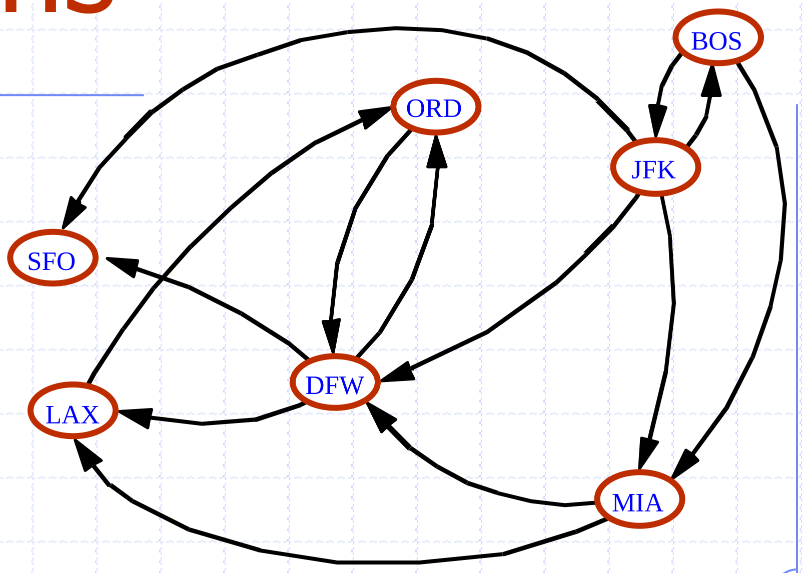
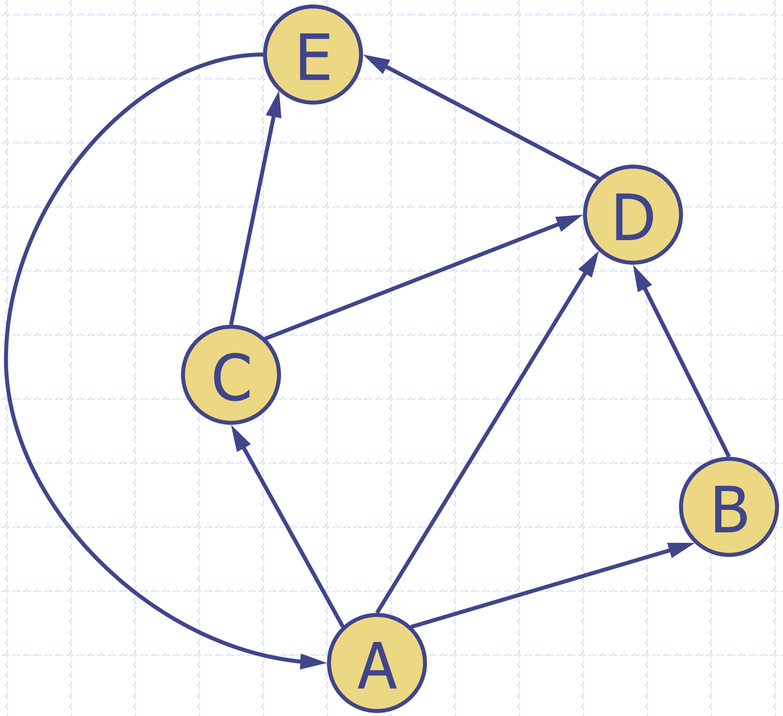
Directed Graphs

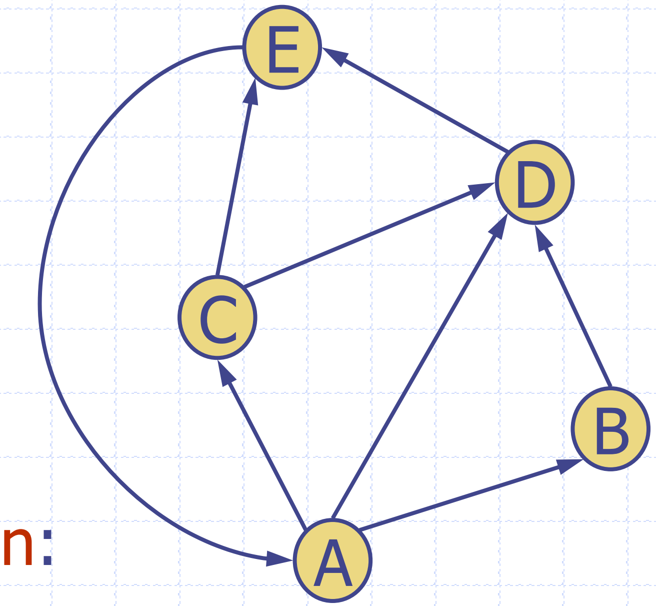


Digraphs

- A digraph is a graph whose edges are all directed
  - Short for "directed graph"
- Applications
  - one-way streets
  - flights
  - task scheduling



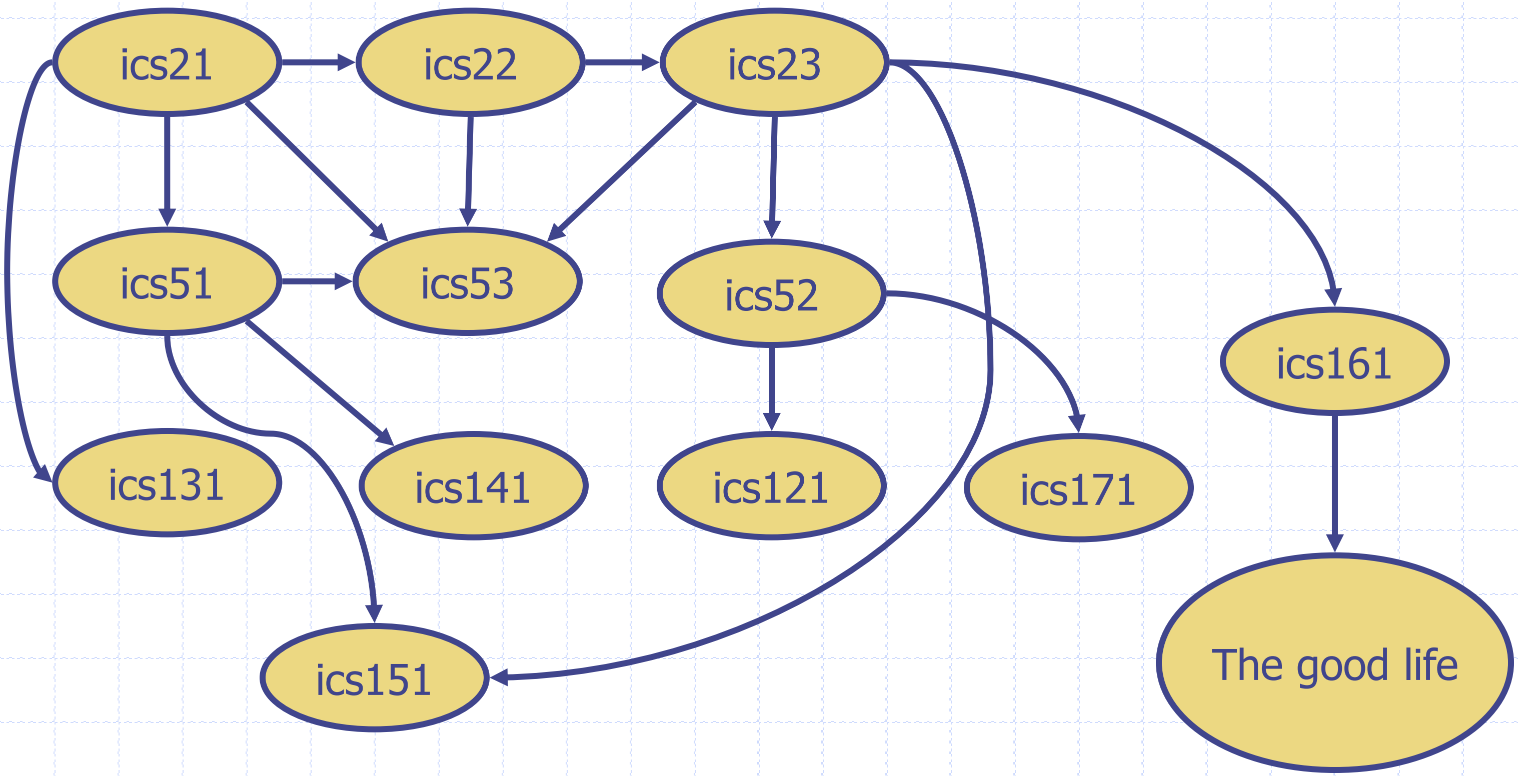
Digraph Properties



- A graph G=(V,E) such that
  - Each edge goes in one direction:
  - Edge (a,b) goes from a to b, but not b to a
- If G is simple, $m \leq n \cdot (n - 1)$
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of incoming edges and outgoing edges in time proportional to their size

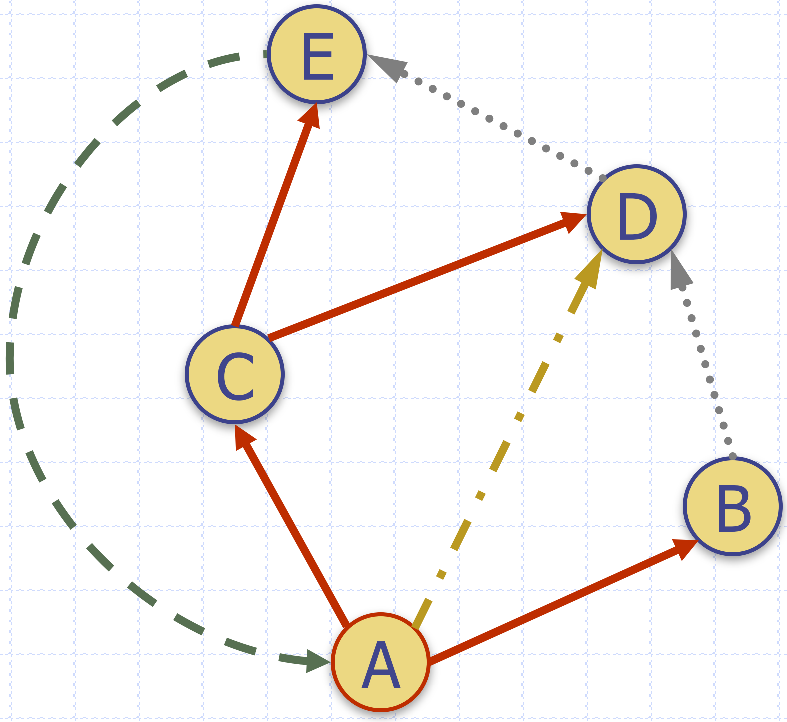
Digraph Application

- Scheduling: edge (a,b) means task a must be completed before b can be started



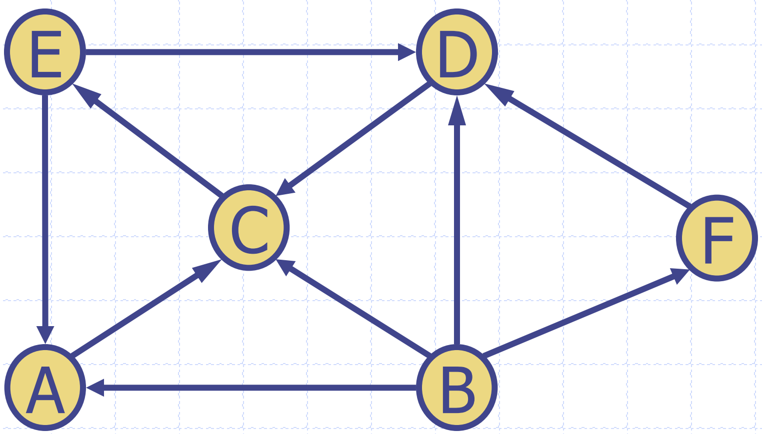
Directed DFS

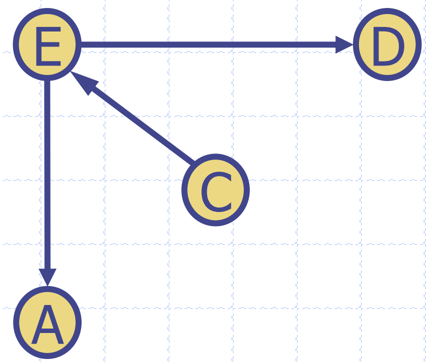
- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex *s* determines the vertices reachable from *s*

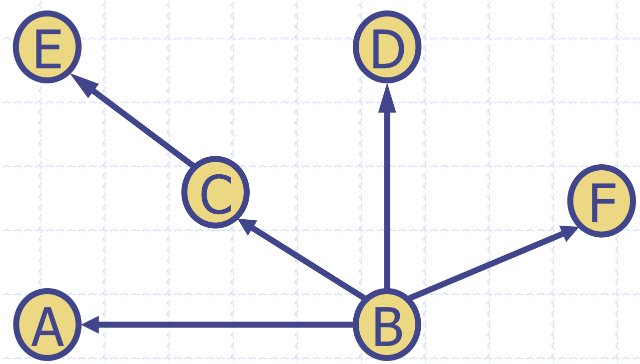


Reachability

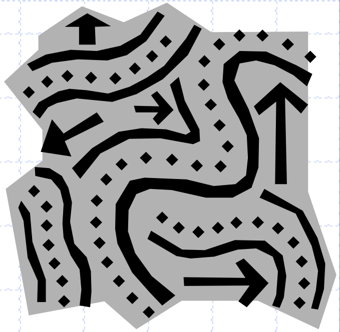
- DFS tree rooted at v: vertices reachable from v via directed paths



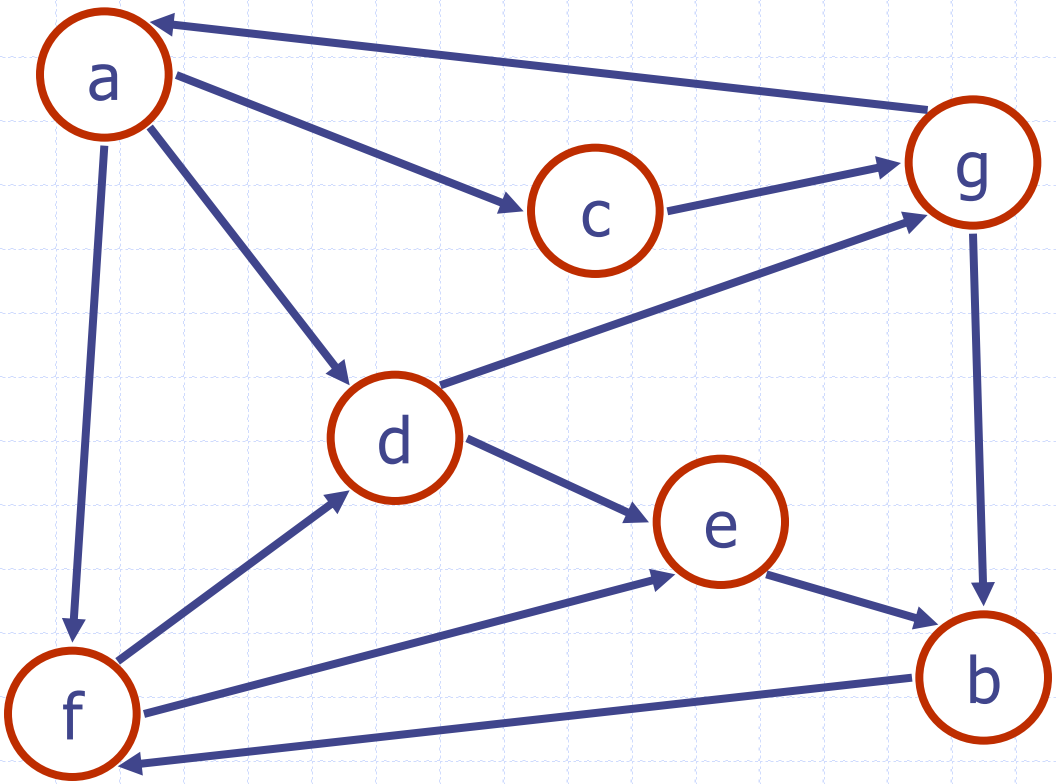




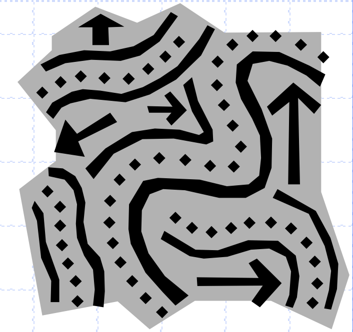
Strong Connectivity



- Each vertex can reach all other vertices

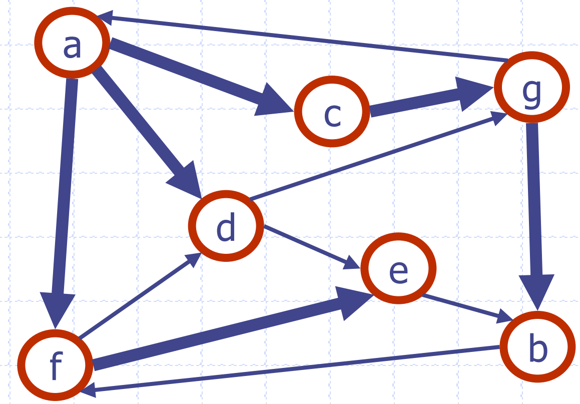


Strong Connectivity Algorithm

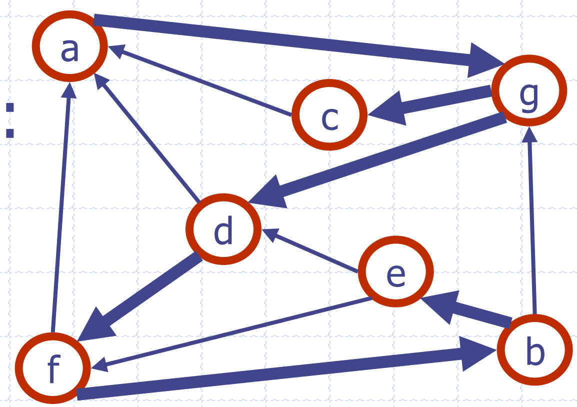


- Pick a vertex v in G
- Perform a DFS from v in G
  - If there’s a w not visited, print “no”
- Let G’ be G with edges reversed
- Perform a DFS from v in G’
  - If there’s a w not visited, print “no”
  - Else, print “yes”
- Running time: O(n+m)

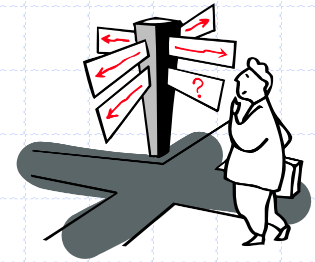
G:



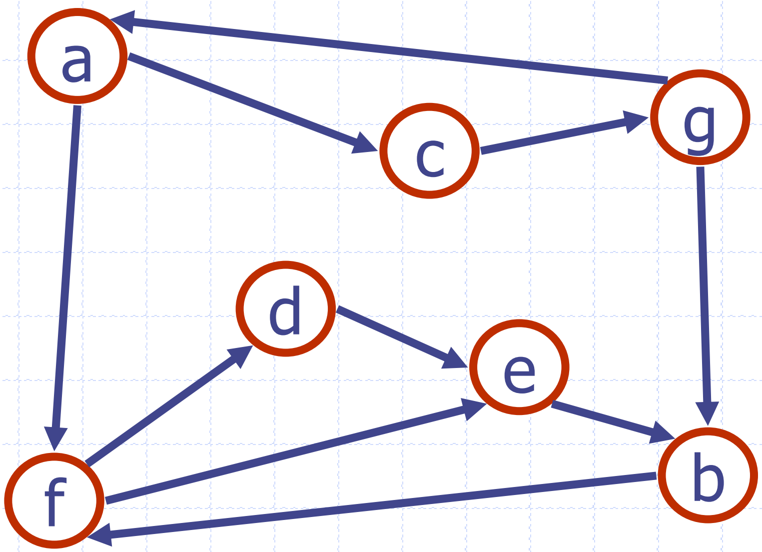
G’ :



Strongly Connected Components



- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in O(n+m) time using DFS, but is more complicated (similar to biconnectivity).

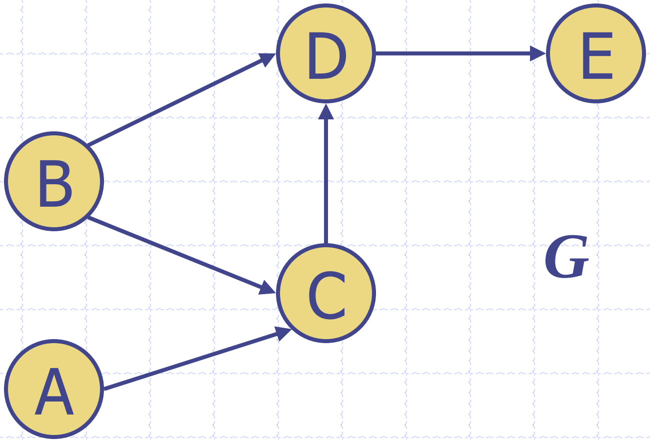


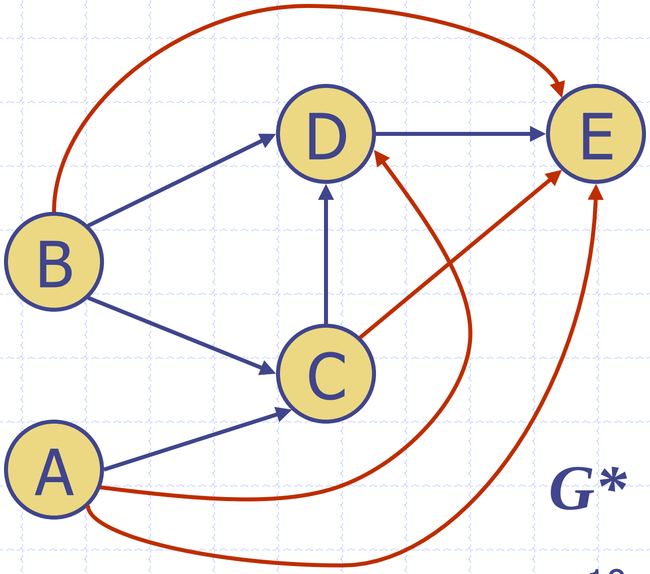
{ a , c , g }

{ f , d , e , b }

Transitive Closure

- Given a digraph G, the transitive closure of G is the digraph G^* such that
  - G^* has the same vertices as G
  - if G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v
- The transitive closure provides reachability information about a digraph

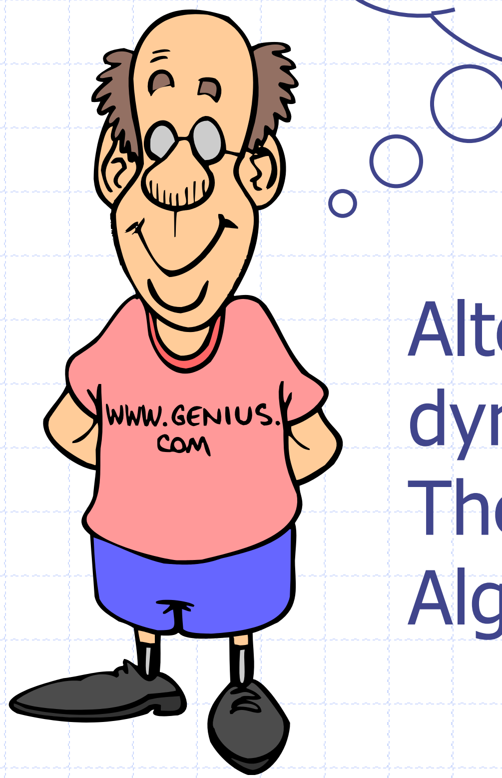




Computing the Transitive Closure

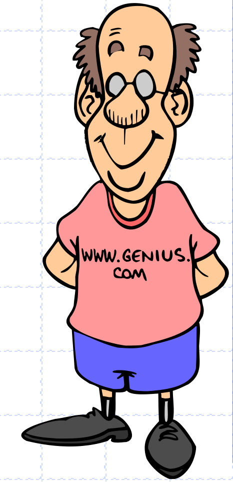
- We can perform DFS starting at each vertex
  - O(n(n+m))

If there's a way to get from A to B and from B to C, then there's a way to get from A to C.

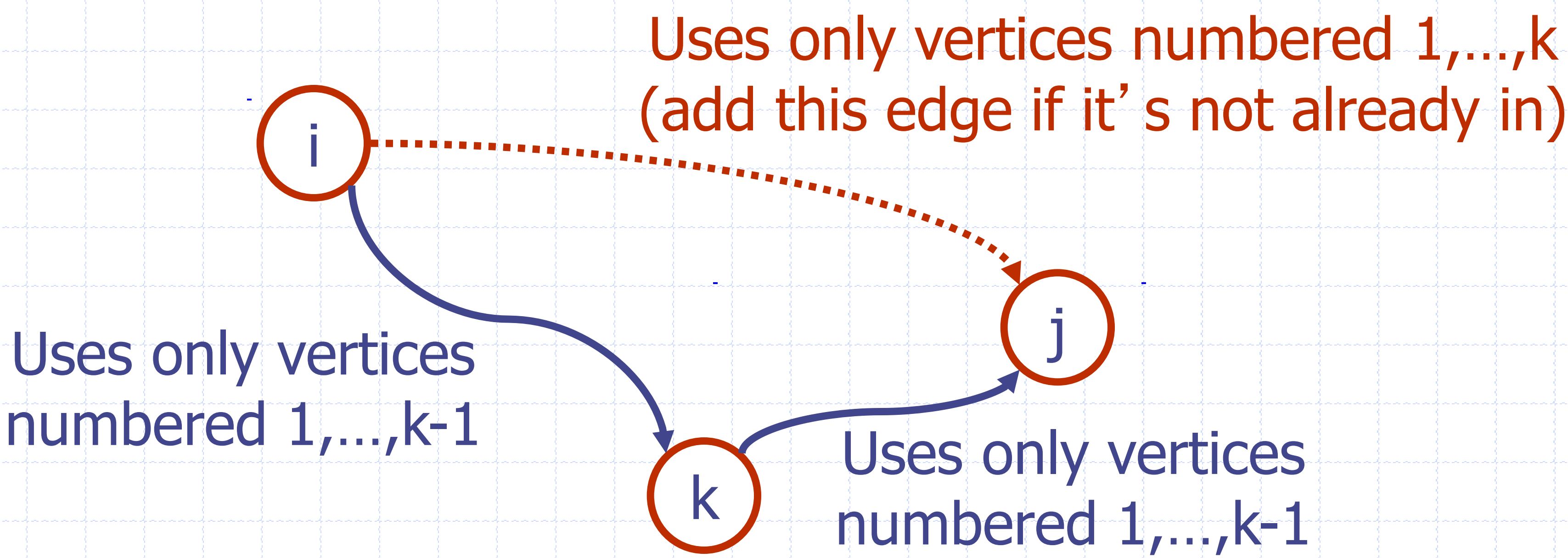


Alternatively ... Use dynamic programming: The Floyd-Warshall Algorithm

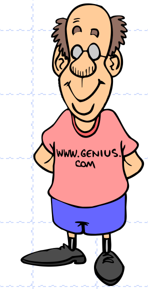
Floyd-Warshall Transitive Closure



- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:



Floyd-Warshall's Algorithm



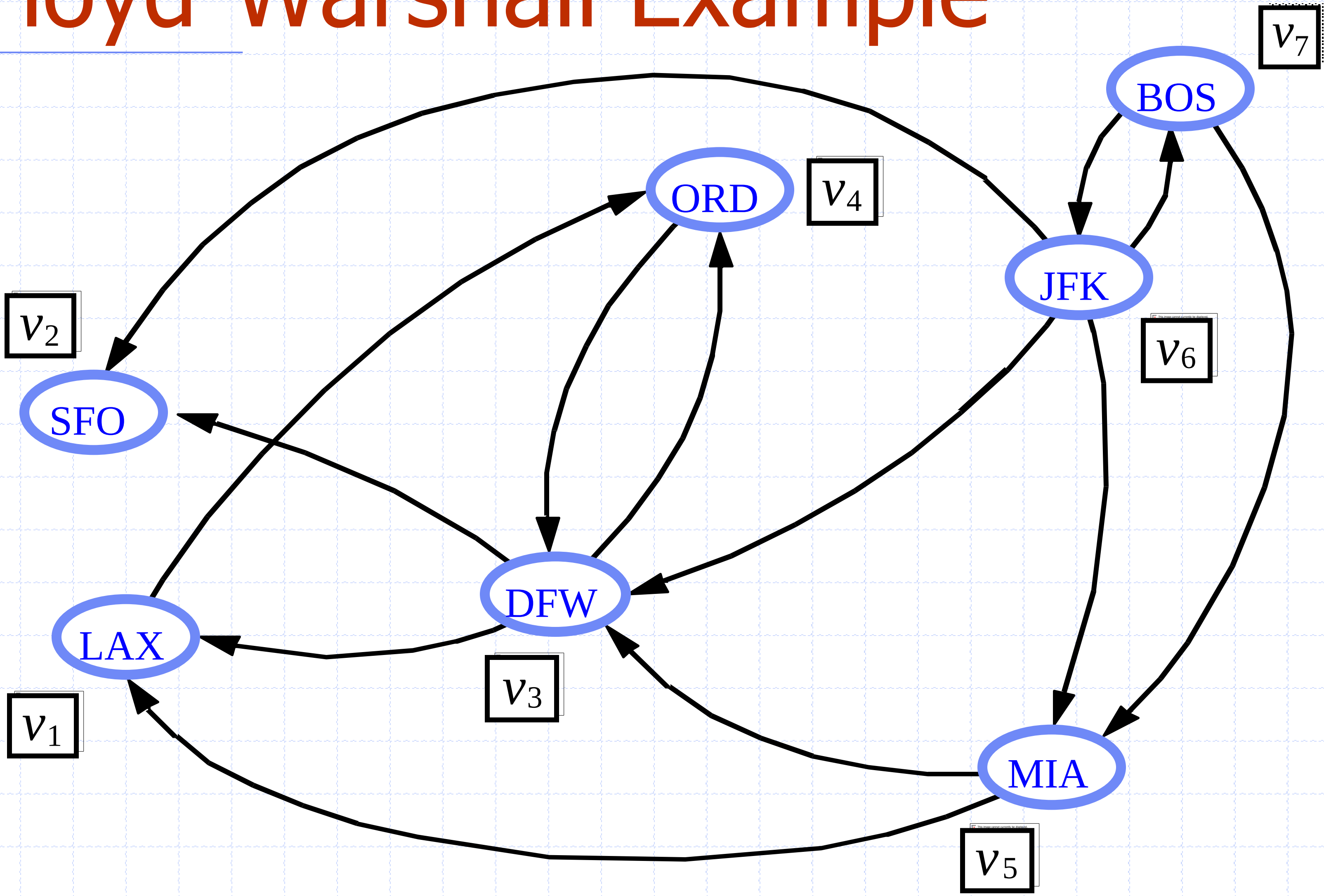
- Number vertices $v_1, \ldots, v_n$
- Compute digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - G_k has directed edge (v_i, v_j) if G has a directed path from v_i to v_j with intermediate vertices in $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase k, digraph G_k is computed from G_{k-1}
- Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

```
Algorithm FloydWarshall(G)
  Input digraph G
  Output transitive closure G* of G
  i ← 1
  for all v ∈ G.vertices()
     denote v as v_i
     i ← i + 1
  G_0 ← G
  for k ← 1 to n do
     G_k ← G_(k−1)
      for i ← 1 to n (i ≠ k) do
         for j ← 1 to n (j ≠ i, k) do
            if G_(k−1).areAdjacent(v_i, v_k) ∧
                  G_(k−1).areAdjacent(v_k, v_j)
               if ¬G_k.areAdjacent(v_i, v_j)
                  G_k.insertDirectedEdge(v_i, v_j, k)
     return G_n
```

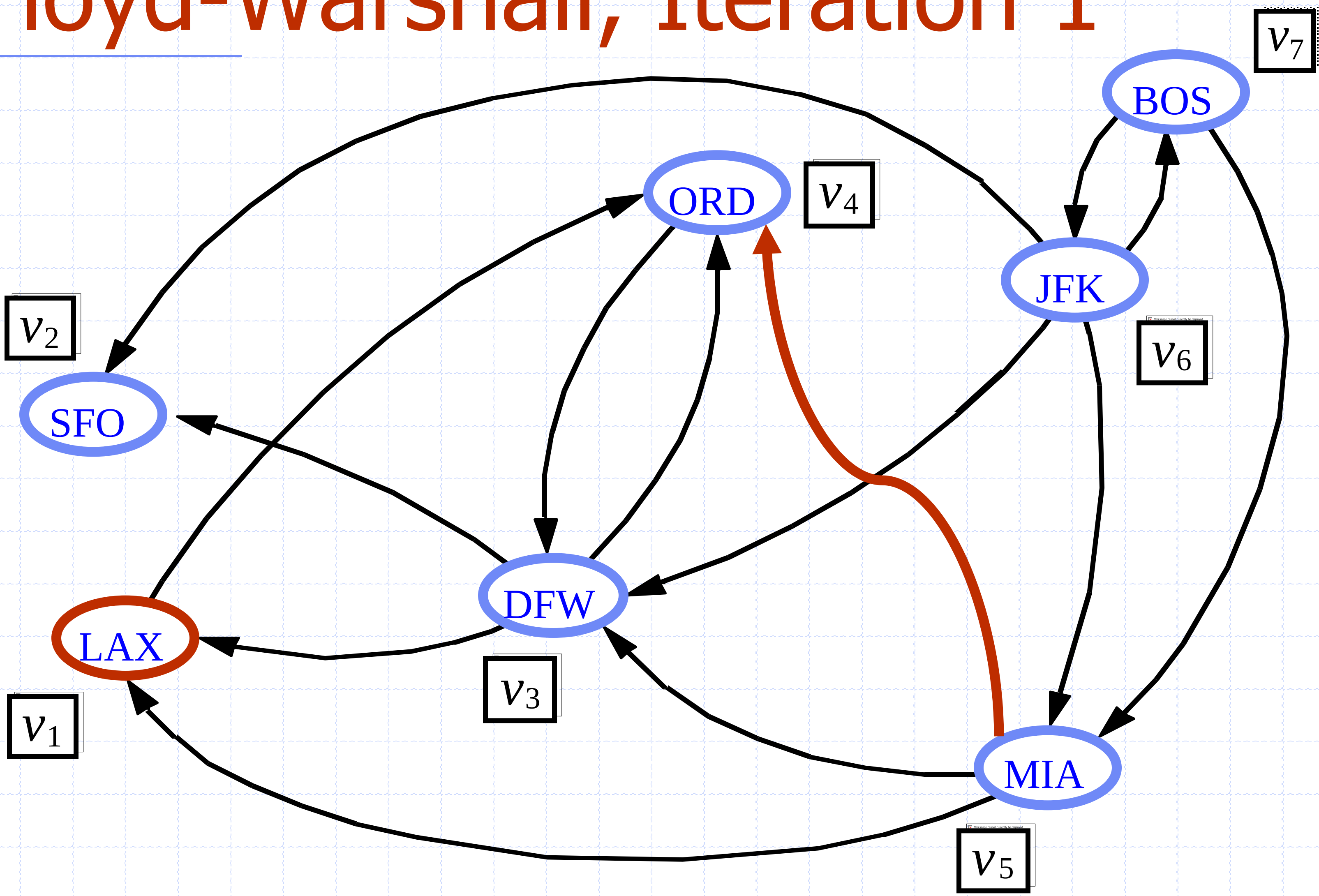
Python Implementation

```
def floyd_warshall(g):
  """Return a new graph that is the transitive closure of g."""
  closure = deepcopy(g)                    # imported from copy module
  verts = list(closure.vertices())         # make indexable list
  n = len(verts)
  for k in range(n):
    for i in range(n):
      # verify that edge (i,k) exists in the partial closure
      if i != k and closure.get_edge(verts[i],verts[k]) is not None:
        for j in range(n):
          # verify that edge (k,j) exists in the partial closure
          if i != j != k and closure.get_edge(verts[k],verts[j]) is not None:
            # if (i,j) not yet included, add it to the closure
            if closure.get_edge(verts[i],verts[j]) is None:
              closure.insert_edge(verts[i],verts[j])
  return closure
```

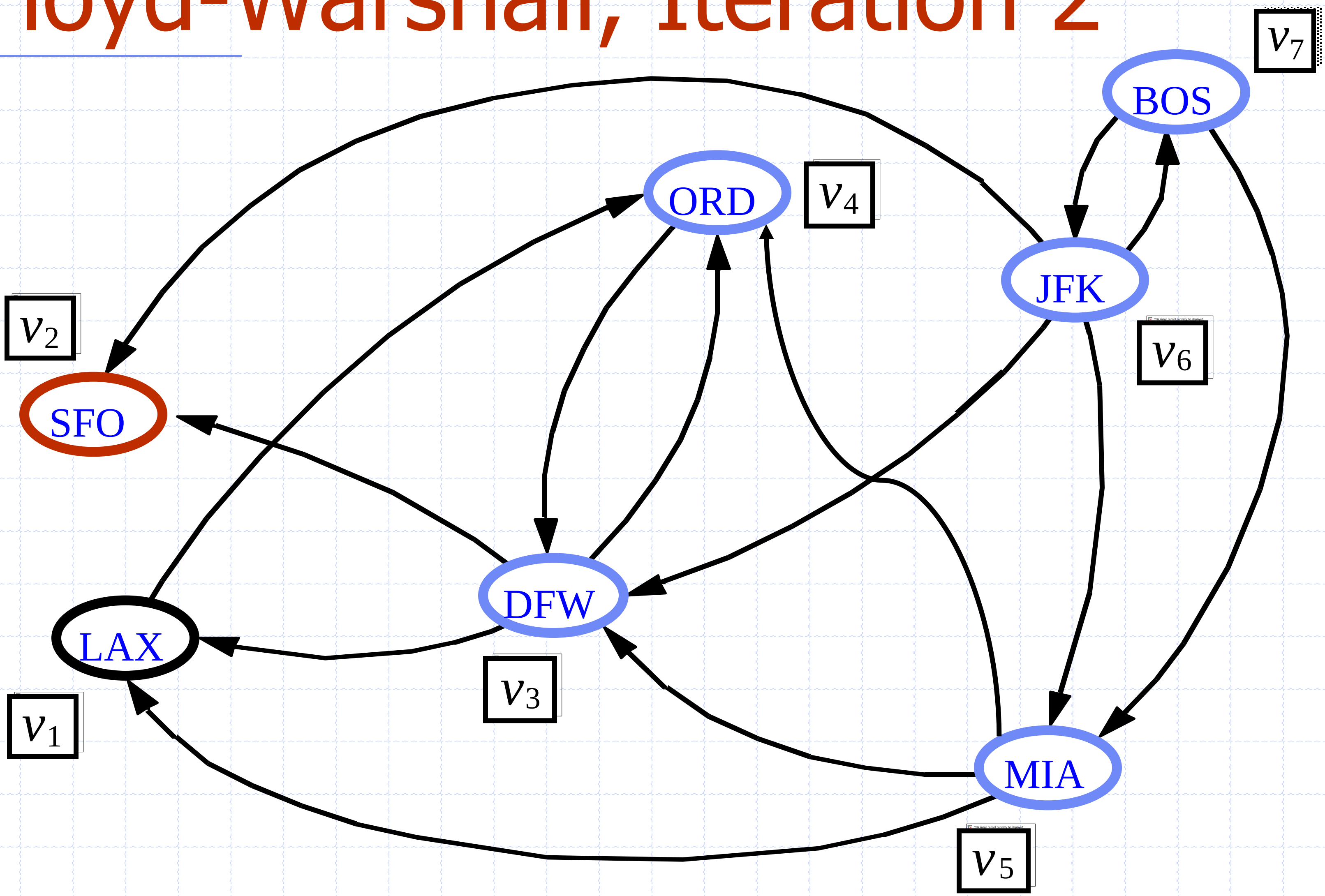
Floyd-Warshall Example



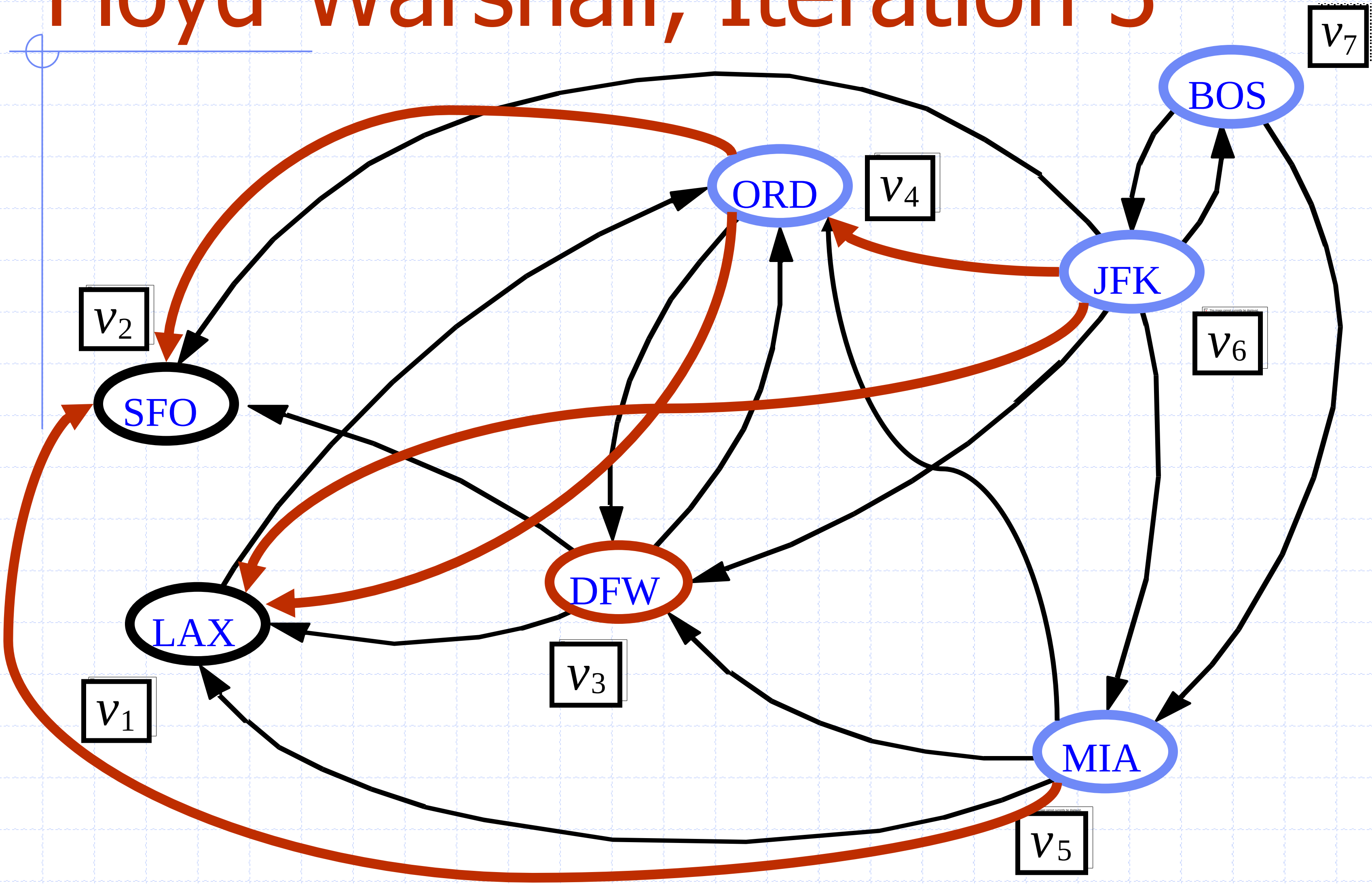
Floyd-Warshall, Iteration 1



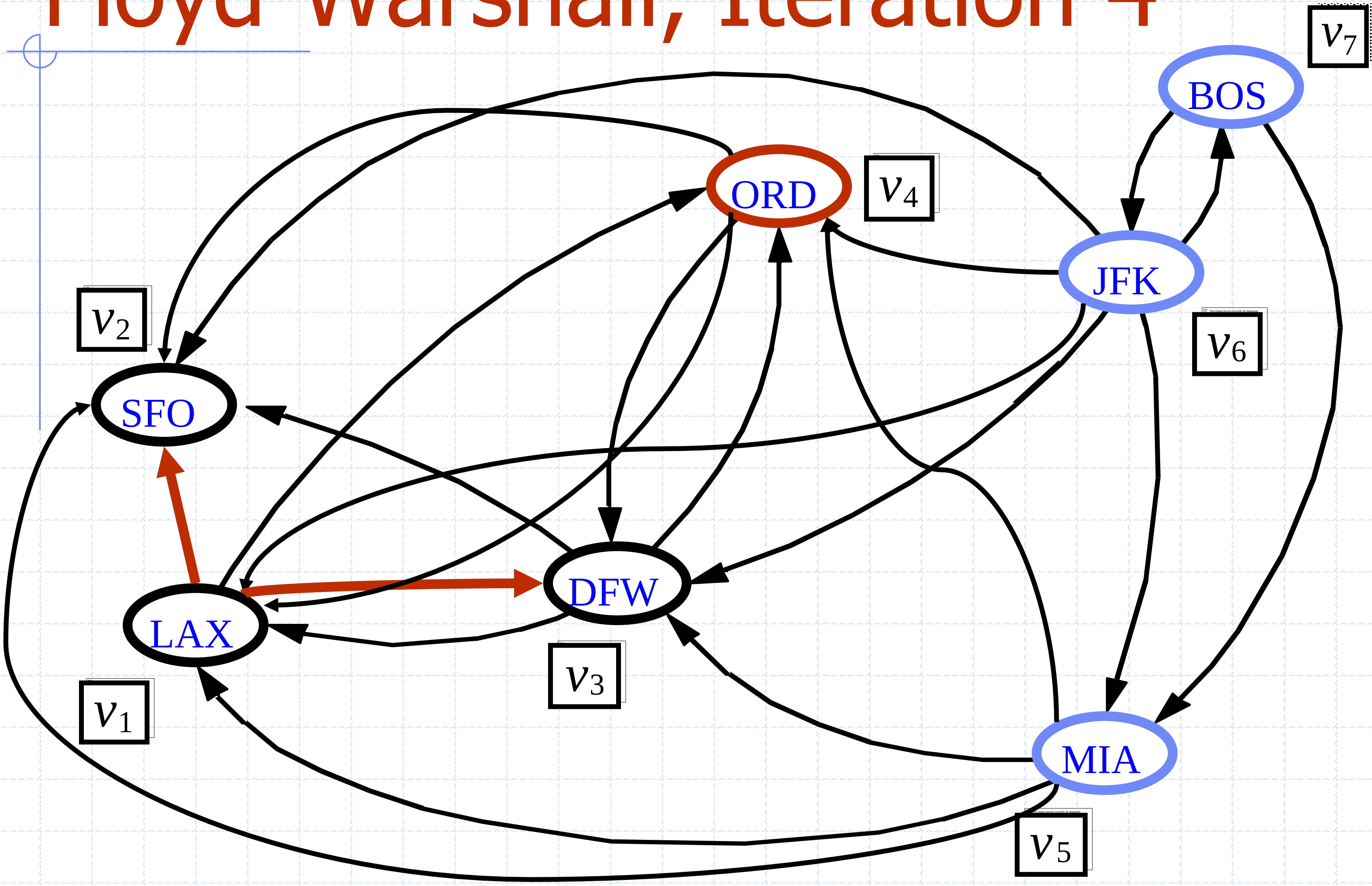
Floyd-Warshall, Iteration 2



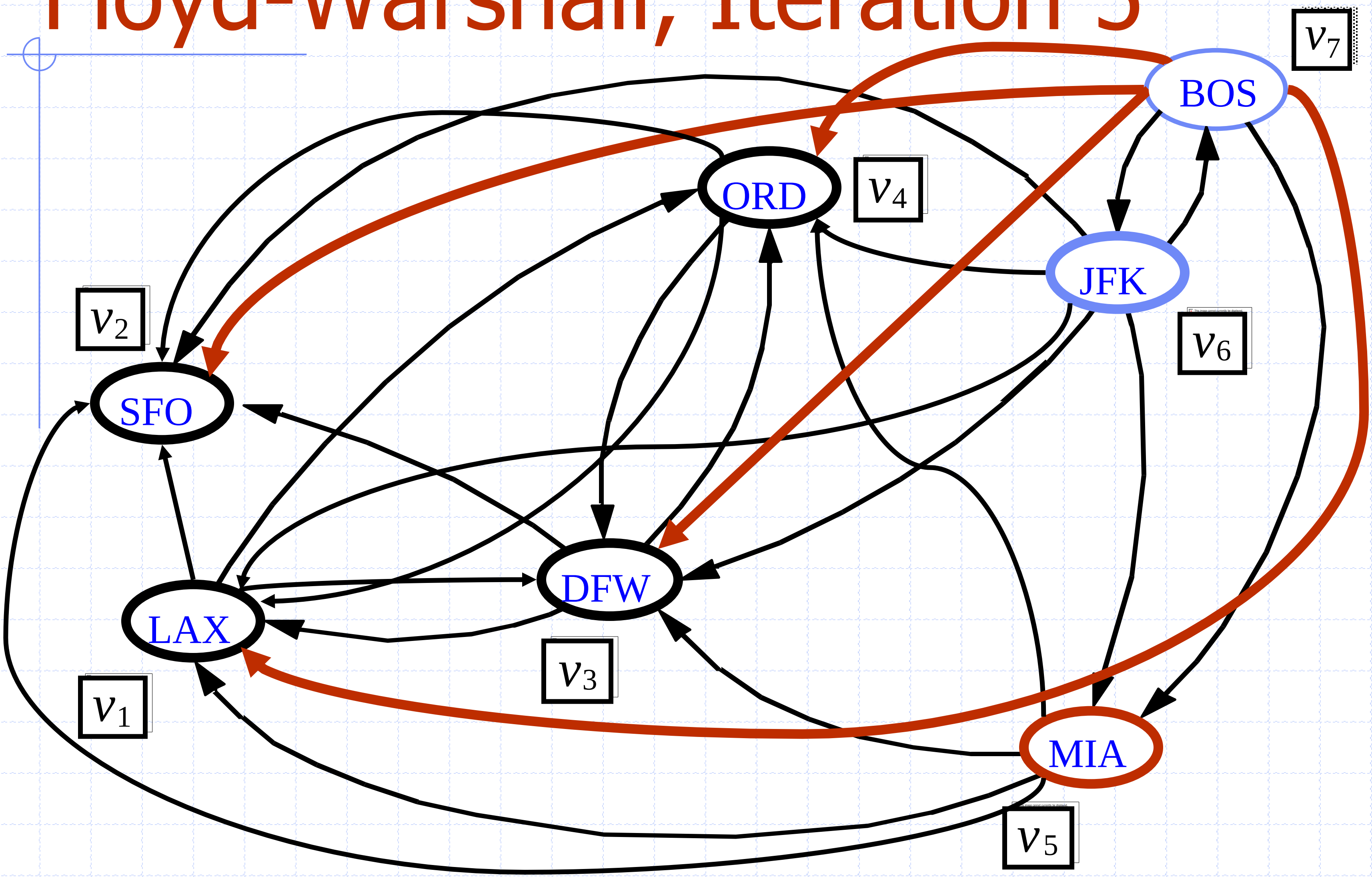
Floyd-Warshall, Iteration 3



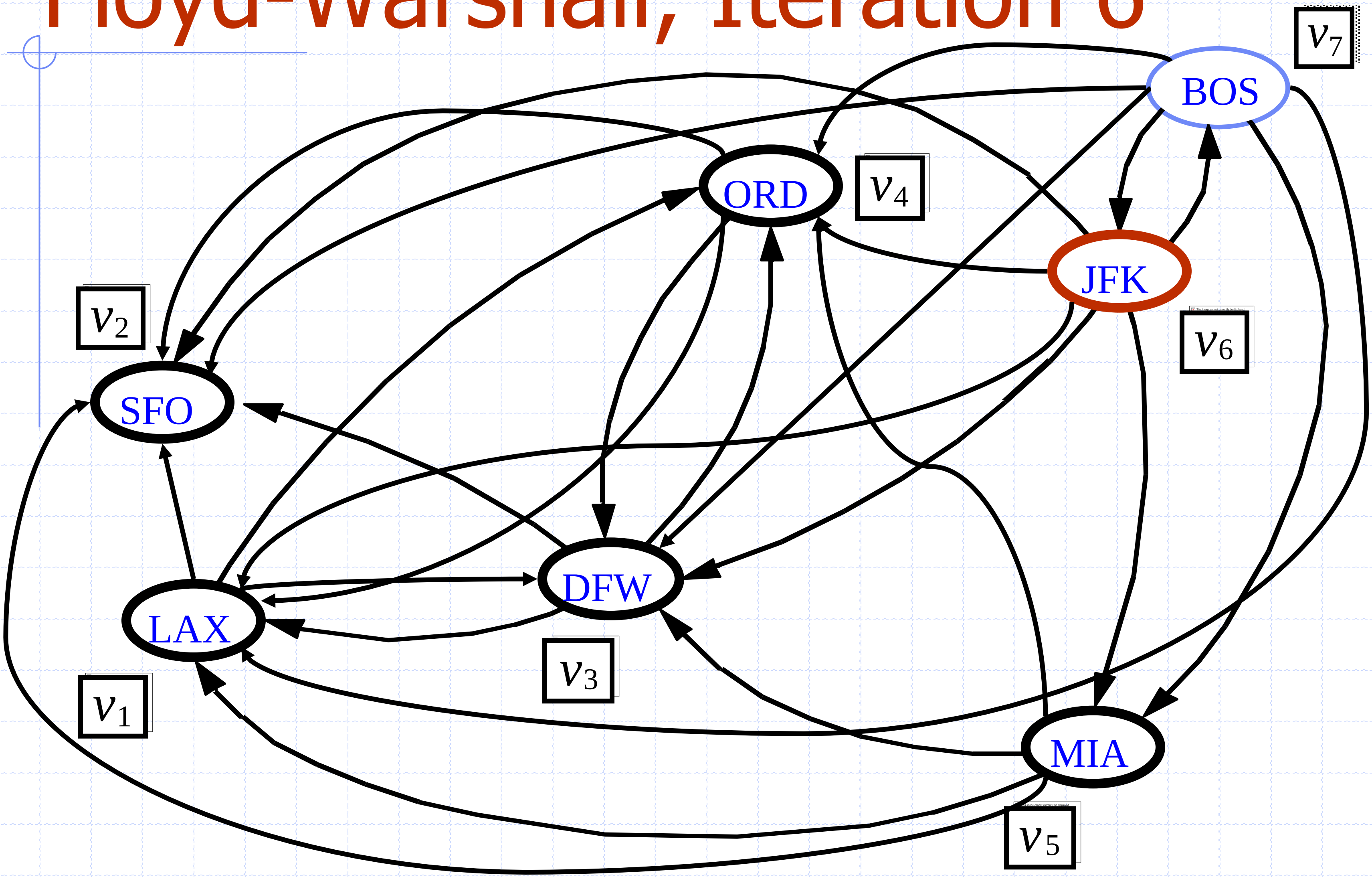
Floyd-Warshall, Iteration 4



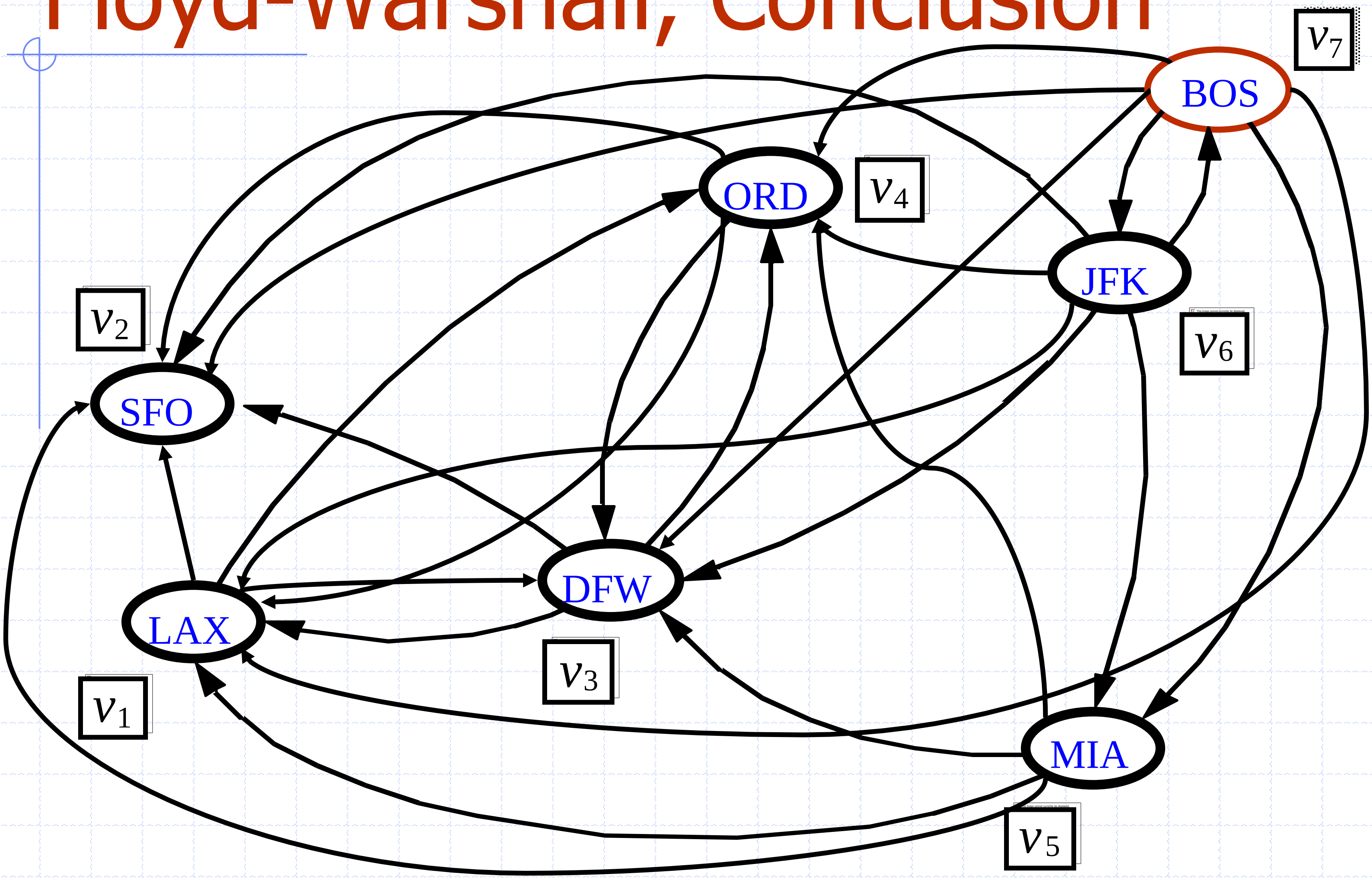
Floyd-Warshall, Iteration 5



Floyd-Warshall, Iteration 6



Floyd-Warshall, Conclusion



DAGs and Topological Ordering

- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering

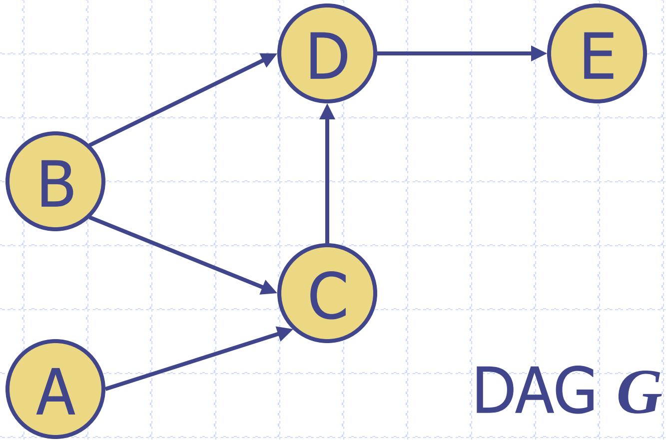
$$v_1, \ldots, v_n$$

of the vertices such that for every edge (v_i, v_j), we have $i < j$

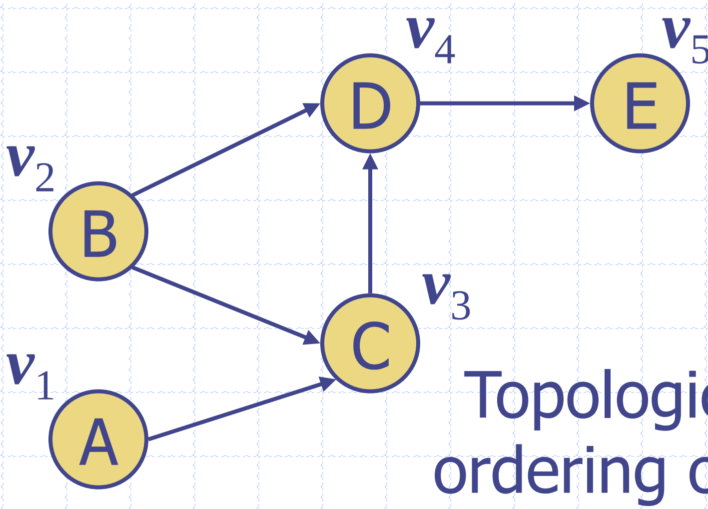
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints

Theorem

A digraph admits a topological ordering if and only if it is a DAG

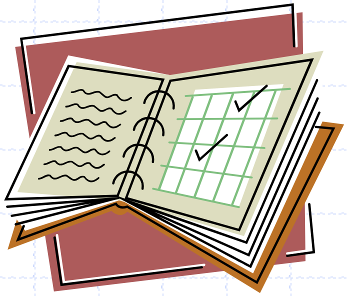


DAG G

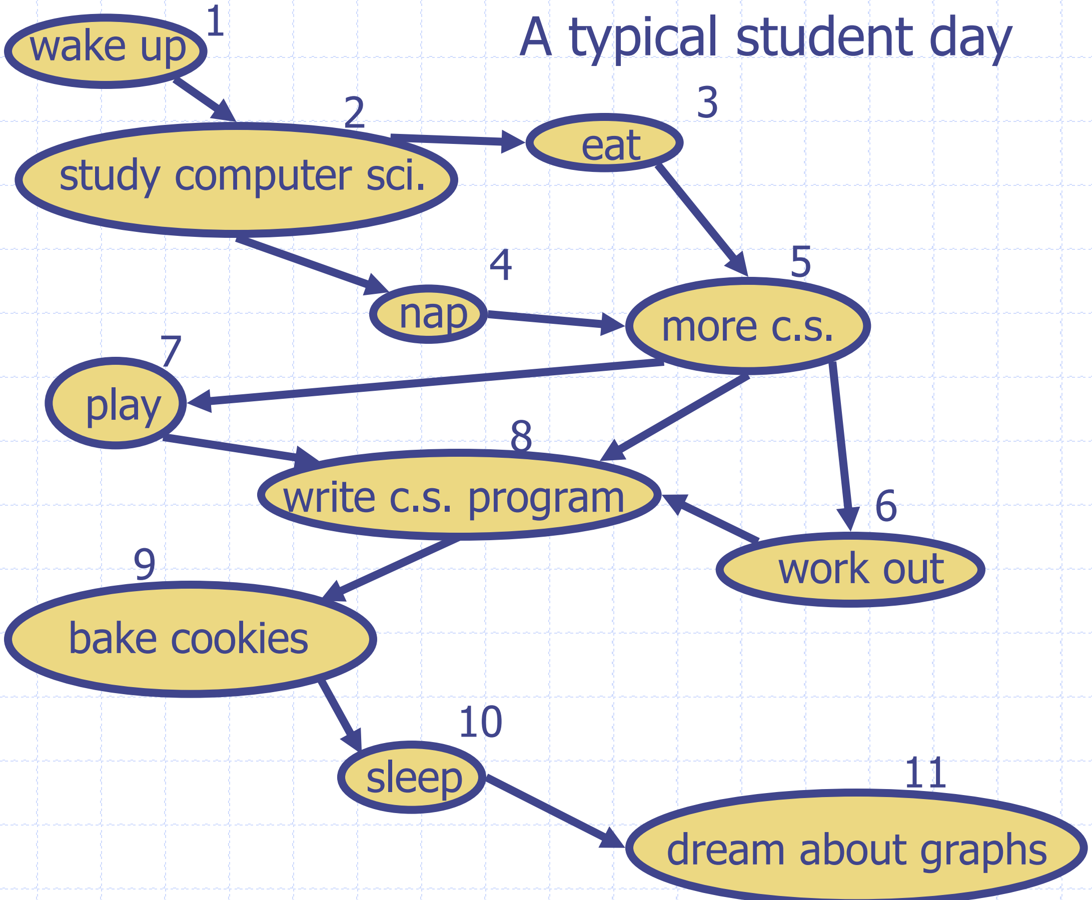


Topological ordering of G

Topological Sorting



- Number vertices, so that (u,v) in E implies u < v



Algorithm for Topological Sorting

- Note: This algorithm is different than the one in the book

```
Algorithm TopologicalSort(G)
    H ← G            // Temporary copy of G
    n ← G.numVertices()
    while H is not empty do
        Let v be a vertex with no outgoing edges
        Label v ← n
        n ← n – 1
        Remove v from H
```

- Running time: $O(n + m)$

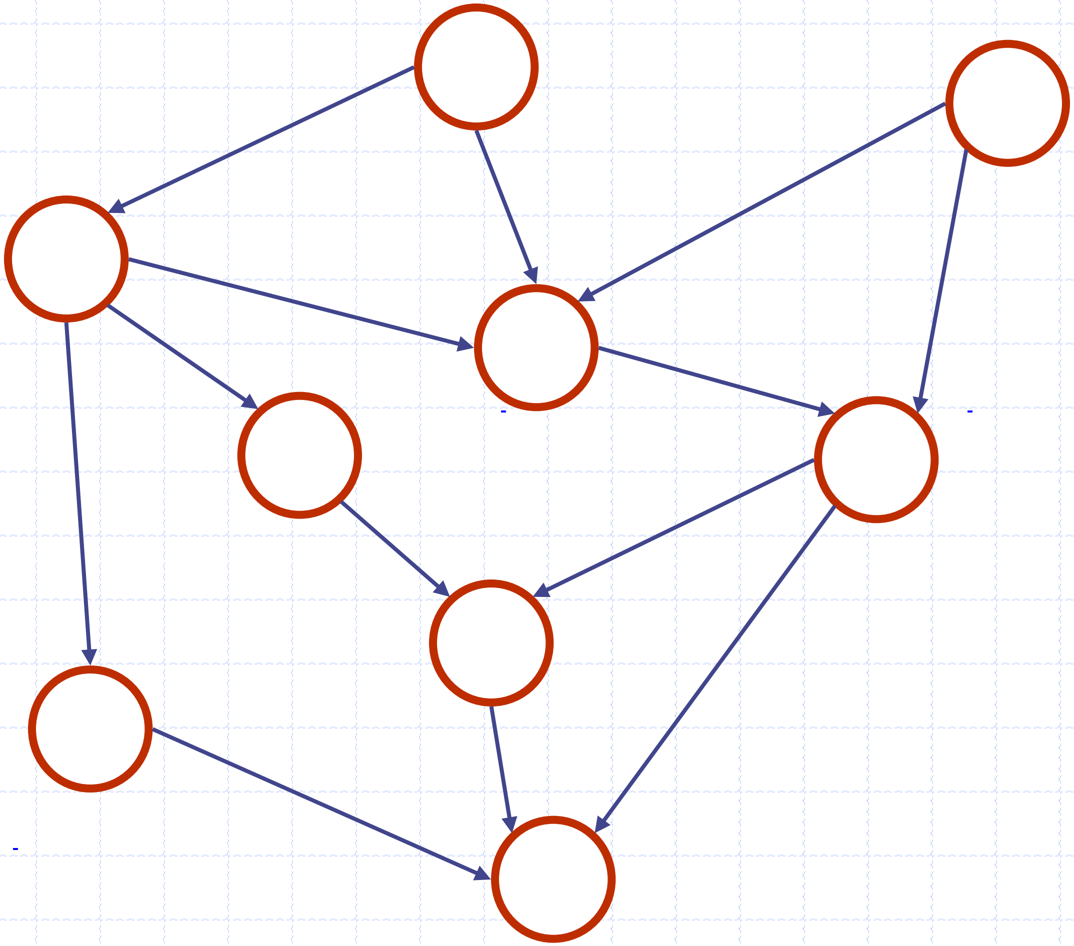
Implementation with DFS

- Simulate the algorithm by using depth-first search
- O(n+m) time.

Algorithm *topologicalDFS*(*G*)
  Input dag *G*
  Output topological ordering of *G*
 $n \leftarrow G.numVertices()$
 for all $u \in G.vertices()$
  setLabel(*u*, *UNEXPLORED*)
 for all $v \in G.vertices()$
  if *getLabel*(*v*) = *UNEXPLORED*
   topologicalDFS(*G*, *v*)

Algorithm *topologicalDFS*(*G*, *v*)
  Input graph *G* and a start vertex *v* of *G*
  Output labeling of the vertices of *G* in the connected component of *v*
 setLabel(*v*, *VISITED*)
 for all $e \in G.outEdges(v)$
  { outgoing edges }
  $w \leftarrow opposite(v,e)$
  if *getLabel*(*w*) = *UNEXPLORED*
   { *e* is a discovery edge }
   topologicalDFS(*G*, *w*)
  else
   { *e* is a forward or cross edge }
 Label *v* with topological number *n*
 $n \leftarrow n - 1$

Topological Sorting Example



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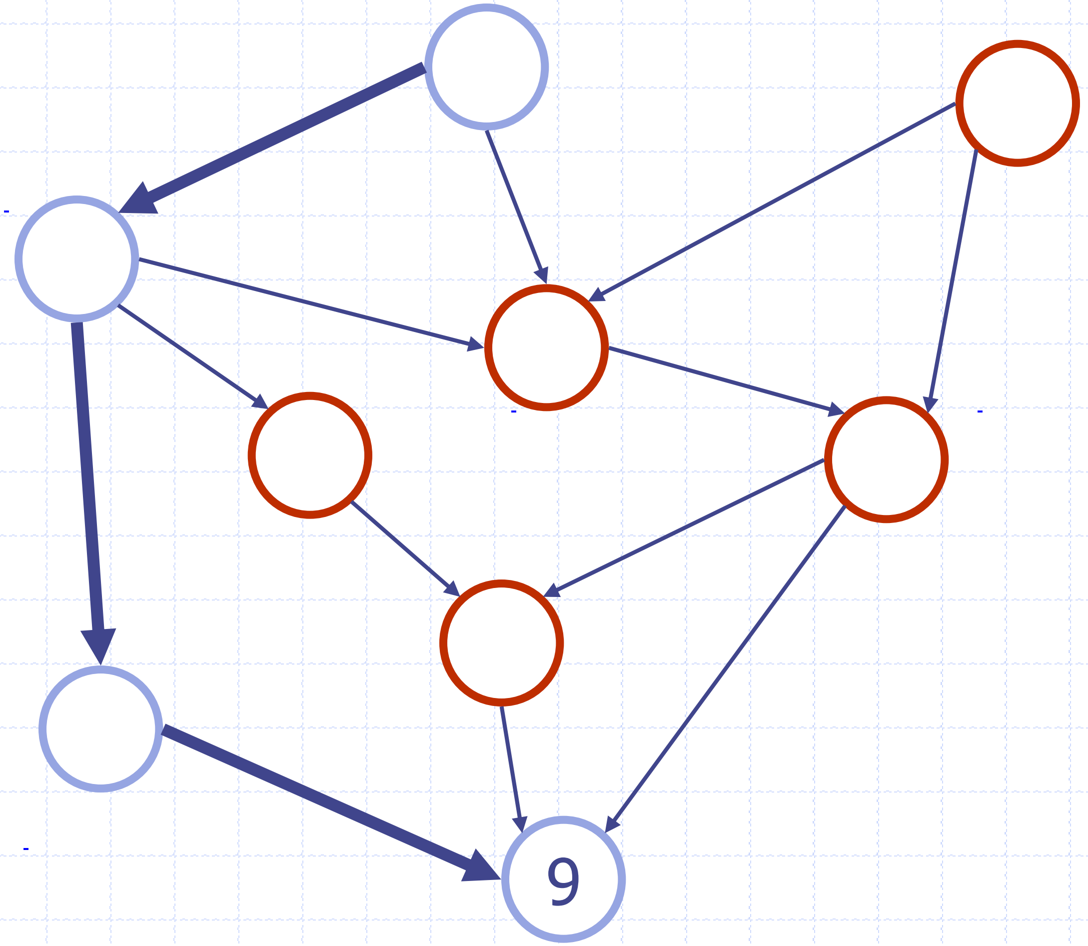

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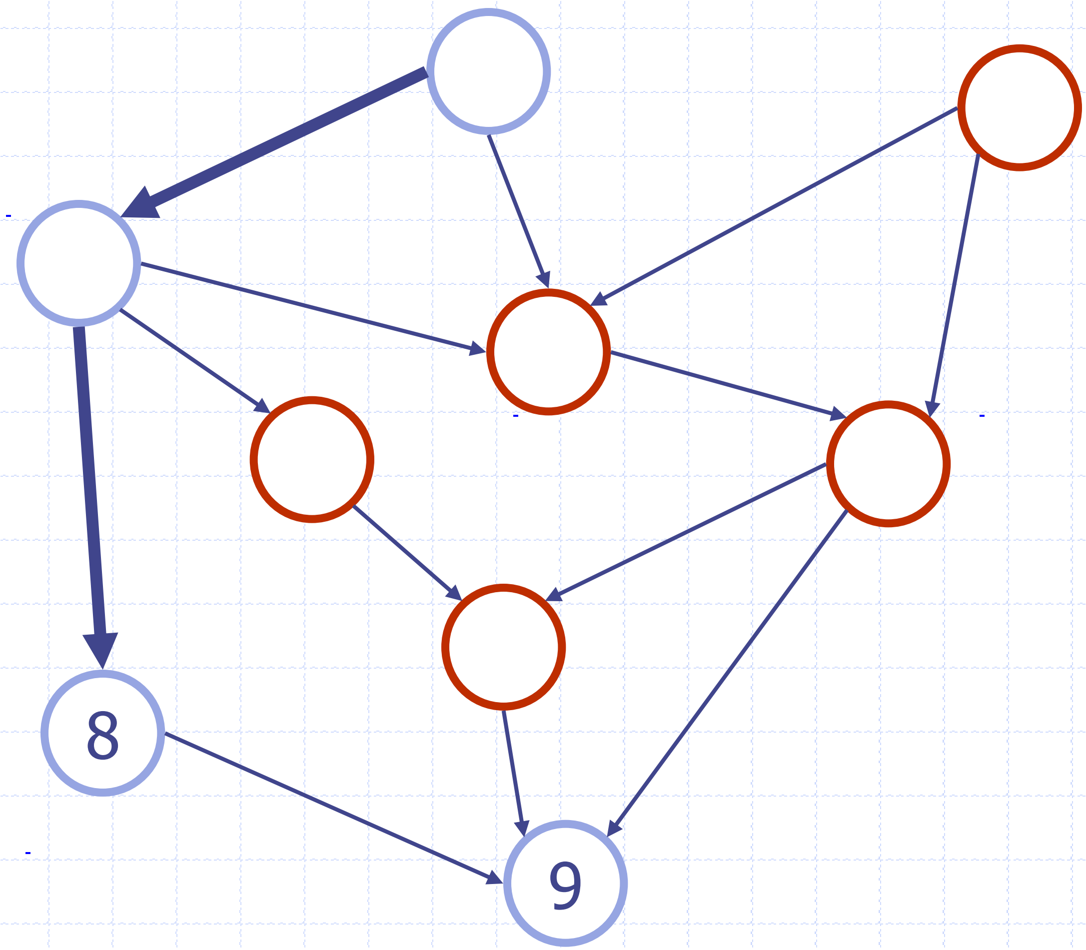

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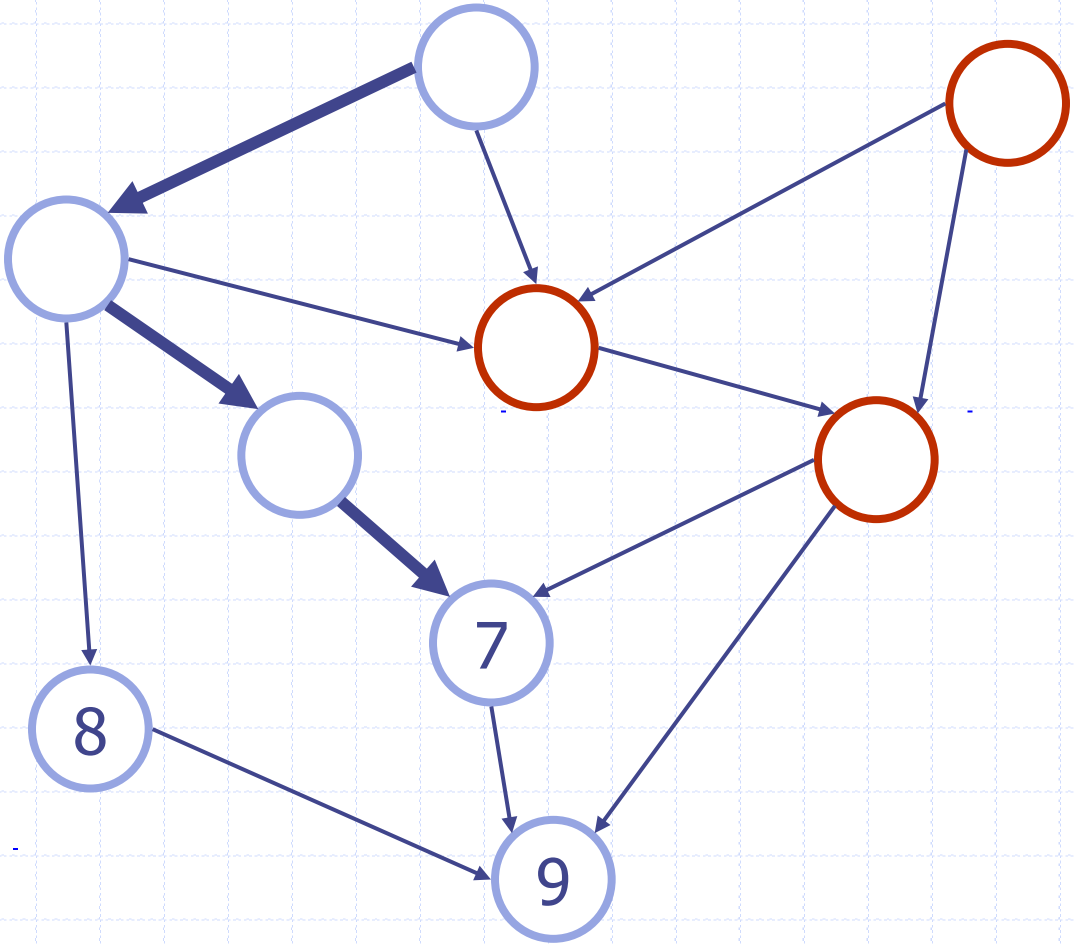
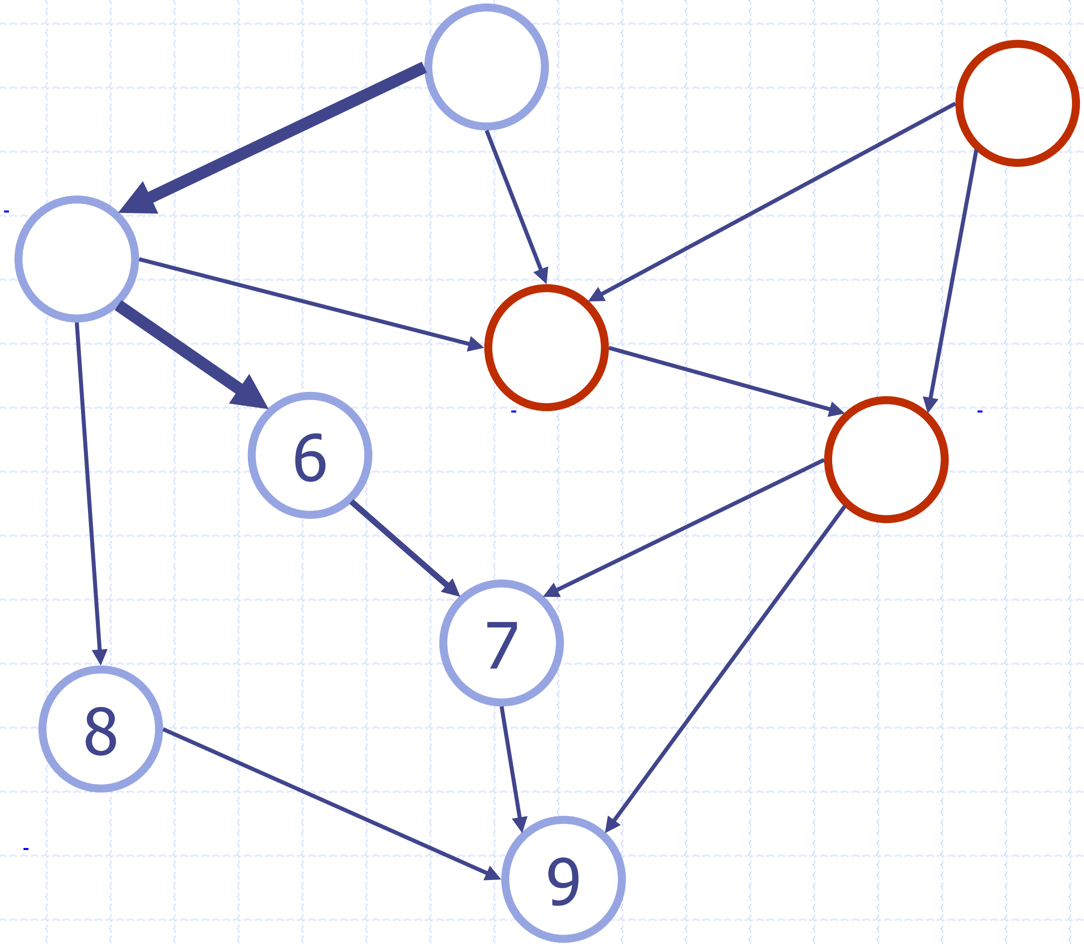


Topological Sorting Example



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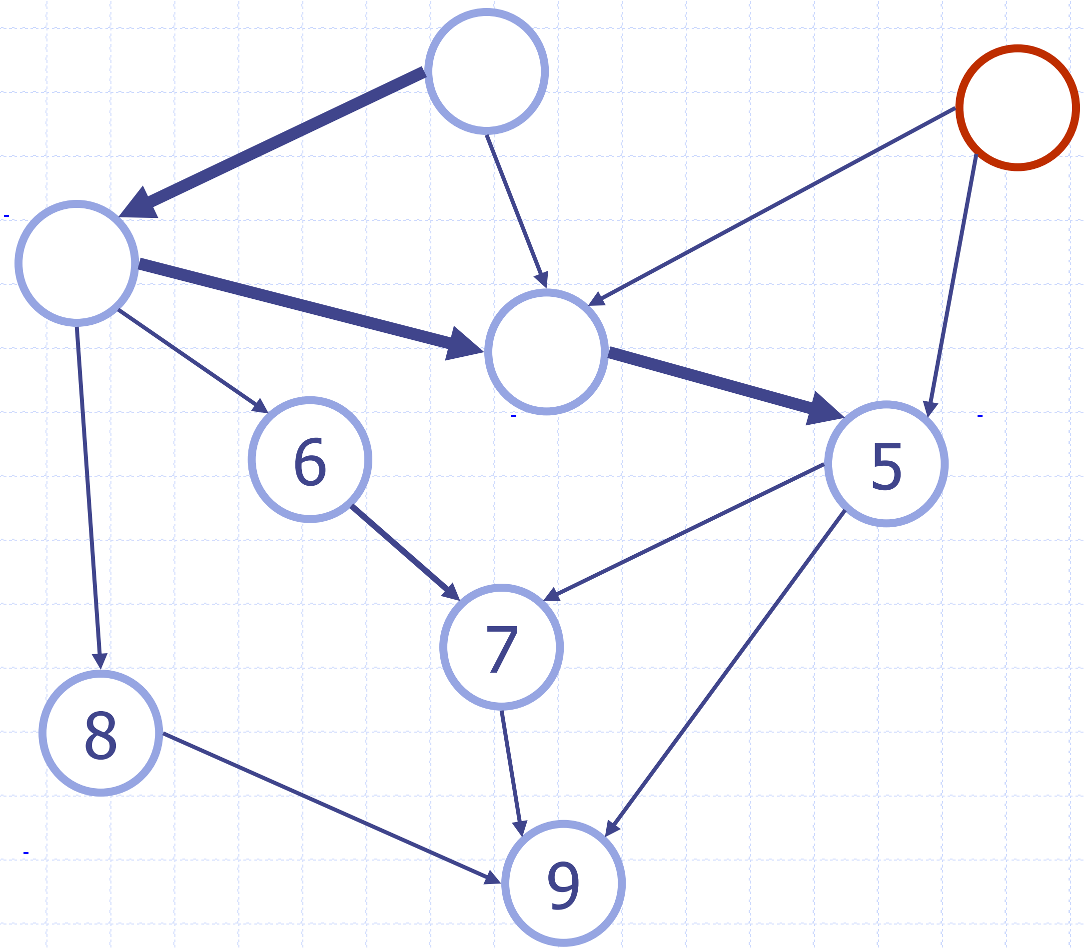

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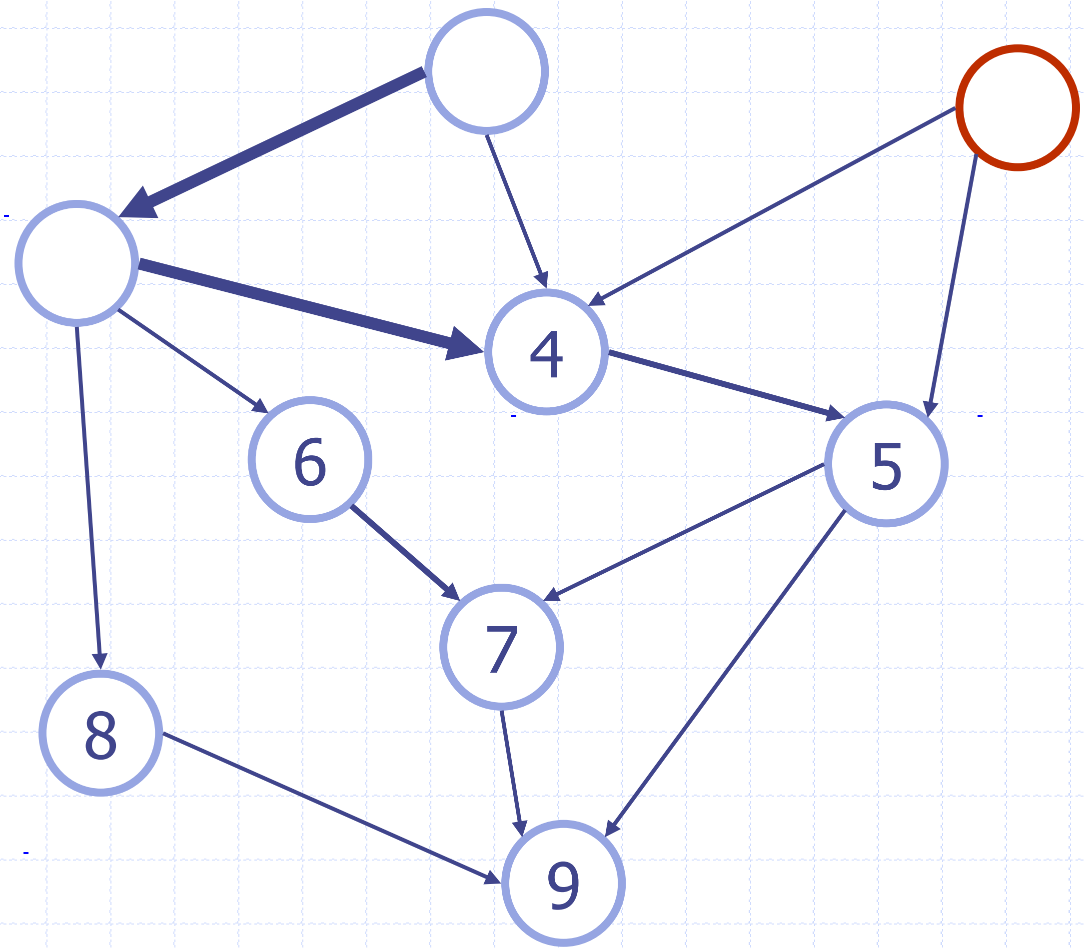

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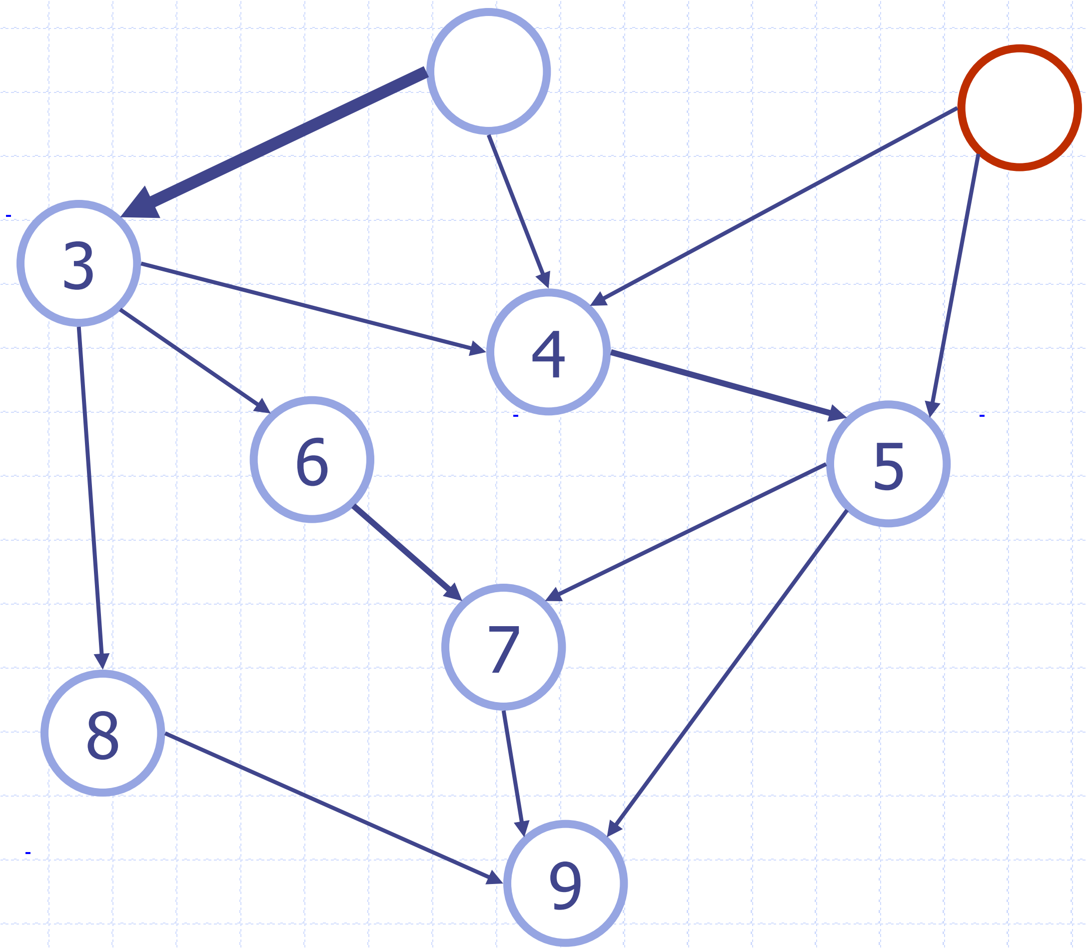

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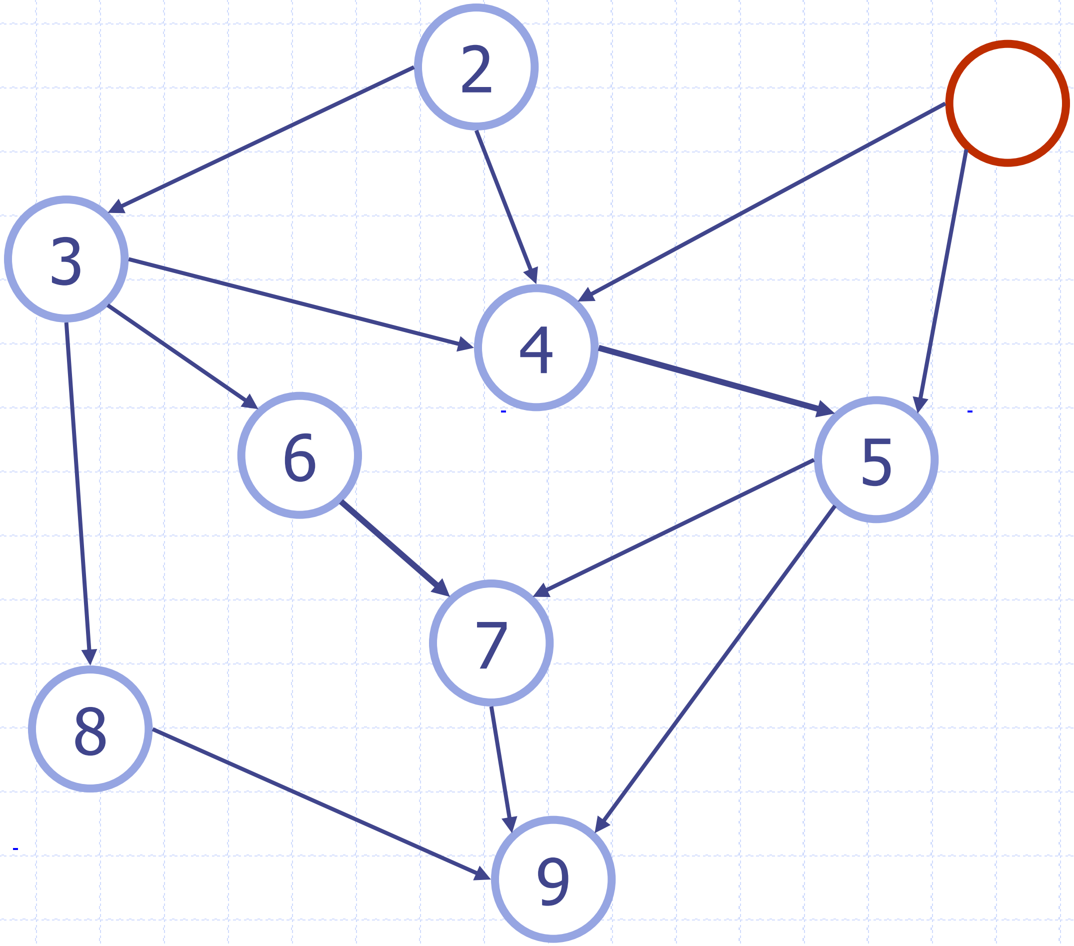

Topological Sorting Example