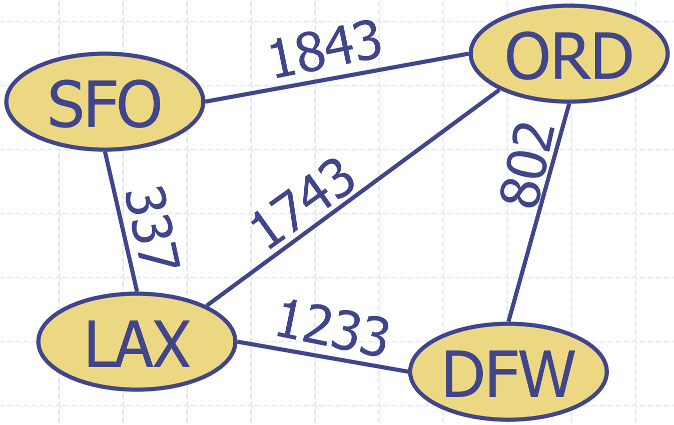
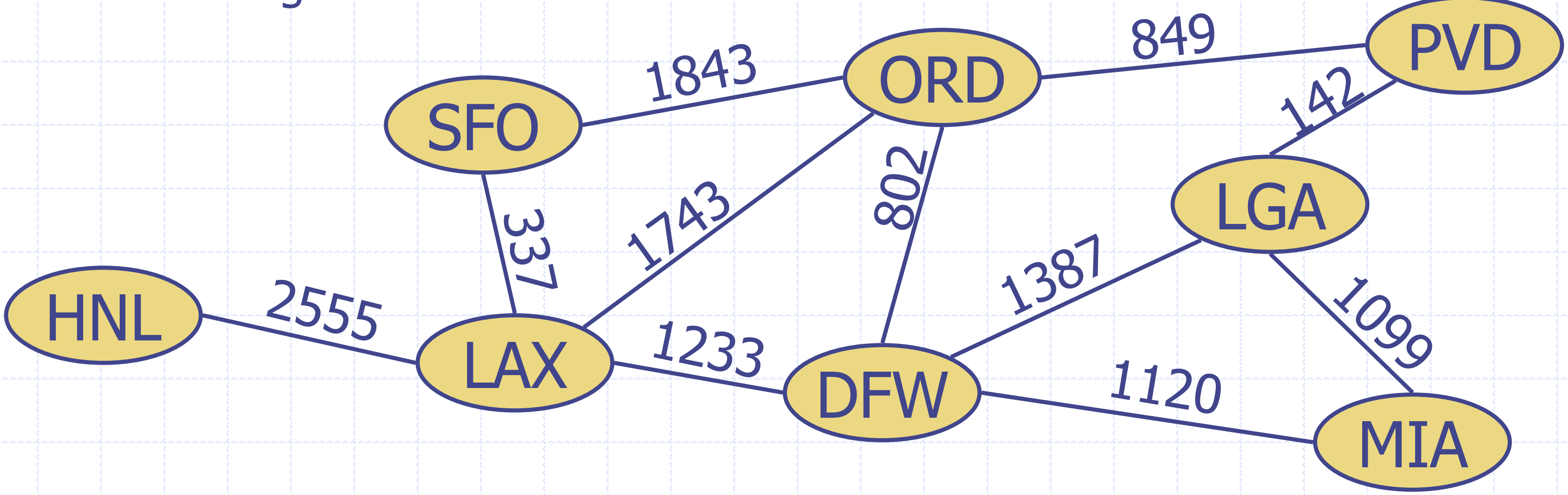
Graphs



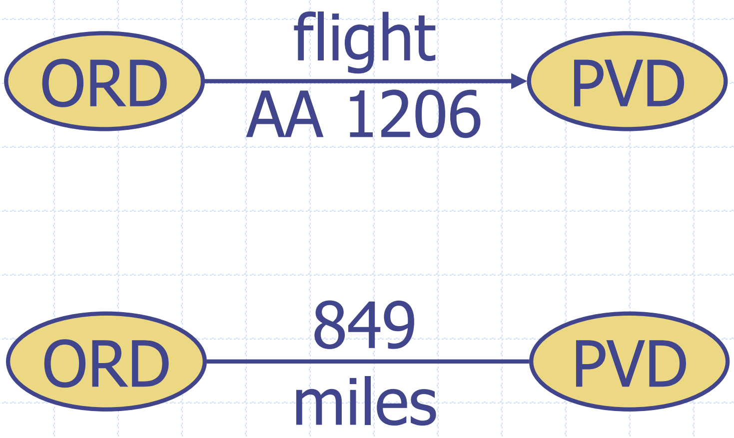
Graphs

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



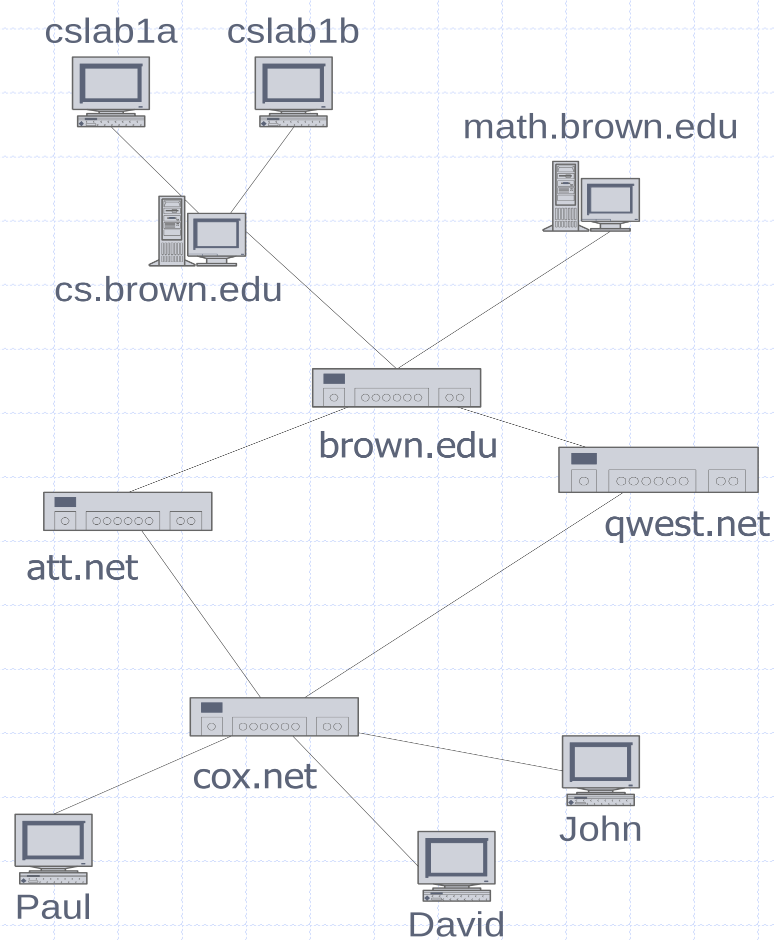
Edge Types

- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a flight route
- Directed graph
  - all the edges are directed
  - e.g., route network
- Undirected graph
  - all the edges are undirected
  - e.g., flight network



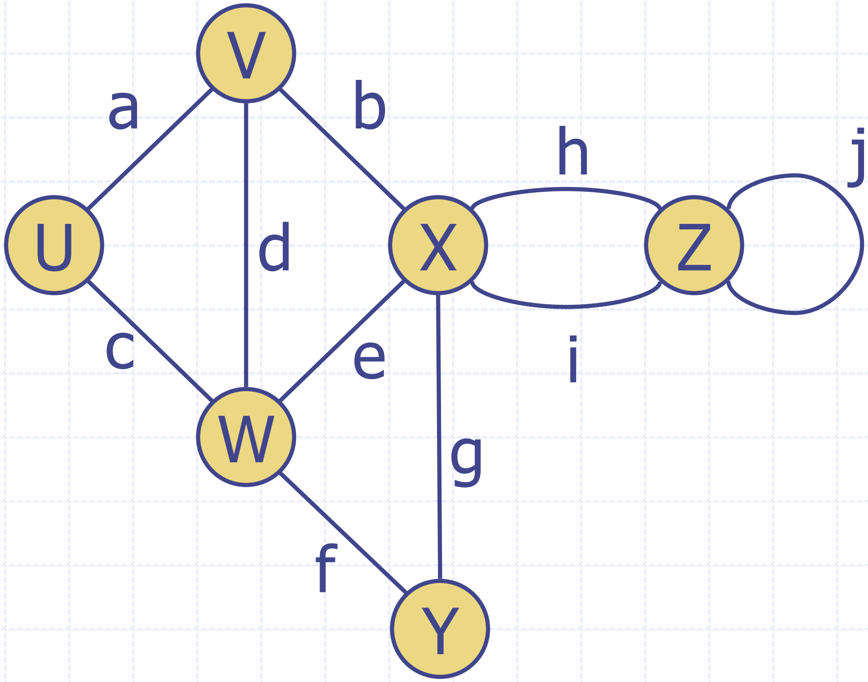
Applications

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



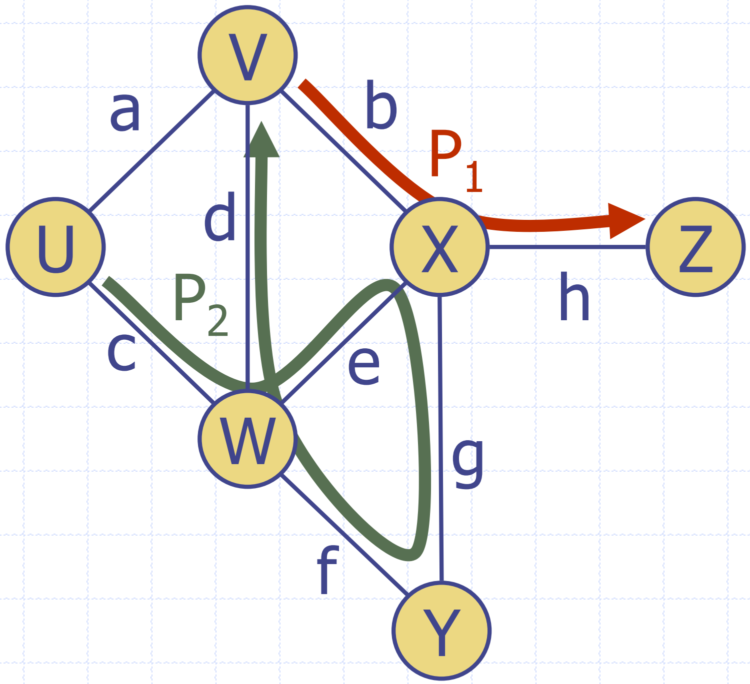
Terminology

- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop



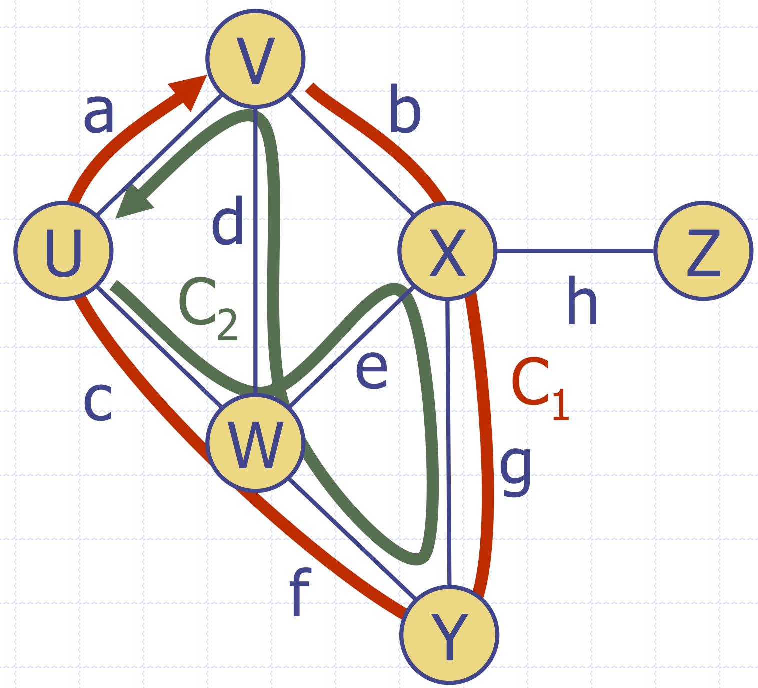
Terminology (cont.)

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



Terminology (cont.)

- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,↵) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,↵) is a cycle that is not simple



Properties

Property 1

$\sum_v \deg(v) = 2m$

Proof: each edge is counted twice

Property 2

In an undirected graph with no self-loops and no multiple edges

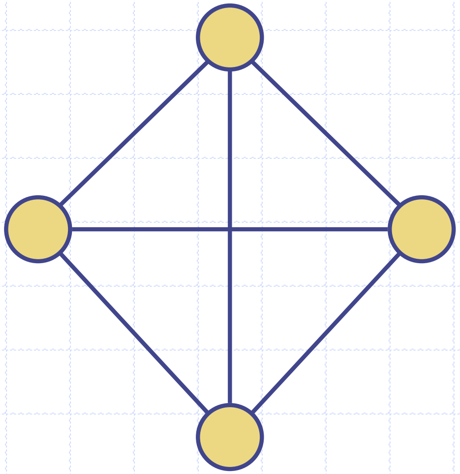
$m \leq n\,(n-1)/2$

Proof: each vertex has degree at most $(n-1)$

What is the bound for a directed graph?

Notation

n	number of vertices
m	number of edges
$\deg(v)$	degree of vertex v



Example

- $n = 4$
- $m = 6$
- $\deg(v) = 3$

Vertices and Edges

- A **graph** is a collection of **vertices** and **edges**.
- We model the abstraction as a combination of three data types: Vertex, Edge, and Graph.
- A **Vertex** is a lightweight object that stores an arbitrary element provided by the user (e.g., an airport code)
  - We assume it supports a method, element(), to retrieve the stored element.
- An **Edge** stores an associated object (e.g., a flight number, travel distance, cost), retrieved with the element( ) method.
- In addition, we assume that an Edge supports the following methods:

endpoints( ): Return a tuple (u, v) such that vertex u is the origin of the edge and vertex v is the destination; for an undirected graph, the orientation is arbitrary.

opposite(v): Assuming vertex v is one endpoint of the edge (either origin or destination), return the other endpoint.

Graph ADT

vertex_count( ):	Return the number of vertices of the graph.
vertices( ):	Return an iteration of all the vertices of the graph.
edge_count( ):	Return the number of edges of the graph.
edges( ):	Return an iteration of all the edges of the graph.
get_edge(u,v):	Return the edge from vertex u to vertex v, if one exists; otherwise return None. For an undirected graph, there is no difference between get_edge(u,v) and get_edge(v,u).
degree(v, out=True):	For an undirected graph, return the number of edges incident to vertex v. For a directed graph, return the number of outgoing (resp. incoming) edges incident to vertex v, as designated by the optional parameter.
incident_edges(v, out=True):	Return an iteration of all edges incident to vertex v. In the case of a directed graph, report outgoing edges by default; report incoming edges if the optional parameter is set to False.
insert_vertex(x=None):	Create and return a new Vertex storing element x.
insert_edge(u, v, x=None):	Create and return a new Edge from vertex u to vertex v, storing element x (None by default).
remove_vertex(v):	Remove vertex v and all its incident edges from the graph.
remove_edge(e):	Remove edge e from the graph.

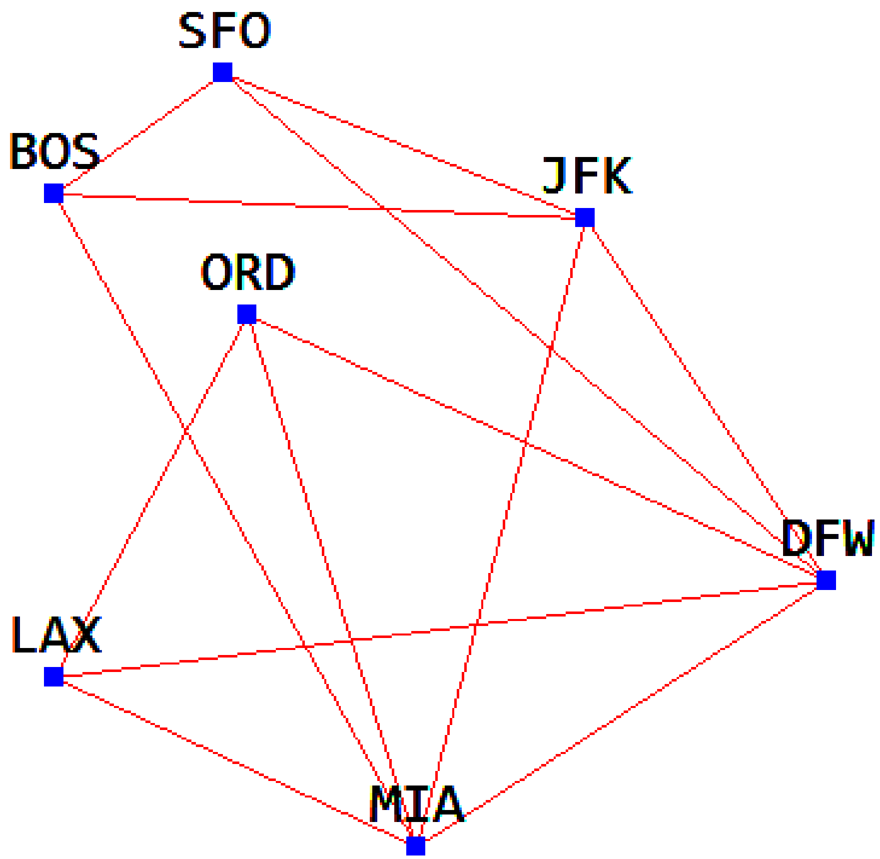
Graph ADT: Basic Usage

```
def basic_graph_example_1():
    g = Graph()
    v1 = g.insert_vertex(1)
    v2 = g.insert_vertex(2)
    v3 = g.insert_vertex(3)
    v4 = g.insert_vertex(4)
    v5 = g.insert_vertex(5)

    e1 = g.insert_edge(v1,v4)
    e2 = g.insert_edge(v3,v1)
    e3 = g.insert_edge(v5,v3)
    e4 = g.insert_edge(v2,v5)

    print "Vertices:"
    for v in g.vertices():
        print v.element()

    print "Edges:"
    for e in g.edges():
        a,b = e.endpoints()
        print a.element(), b.element()
```

Graph ADT: Airport Map Example

```
loc = {
        'BOS':  (80,90),        # BASCO Airport
        'SFO':  (150,40),       # San Francisco International Airport
        'JFK':  (300,100),      # John F. Kennedy Airport, NY
        'MIA':  (230,360),      # Miami Airport, Florida
        'DFW':  (400,250),      # Dallas/Fort Worth International Airport
        'ORD':  (160,140),      # Chicago O'Hare International Airport
        'LAX':  (80,290),       # Los Angeles International Airport
      }

E = (   # Airport connections
        ('BOS','SFO'), ('BOS','JFK'), ('BOS','MIA'), ('JFK','BOS'),
        ('JFK','DFW'), ('JFK','MIA'), ('JFK','SFO'), ('ORD','DFW'),
        ('ORD','MIA'), ('LAX','ORD'), ('DFW','SFO'), ('DFW','ORD'),
        ('DFW','LAX'), ('MIA','DFW'), ('MIA','LAX'),
    )
```

Graph ADT: Graphical View



Graph ADT: Code

```
def draw_airport_map():
    g = Graph(True)  # directed graph !
    vert = dict()    # dictionary from label to vertex object
    for a in loc:
        vert[a] = g.insert_vertex(a)

    for a,b in E:
        g.insert_edge(vert[a], vert[b])

    for v in g.vertices():
        airport = v.element()
        p = Point(*loc[airport])
        p.draw()
        p.text(airport)

    for e in g.edges():
        a, b = e.endpoints()
        x1, y1 = loc[a.element()]
        x2, y2 = loc[b.element()]
        l = Line.from_coords(x1, y1, x2, y2)
        l.draw(fill="red", width=1, arrow="last", arrowshape=[10,14,4])
```

```
loc = {
        'BOS':  (80,90),        # BASCO Airport
        'SFO':  (150,40),       # San Francisco International Airport
        'JFK':  (300,100),      # John F. Kennedy Airport, NY
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        'LAX':  (80,290),       # Los Angeles International Airport
      }

E = (   # Airport connections
        ('BOS','SFO'), ('BOS','JFK'), ('BOS','MIA'), ('JFK','BOS'),
        ('JFK','DFW'), ('JFK','MIA'), ('JFK','SFO'), ('ORD','DFW'),
        ('ORD','MIA'), ('LAX','ORD'), ('DFW','SFO'), ('DFW','ORD'),
        ('DFW','LAX'), ('MIA','DFW'), ('MIA','LAX'),
    )
```
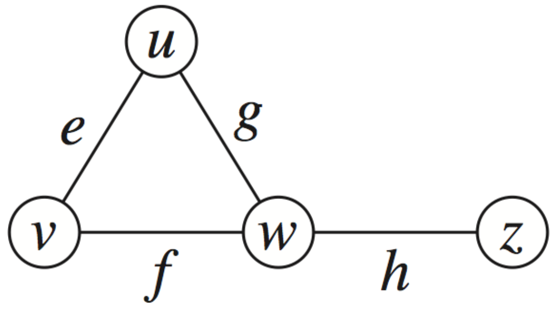
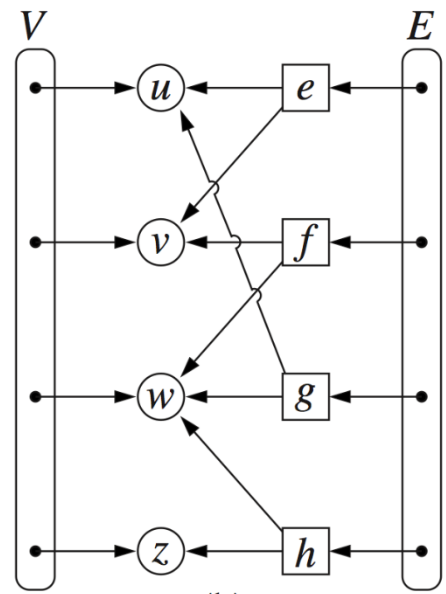

Edge List Structure

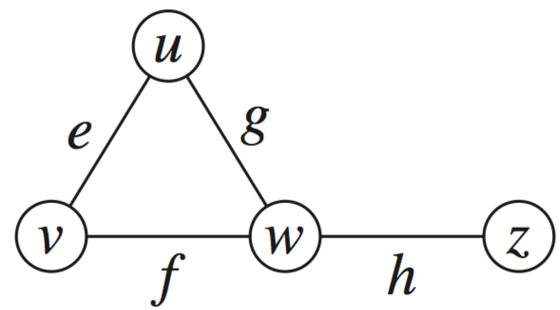
- Vertex object
  - element
  - reference to position in vertex sequence
- Edge object
  - element
  - origin vertex object
  - destination vertex object
  - reference to position in edge sequence
- Vertex sequence
  - sequence of vertex objects
- Edge sequence
  - sequence of edge objects

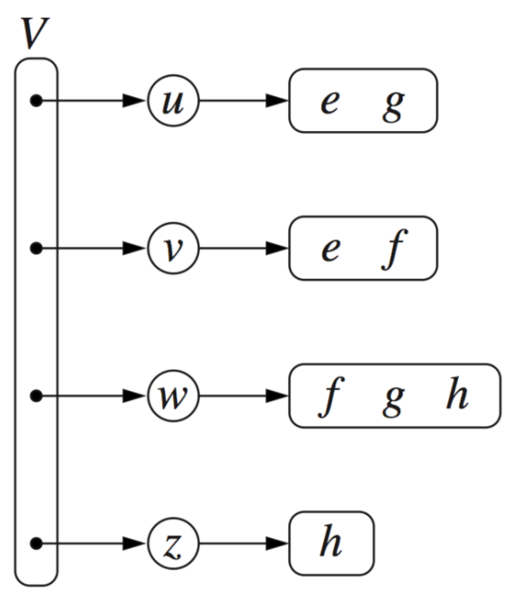




Adjacency List Structure

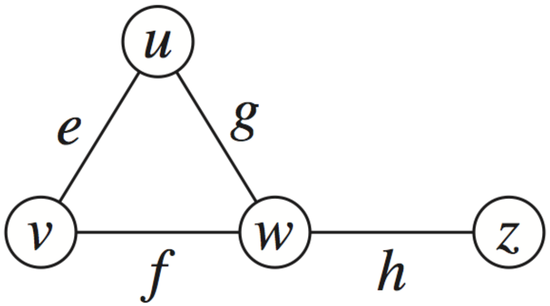
- Incidence sequence for each vertex
  - sequence of references to edge objects of incident edges
- Augmented edge objects
  - references to associated positions in incidence sequences of end vertices





Adjacency Matrix Structure

- Edge list structure
- Augmented vertex objects
  - Integer key (index) associated with vertex
- 2D-array adjacency array
  - Reference to edge object for adjacent vertices
  - Null for non nonadjacent vertices
- The "old fashioned" version just has 0 for no edge and 1 for edge



		0	1	2	3
u →	0		e	g	
v →	1	e		f	
w →	2	g	f		h
z →	3			h	

Performance

▪ n vertices, m edges ▪ no parallel edges ▪ no self-loops	Edge List	Adjacency List	Adjacency Matrix
Space	$n + m$	$n + m$	n^2
incidentEdges(v)	m	$\deg(v)$	n
areAdjacent (v, w)	m	$\min(\deg(v), \deg(w))$	1
insertVertex(o)	1	1	n^2
insertEdge(v, w, o)	1	1	1
removeVertex(v)	m	$\deg(v)$	n^2
removeEdge(e)	1	1	1

Python Graph Implementation

- We use a variant of the ***adjacency map*** representation.
- For each vertex v, we use a Python dictionary to represent the secondary incidence map $I(v)$.
- The list V is replaced by a top-level dictionary D that maps each vertex v to its incidence map $I(v)$.
  - Note that we can iterate through all vertices by generating the set of keys for dictionary D.
- A vertex does not need to explicitly maintain a reference to its position in D, because it can be determined in $O(1)$ expected time.
- Running time bounds for the adjacency-list graph ADT operations, given above, become ***expected*** bounds.

Vertex Class

```
#------------------------- nested Vertex class -------------------------
class Vertex:
  """Lightweight vertex structure for a graph."""
  __slots__ = '_element'

  def __init__(self, x):
    """Do not call constructor directly. Use Graph's insert_vertex(x)."""
    self._element = x

  def element(self):
    """Return element associated with this vertex."""
    return self._element

  def __hash__(self):          # will allow vertex to be a map/set key
    return hash(id(self))
```

Edge Class

```
#------------------------ nested Edge class ------------------------
class Edge:
  """Lightweight edge structure for a graph."""
  __slots__ = '_origin', '_destination', '_element'

  def __init__(self, u, v, x):
    """Do not call constructor directly. Use Graph's insert_edge(u,v,x)."""
    self._origin = u
    self._destination = v
    self._element = x

  def endpoints(self):
    """Return (u,v) tuple for vertices u and v."""
    return (self._origin, self._destination)

  def opposite(self, v):
    """Return the vertex that is opposite v on this edge."""
    return self._destination if v is self._origin else self._origin

  def element(self):
    """Return element associated with this edge."""
    return self._element

  def __hash__(self):              # will allow edge to be a map/set key
    return hash( (self._origin, self._destination) )
```

Graph, Part 1

```
class Graph:
  """Representation of a simple graph using an adjacency map."""

  def __init__(self, directed=False):
    """Create an empty graph (undirected, by default).

    Graph is directed if optional paramter is set to True.
    """
    self._outgoing = { }
    # only create second map for directed graph; use alias for undirected
    self._incoming = { } if directed else self._outgoing

  def is_directed(self):
    """Return True if this is a directed graph; False if undirected.

    Property is based on the original declaration of the graph, not its contents.
    """
    return self._incoming is not self._outgoing # directed if maps are distinct

  def vertex_count(self):
    """Return the number of vertices in the graph."""
    return len(self._outgoing)

  def vertices(self):
    """Return an iteration of all vertices of the graph."""
    return self._outgoing.keys()

  def edge_count(self):
    """Return the number of edges in the graph."""
    total = sum(len(self._outgoing[v]) for v in self._outgoing)
    # for undirected graphs, make sure not to double-count edges
    return total if self.is_directed() else total // 2

  def edges(self):
    """Return a set of all edges of the graph."""
    result = set()          # avoid double-reporting edges of undirected graph
    for secondary_map in self._outgoing.values():
      result.update(secondary_map.values())        # add edges to resulting set
    return result
```

Graph, end

```
def get_edge(self, u, v):
  """Return the edge from u to v, or None if not adjacent."""
  return self._outgoing[u].get(v)          # returns None if v not adjacent

def degree(self, v, outgoing=True):
  """Return number of (outgoing) edges incident to vertex v in the graph.

  If graph is directed, optional parameter used to count incoming edges.
  """
  adj = self._outgoing if outgoing else self._incoming
  return len(adj[v])

def incident_edges(self, v, outgoing=True):
  """Return all (outgoing) edges incident to vertex v in the graph.

  If graph is directed, optional parameter used to request incoming edges.
  """
  adj = self._outgoing if outgoing else self._incoming
  for edge in adj[v].values():
    yield edge

def insert_vertex(self, x=None):
  """Insert and return a new Vertex with element x."""
  v = self.Vertex(x)
  self._outgoing[v] = { }
  if self.is_directed():
    self._incoming[v] = { }          # need distinct map for incoming edges
  return v

def insert_edge(self, u, v, x=None):
  """Insert and return a new Edge from u to v with auxiliary element x."""
  e = self.Edge(u, v, x)
  self._outgoing[u][v] = e
  self._incoming[v][u] = e
```