number N such that, for every $n \geq N$, the polynomial

$$P(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}$$

has no zeros in the disk $\mathbb{D}(0, r)$.



57. Prove: If all the zeros of the polynomial $P(z)$ lie in a given half-plane, then the zeros of $P'(z)$ also lie there.



58. Let $f(z)$ be an analytic function on a domain D . Prove: If f is one-to-one on D , then its derivative $f'(z)$ has no zeros in D .

59. Determine the number of solutions of the equation $z - 2 - e^{-z} = 0$ in the right half-plane $\text{Re}(z) > 0$.

Hint: Prove that it is enough to consider the right half of the disk $|z| < 3$, and then apply Rouché's theorem.



60. Let $f(z)$ be a meromorphic function that has no analytic continuation to an entire function. Prove that the function $g(z) = e^{f(z)}$ is not meromorphic.



Hint: It must be proved that the function g has an essential singular point.



- 61.** Let $\{a_n\}_{n=0}^{\infty}$ be a strictly decreasing sequence of positive real numbers.
- (a) Prove that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function on the unit disk $|z| < 1$.
 - (b) Prove that every partial sum $\sum_{n=0}^m a_n z^n$ has no zeros in the unit disk other than $z = 0$.
 - (c) Prove that the function $f(z)$ has no zeros in the unit disk.
 - (d) At which points on the circle $|z| = 1$ does the series converge?

Hint: Part d may be difficult. One must distinguish between the case in which the series $\sum_{n=0}^{\infty} a_n$ converges and the case in which the series diverges.

- 62.** Let $c \in \mathbb{C}$, $|c| > e$, and let n be a positive natural number. Prove that the function $f(z) = cz^n - e^z$ has exactly n distinct zeros inside the disk $|z| < 1$.
- 63.** Let $a \neq 0$ be a real number. Prove that the equation $ze^z = a$ has infinitely many solutions.
- 64.** Prove that if $0 < |a| < 1$ and n is a positive natural number, then the equation $(z-1)^n e^z = a$ has exactly n zeros in the right half-plane $\operatorname{Re}(z) > 0$, each of multiplicity 1.
- 65.** Prove that the function $f(z) = z + \frac{1}{z}$ takes every nonreal value exactly once on the disk $|z| \leq 1$. Which real values are attained more than once?



- 66.** Let $f(z)$ and $g(z)$ be two entire functions satisfying

$$f\left(\frac{1}{n+1}\right) g\left(\frac{1}{2n+3}\right) = 0$$

for every natural number n . Prove that either f is identically zero or g is identically zero.

- 67.** Let $f(z)$ be a meromorphic function on $\mathbb{C}$ with finitely many poles $z_1, z_2, \dots, z_n$. Assume that $\lim_{|z| \rightarrow \infty} f(z) = 0$. Prove that

$$\sum_{k=1}^n \operatorname{Res}(z_k, f) = \lim_{z \rightarrow \infty} z f(z)$$



- 68.** Let z_0 be a zero of multiplicity 2 of an analytic function $g(z)$, and let $f(z)$ be a function analytic at the point z_0 , with $f'(z_0) \neq 0$. Prove the following formula:

$$\operatorname{Res}\left(\frac{f}{g}, z_0\right) = \frac{2f'(z_0)}{g''(z_0)} \frac{2f(z_0)g'''(z_0)}{3[g''(z_0)]^2}$$

- 69.** Use the formula you obtained in Exercise 68 to compute the residue of $f(z) = \frac{\cos z}{[\operatorname{Log}(z)-1]^2}$ at



the point $z = e$.



70. Prove the fundamental theorem of algebra using Rouché's theorem.

Hint: If $P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ is a polynomial of degree n , to use Rouché's theorem, take $f(z) = a_nz^n$, $g(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{n-1}z^{n-1}$, and find a sufficiently large $R > 0$ such that the conditions of Rouché's theorem hold on the circle $|z| = R$. To compare the magnitudes of polynomials, it is recommended to review the proofs of Propositions 3.18, 3.19 (page 102).

71. Let $f: D \rightarrow \mathbb{C}$ be a nonconstant analytic function on an open set D . Let $z_0 \in D$ be a zero of f . Prove that there exists a real number $r > 0$ such that $f(z) \neq 0$ for every z on the circle $|z - z_0| = r$. That is: around every zero of f , there exists a circle on which f does not vanish.

Hint: Suppose, to the contrary, that there is no such circle, and then, for every sufficiently large n , the circle $|z - z_0| = \frac{1}{n}$ contains a point z_n at which $f(z_n) = 0$. Obtain a contradiction to Theorem 7.4.



72. Let $f(z)$ be an analytic function on a domain D , and let $z_0 \in D$ be such that $f'(z_0) = 0$. Prove that f is not one-to-one in any neighborhood of z_0 .

Hint: In other words, for every $r > 0$, there exist $z_1, z_2 \in \mathbb{D}(z_0, r)$ such that $f(z_1) = f(z_2)$. Use Exercise 58.



73. Prove that for every $\varepsilon > 0$, the function $f(z) = \frac{1}{z+i} + \sin z$ has infinitely many zeros in the strip $|\operatorname{Im}(z)| < \varepsilon$.



74. Does there exist a function analytic on the unit disk $|z| < 1$ such that $\lim_{|z| \rightarrow 1} |f(z)| = \infty$?

Hint: The function f has finitely many zeros inside the disk (why?). Define $g(z) = f(z)/(z - z_1)(z - z_2) \cdots (z - z_n)$ so that g has no zeros, and still $\lim_{|z| \rightarrow 1} |g(z)| = \infty$. Investigate the function $1/g(z)$. Use the maximum principle.



75. Prove the **Open Mapping Theorem**: If $f: D \rightarrow \mathbb{C}$ is a nonconstant analytic function on an open set D , then the image $f(D)$ is also an open set.

Hint: Let $z_0 \in D$. It must be proved that $f(z_0)$ is an interior point of $f(D)$. That is, there exists $t > 0$ such that $\mathbb{D}(f(z_0), t) \subseteq f(D)$. Without loss of generality, one may assume that $f(z_0) = 0$ (otherwise replace f by the function $g(z) = f(z) - f(z_0)$). By Exercise 71, there exists $r > 0$ such that f does not vanish on the circle $|z - z_0| = r$. Take $r = \frac{1}{2} \min_{|z - z_0| = r} |f(z)|$. Prove that $\mathbb{D}(0, r) \subseteq f(D)$ (the minimum principle should be used).



76. By using the open mapping theorem from Exercise 75, prove the maximum principle for a real harmonic function: A real harmonic function $u(x, y)$ on an open set $S \subseteq \mathbb{R}^2$ cannot attain a local maximum at an interior point of S .

Hint: Suppose, to the contrary, that u attains a local maximum M at an interior point $z_0 = x_0 + iy_0$. By Theorem 4.13 (page 131), there exists an analytic function $f: S \rightarrow \mathbb{C}$ such that $\operatorname{Re}(f) = u(x, y)$. Show that f is not an open mapping because $f(z_0) = M + iv(x_0, y_0)$, and no neighborhood of the point $w = M + iv(x_0, y_0)$ is contained in the image $f(S)$.



77. Prove: If $f: D \rightarrow \mathbb{C}$ is a nonconstant analytic function on an open and connected set D , then the image $f(D)$ is also an open and connected set.

Hint: The first part follows from Exercise 75. It is enough to prove path connectedness: Every path in D maps to a path in $f(D)$.





- 78.** Prove: There is no one-to-one and onto conformal mapping from the punctured unit disk $0 < |z| < 1$ to the annulus $1 < |z| < 2$.

Hint: Denote $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Suppose, to the contrary, that there exists a map $f: \dot{\mathbb{D}}(0, 1) \rightarrow A$ as above. Prove that $z = 0$ is a removable singular point (because $\lim_{z \rightarrow 0} z f(z) = 0$). Investigate the extension of f , $g: \overline{\mathbb{D}}(0, 1) \rightarrow \overline{A}$.



- 79.** Let $f(z)$ be a function analytic in a neighborhood of the closed unit disk $|z| \leq 1$. Assume that f is one-to-one on the circle $|z| = 1$. Prove that f is one-to-one inside the disk $|z| < 1$. State a more general proposition for every simply connected domain D , and examine whether your proof can also be generalized to the general case.

Hint: Step 1: Show that the image of the unit circle σ is a Jordan path. Step 2: Use the open mapping theorem (Exercise 75) to show that $f(\mathbb{D}(0, 1))$ must be the interior of $f(\sigma)$. At the point of despair, a solution can be found at the following link: [hw9sol.pdf](#).



- 80.** Prove the **Casorati-Weierstrass Theorem**: If z_0 is an essential singular point of a function $f(z)$, then the image of every punctured neighborhood $\dot{\mathbb{D}}(z_0, r)$ is a dense subset of $\mathbb{C}$.

Hint: Reminder: A set D is **dense** in a set E if every point $z \in E$ is the limit of a sequence of points $z_n \in D$. Idea: Suppose, to the contrary, that the image $U = f(\dot{\mathbb{D}}(z_0, r))$ is not dense in $\mathbb{C}$. Then there exist a point w_0 and a real number $r > 0$ such that $\mathbb{D}(w_0, r) \cap U = \emptyset$. That is, for every $z \in \dot{\mathbb{D}}(z_0, \varepsilon)$, $|f(z) - w_0| \geq r$. Therefore, for every $z \in \dot{\mathbb{D}}(z_0, \varepsilon)$, $\left| \frac{1}{f(z) - w_0} \right| \leq \frac{1}{r}$. Thus $\frac{1}{f(z) - w_0}$ is bounded in a neighborhood of the point z_0 , and therefore, by Proposition 7.10, z_0 is a removable singular point of $\frac{1}{f(z) - w_0}$.

- 81.** Let $P(z)$ be a nonconstant polynomial. Prove that the equation $P(z) = e^z$ has infinitely many distinct solutions.

Hint: Try to use the Casorati-Weierstrass theorem (see Exercise 80).

- 82.** Let $f(z)$ be an entire function. We say that the point $z = \infty$ is an essential singular point of f if $z = 0$ is an essential singular point of $f(1/z)$. Prove: An entire function $f(z)$ is a polynomial if and only if $z = \infty$ is not an essential singular point of f .

Hint: Define the function $g(z) = f(1/z)$. If $f(z)$ is a polynomial, then $\lim_{z \rightarrow \infty} f(z) = \infty$, and therefore $\lim_{z \rightarrow 0} g(z) = \infty$. By the Casorati-Weierstrass theorem (see Exercise 80), $z = 0$ cannot be an essential singular point of g (why?). Converse direction: If $z = \infty$ is not an essential singular point of f , then $z = 0$ is not an essential singular point of g , and therefore, by Proposition 7.9, the Maclaurin series of g about the point $z = 0$ contains a etinf number of negative powers, and therefore the Taylor series of f is finite (why?). That is, f is a polynomial. It is recommended to solve this exercise together with Exercise 24. It is important to read the illumination for Exercise 24 concerning the duality between the point $z = 0$ and the point $z = \infty$. Additional sources: [math.stackexchange.1](#), [math.stackexchange.2](#).



- 83.** Let $f(z)$ be a meromorphic function on $\mathbb{C}$ such that $\lim_{z \rightarrow \infty} f(z) = \infty$. Prove that f is necessarily a rational function.

Hint: Note that this is a generalization of Exercise 4, which cannot be obtained without the Casorati-Weierstrass theorem (see Exercise 80). The formulation does not state whether the number of poles is finite or infinite, but the existence of the limit $\lim_{z \rightarrow \infty} f(z) = \infty$ requires that there exist $R > 0$ such that $f(z)$ is defined throughout the domain $|z| > R$. (see Definition 3.6 on page 101). Therefore all the poles of f must lie in the disk $|z| \leq R$, and hence their number is finite (why?). Use formula (7.9) of Proposition 7.23 (page 297), and, of course, the Casorati-Weierstrass theorem should be used.



- 84.** Let $f(z)$ be an entire function satisfying $|f(z)| = 1$ on the unit circle $|z| = 1$. Prove that f



must be a polynomial.

Hint: Consider the function $g(z) = 1/\overline{f(1/\bar{z})}$. Prove that g is analytic in the punctured plane $\mathbb{C} - \{0\}$ and that $f(z) = g(z)$ on the unit circle $|z| = 1$. By the identity theorem, $f(z) = g(z)$ for every $z \in \mathbb{C}$. But f is entire, and therefore $z = 0$ must be a removable singular point of g . The function f has no zeros except perhaps at $z = 0$, because if $z \neq 0$ is a zero, then, by the symmetry $f(z) = 1/\overline{f(1/\bar{z})}$, $\bar{z}$ is a pole of f , contradicting the fact that f is entire. If $z = 0$ is not a zero of f , then f must be constant. If $z = 0$ is a zero of f , then prove that $\lim_{z \rightarrow \infty} f(z) = \infty$, and therefore f is a polynomial.



- 85.** Find all functions $f(z)$ satisfying $|f(z)| = 1$ on the unit circle $|z| = 1$.

Hint: By Exercise 84, f must be a polynomial. It remains only to determine what kind of polynomials these are. By the considerations encountered in Exercise 84, every such polynomial has only one zero, $z = 0$, and therefore it must have the form az^n (why?), for a coefficient $a \in \mathbb{C}$. The condition $|f(z)| = 1$ implies that $|a| = 1$.



- 86.** Let p be a real number, and let $f(z)$ be an entire function satisfying

$$\int_0^{2\pi} |f(re^{it})| dt \leq r^p$$

for every $r > 0$. Prove that f must be a polynomial.



- 87.** Let $f(z)$ and $g(z)$ be two entire functions whose composition $f(g(z))$ is a polynomial. Prove that both functions $f(z)$ and $g(z)$ are polynomials. What can you say about the degree of $f(g(z))$ in relation to the degrees of the two functions?

Hint: Use the solution of Exercise 82. After the break point, you may consult the following link:
[proving-fz-and-gz-are-polynomials](#)



- 88.** Prove: If $f(z)$ is an entire and one-to-one function, then f is necessarily a linear polynomial:
 $f(z) = az + b$.

Hint: Without loss of generality, assume that $f(0) = 0$. By the open mapping theorem (Exercises 75, 77), the image of the unit disk $U = f(\mathbb{D}(0, 1))$ is an open set containing $z = 0$ as an interior point. That is, there exists $r > 0$ such that $\mathbb{D}(0, r) \subseteq U$. Therefore, by the one-to-one property of f , for every z outside the unit disk, $|f(z)| > r$. Hence $1/f(z)$ is bounded on the plane with a hole $|z| > 1$. And now Liouville's theorem ... Show that $f(z) = az$ for a constant a .
 Second method: Apply the Casorati-Weierstrass theorem (Exercise 80) to the function $g(z) = f(1/z)$. Also review Exercise 82.



- 89.** Let $f(z)$ be a function analytic in the open disk $|z - z_0| < r$. Suppose that z_0 is an isolated zero of f . Prove that there exists a point z in the disk such that $\operatorname{Re}(f(z)) > 0$.

Hint: On further thought, the skull symbol for this problem may not be justified. This has not yet been checked, but it may be that an approach using harmonic functions will lead to an elegant solution. If $f(x + iy) = u(x, y) + iv(x, y)$, then, by Theorem 4.12, $u(x, y)$ is a harmonic function, and therefore it should attain a minimum and a maximum on the boundary.



[Link to solutions](#)

